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Research Article

On Arc Connectivity of Direct-Product Digraphs

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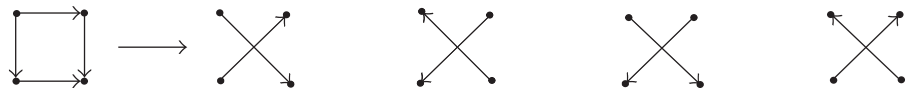
Four natural orientations of the direct product of two digraphs are introduced in this paper. Sufficient and necessary conditions for these orientations to be strongly connected are presented, as well as an explicit expression of the arc connectivity of a class of direct-product digraphs.

1. Introduction

Various product operations are employed for constructing larger networks from smaller ones, among which direct-product operation is the most frequently employed one. The direct product of two graphs G_1 and G_2, denoted by $G_1 \times G_2$, is defined on vertex set $V(G_1) \times V(G_2)$, where two vertices (x_1, x_2) and (y_1, y_2) are adjacent to each other in $G_1 \times G_2$ if and only if $x_1 y_1 \in E(G_1)$ and $x_2 y_2 \in E(G_2)$. Other names for direct product are tensor product, categorical product, Kronecker product, cardinal product, relational product, and weak direct product [1]. Some basic connectivity properties of direct-product graphs are presented in [2, 3] and elsewhere. Specially, the authors characterize connected product graphs by presenting the following Theorem 1.1 in [1]; an explicit expression of the connectivity of a direct-product graph is presented in [4].

Theorem 1.1 (see [1, Theorem 5.29]). *Let G and H be connected nonempty graphs. Then $G \times H$ is connected if and only if at least one of them is nonbipartite. Furthermore, if both G and H are bipartite, then $G \times H$ has exactly two components.*

Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be two digraphs. It is naturally to define the vertex set and adjacency relationship between vertices of direct product of digraph as those of undirected graphs. But the orientation is not unique, for every pair of arcs $(x_1, y_1) \in A_1$ and $(x_2, y_2) \in A_2$, there are four natural orientations of the two edges $(x_1, x_2)(y_1, y_2)$ and



$\vec{K}_2 \square \vec{K}_2$ (1) (2) (3) (4)

Figure 1: Orientations of direct-product digraph $\vec{K}_2 \times \vec{K}_2$.

$(y_1, x_2)(x_1, y_2)$. For clarity and comparison, we depicture the Cartesian product $\vec{K}_2 \square \vec{K}_2$ and the four orientations of direct-product digraph $\vec{K}_2 \times \vec{K}_2$ in Figure 1.

A digraph is strongly connected or disconnected if any vertex is reachable from any other vertex, where a vertex u is said to be reachable form another vertex v if there is a directed path from v to u. The minimum number $\lambda(D)$ of arcs needed to be removed for destroying the strong connectivity of a digraph is called its arc connectivity. This work characterizes strongly connected direct-product digraphs of the above four orientations. We follow [5] for symbols and terminology not specified in this work.

2. Strongly Connected Direct-Product Digraphs

As will be shown in next section, the above four orientations can be transformed by one another to some extent. So we assume in this section that $D_1 \times D_2$ contains an arc from vertex (x_1, x_2) to (y_1, y_2) if and only if arc $(x_2, y_2) \in A(D_2)$, and $(x_1, y_1) \in A(D_1)$ or $(y_1, x_1) \in A(D_1)$.

Lemma 2.1. *Let D be a digraph. Then $\vec{K}_2 \times D$ is strongly connected if and only if D is nonbipartite and strongly connected.*

Proof. Necessity. If D is bipartite, then by Theorem 1.1 the underlying graph of $\vec{K}_2 \times D$ is disconnected. This contradiction shows that D is nonbipartite. Let $\{a, b\}$ be the vertex set of $\vec{K}_2$. Since $\vec{K}_2 \times D$ is strongly connected, it follows that for any two vertices $u, v \in V(D)$, $\vec{K}_2 \times D$ contains a directed walk W_1 from (a, u) to (a, v). Let $W_1 = (a, u)(b, v_1)(a, v_2) \cdots (b, v_k)(a, v)$. Then it yields a directed walk $uv_1v_2 \cdots v_kv$ of D. The necessity follows from this observation.

Sufficiency. Since D is strongly connected, it contains a spanning closed directed walk $W = u_0v_1v_2 \cdots v_ku_0$, which corresponds two directed walks $W' = (a, u_0)(b, v_1)(a, v_2) \cdots (x, u_0)$ and $W'' = (b, u_0)(a, v_1)(b, v_2) \cdots (y, u_0)$ of $\vec{K}_2 \times D$, where $x, y \in \{a, b\}$. It is obvious that $W' \cup W''$ is a spanning subgraph of $V(\vec{K}_2 \times D)$.

Case 1. The length of $W = u_0v_1v_2 \cdots v_ku_0$ is odd, say, $k = 2n$.

In this case, $W' = (a, u_0)(b, v_1)(a, v_2) \cdots (a, v_{2n})(b, u_0)$ and $W'' = (b, u_0)(a, v_1)(b, v_2) \cdots (b, v_{2n})(a, u_0)$. And so $W' \cup W''$ is a spanning closed directed walk of $\vec{K}_2 \times D$. Hence, $\vec{K}_2 \times D$ is strongly connected.

Case 2. The length of $W = u_0v_1v_2 \cdots v_ku_0$ is even, say, $k = 2n + 1$.

In this case, both $W' = (a, u_0)(b, v_1)(a, v_2) \cdots (a, v_{2n})(a, u_0)$ and $W'' = (b, u_0)(a, v_1)(b, v_2) \cdots (b, v_{2n})(b, u_0)$ are closed directed walks. Since D is nonbipartite, it contains an arc joining two vertices of W whose suffixes have same parity. Without loss of generality, let $(u_0, v_{2l}) \in A(D)$. This arc corresponds to two arcs $((a, u_0), (b, v_{2l}))$, $((b, u_0), (a, v_{2l}))$ of $\vec{K}_2 \times D$. These two arcs make W' and W'' reachable from each other. Therefore, $\vec{K}_2 \times D$ is strongly connected. □

Theorem 2.2. *Let D_1 and D_2 be two digraphs. Then $D_1 \times D_2$ is strongly connected if and only if D_2 is strongly connected and D_1 or D_2 is nonbipartite.*

Proof. Sufficiency. If D_2 is nonbipartite and strongly connected, by Lemma 2.1, $D_1 \times D_2$ is strongly connected.

If D_2 is strongly connected and bipartite, but D_1 is nonbipartite, then D_2 contains a spanning closed directed walk $W_2 = u_0u_1u_2 \cdots u_k u_0$. Since D_2 is bipartite, it follows that k is odd. Let $W_1 = v_0v_1v_2 \cdots v_i \cdots v_{2m+i}v_i \cdots v_s v_0$ be a spanning closed walk of D_1, where $C = v_i \cdots v_{2m+i}v_i$ is an odd cycle. For any vertex $v_j \in V(W_1)$ and any one of its neighbor v_l in W_1, $W'_j = (v_j, u_0)(v_l, u_1)(v_j, u_2) \cdots (v_l, u_k)(v_j, u_0) \cong W_2$ and $V(D) = \cup_{i=0}^{s} V(W'_i)$. Since v_j and v_l are adjacent in W_1, it follows that $l = j + 1$ or $2m + j$. Let

$$\begin{aligned} W'' &= (v_j, u_0)(v_{j+1}, u_1)(v_j, u_2) \cdots (v_{j+1}, u_k)(v_j, u_0), \\ W''' &= (v_j, u_0)(v_{2m+j}, u_1)(v_j, u_2) \cdots (v_{2m+j}, u_k)(v_j, u_0). \end{aligned} \tag{2.1}$$

The sum of the suffixes of each vertex in W'' has the same parity as the integer j. Since v_j may be any vertex of W_1, it follows that the vertices of $V(D_1) \times V(D_2)$ whose suffix sum has same parity induce a strong component (a strongly connected vertex-induced subgraph with as many as possible vertices). Since W''' contains vertices with odd suffix sum as well vertices with even suffix sum, it follows that $D_1 \times D_2$ is strongly connected.

Necessity. Since $D_1 \times D_2$ is strongly connected, every two vertices (u_i, v_j) and (u_k, v_l) are reachable from each other in $D_1 \times D_2$. It follows that v_j and v_l are reachable from each other in D_2. Hence, D_2 is strongly connected. If D_2 is strongly connected, but both D_1 and D_2 are bipartite, then $D_1 \times D_2$ is not strongly connected by Lemma 2.1. The necessity follows from this contradiction. □

3. Relationship of the Four Orientations

The last three orientations of Figure 1 can be defined as follows, respectively.

Orientation 2. Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be two digraphs. $D_1 \times D_2$ contains an arc from (x_1, x_2) to (y_1, y_2) if and only if $(y_2, x_2) \in A_2$, and (y_1, x_1) or $(x_1, y_1) \in A_1$.

Orientation 3. Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be two digraphs. $D_1 \times D_2$ contains an arc from (x_1, x_2) to (y_1, y_2) if and only if $(x_1, y_1) \in A_1$, and (x_2, y_2) or $(y_2, x_2) \in A_2$.

Orientation 4. Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be two digraphs. $D_1 \times D_2$ has an arc from (x_1, x_2) to (y_1, y_2) if and only if $(y_1, x_1) \in A_1$, and (x_2, y_2) or $(y_2, x_2) \in A_2$.

The readers are suggested to refer to (2), (3), and (4) of Figure 2 for Orientations 2, 3, and 4 respectively. It is not difficult to see that Orientation 2 is the converse of the first orientation in Figure 1. And so, the following Corollary 3.1 follows directly from Theorem 2.2. Similarly, if $D_2 \times D_1$ has Orientation 1 (refer to (2) of Figure 1) and $D_1 \times D_2$ has Orientation 3 then $D_2 \times D_1 \cong D_1 \times D_2$. From this observation and Corollary 3.1, Corollary 3.2 follows directly.

Corollary 3.1. *Let $D_1 \times D_2$ be oriented as Orientation 2. Then it is strongly connected if and only if D_2 is strongly connected and D_1 or D_2 is nonbipartite.*

Corollary 3.2. *Let $D_1 \times D_2$ be oriented as in Orientations 3 or 4. Then it is strongly connected if and only if D_1 is strongly connected and D_1 or D_2 is nonbipartite.*

4. Strong Arc Connectivity

Let D be a strongly arc connected digraph and $X \subsetneq V(D)$ be a nonempty vertex set. Denote by $(X, V(D) - X)$ the set of arcs with tail in X and head in $V(D) - X$, which is called a *directed cut* of D. If T is a directed cut of strongly connected digraph D, then $D - T$ is connected but is not strongly connected. The size $\lambda(D)$ of minimum directed cuts of digraph D is called its strong arc connectivity. Let $\beta(D) = \min\{|S| : S \subset A(D)$ such that $D - S$ is a bipartite graph$\}$, $\delta(D) = \min\{\delta^+(D), \delta^-(D)\}$ be the minimum degree of D. For every integer $\delta(D) \geq j \geq \lambda(D)$, let $\beta_j = \min\{\beta(C) : C$ be any strongly connected component of $D - T$ and T is an arbitrary directed cut of D of size $j\}$.

Let $D_1 \times D_2$ be oriented according to the first orientation of Figure 1. For every arc $((x_1, x_2), (y_1, y_2))$ of $D_1 \times D_2$, the arc $(x_2, y_2) \in A(D_2)$ is called its projection on D_2. The following Lemma 4.1 is immediate, so we omit its proof herein.

Lemma 4.1. *If directed graph D is not strongly connected, then it contains two strongly connected components, one of which has no outer neighbors and the other has no inner neighbors.*

Theorem 4.2. *If $\vec{K}_2 \times D$ is strongly connected, then $\lambda(\vec{K}_2 \times D) = \min\{2\lambda(D), \beta(D), \min\{j + \beta_j : \lambda(D) \leq j \leq \delta\}\}$.*

Proof. By Lemma 2.1, Theorem 4.2 is clearly true in the case when D is bipartite, and so we assume in what follows that $\beta(D) \geq 1$ and D is nonbipartite. Let T be a minimum direct cut of D and let F be a minimum arc-set of D such that $D - F = (X, Y; E)$ is a bipartite graph with bipartition X and Y. Then $\beta(D) = |F|$ and, $\lambda(D) = |T| \geq 1$ by Lemma 2.1.

Claim 1. $\lambda(\vec{K}_2 \times D) \leq \min\{2\lambda(D), \beta(D), \min\{j + \beta_j : \lambda(D) \leq j \leq \delta\}$.

Let $A(\vec{K}_2) = \{(a, b)\}$. By Theorem 1.1, $\vec{K}_2 \times (D - F)$ consists of two components. The vertex sets of these two components are $\{(a, x) : x \in X\} \cup \{(b, y) : y \in Y\}$ and $\{(b, x) : x \in X\} \cup \{(a, y) : y \in Y\}$. It is not difficult to see that $\{((a, x), (b, y)) : (x, y) \in F \cap A(D[X])\} \cup \{((b, x), (a, y)) : (x, y) \in F \cap A(D[Y])\}$ is a directed cut of $\vec{K}_2 \times D$, since its removal makes $\{(b, x) : x \in X\} \cup \{(a, y) : y \in Y\}$ not reachable from $\{(a, x) : x \in X\} \cup \{(b, y) : y \in Y\}$. And so, $\lambda(\overleftarrow{K}_2 \times D) \leq |F| = \beta(D)$.

By Lemma 4.1, $D - T$ contains a strongly connected component D_1 that has no outer neighbors. By the minimality of T, we have $(D_1, \overline{D_1}) = T$. Obviously, $\{((a, x), (b, y)) : (x, y) \in T\} \cup \{((a, y), (b, x)) : (x, y) \in T\}$ is a directed cut of $\vec{K}_2 \times D$. Hence $\lambda(\vec{K}_2 \times D) \leq 2|T| = 2\lambda(D)$.

Let T_j be a directed cut of D that has size j, C_j be a strongly connected component of $D - T_j$ with $(C_j, D - T_j - C_j) = \emptyset$ (by Lemma 4.1 such components exist), F_j be an arc set of C_j such that $C_j - F_j = (X_j, Y_j; E_j)$ is a bipartite subgraph with bipartition (X_j, Y_j). Then $\vec{K}_2 \times (C_j - F_j)$ consists of two components, whose vertex sets are $\{(a, x) : x \in X_j\} \cup \{(b, y) : y \in Y_j\}$ and $\{(b, x) : x \in X_j\} \cup \{(a, y) : y \in Y_j\}$. It's not difficult to see that the union of $\{((a, x), (b, y)) : (x, y) \in F_j \cap A(D[X_j])\}$, $\{((b, x), (a, y)) : (x, y) \in F_j \cap A(D[Y_j])\}$, $\{((a, x), (b, y)) : (x, y) \in (X_j, D - C_j)\}$ and $\{((b, x), (a, y)) : (x, y) \in (Y_j, D - C_j)\}$ is a directed cut of $\vec{K}_2 \times D$, since its removal makes $\{(a,y) : y \in Y_j\} \cup \{(b,x) : x \in X_j\}$ not reachable from $\{(a,x) : x \in X_j\} \cup \{(b,y) : y \in Y_j\}$. Noticing that $|\{((a,x),(b,y)) : (x,y) \in F_j \cap A(D[X_j])\} \cup \{((b,x),(a,y)) : (x,y) \in F_j \cap A(D[Y_j])\}| = |F_j|$, $|\{((a,x),(b,y)) : (x,y) \in (X_j, D - C_j)\} \cup \{((b,x),(a,y)) : (x,y) \in (Y_j, D - C_j)\}| = |T_j|$ and $|T_j \cup F_j| = |T_j| + |F_j| = j + \beta_j$, we deduce that $\lambda(\vec{K}_2 \times D) \leq \min\{j + \beta_j : \lambda(D) \leq j \leq \delta\}$. And so, Claim 1 follows.

Claim 2. $\lambda(\vec{K}_2 \times D) \geq \min\{2\lambda(D), \beta(D), \min\{j + \beta_j : \lambda(D) \leq j \leq \delta\}$.

Let S be a minimum direct cut of $\vec{K}_2 \times D$, $V_1 = \{x \in V(D)$: vertices (a,x) and (b,x) lie in common strongly connected component of $\vec{K}_2 \times D - S\}$, $V_2 = \{x \in V(D)$: vertices (a,x) and (b,x) lie in different components of $\vec{K}_2 \times (D - S)\}$. Then (V_1, V_2) is a partition of $V(D)$. From the minimality of S it follows that $\vec{K}_2 \times D - S$ consists of two strongly connected components, say C_1 and C_2. Noticing that $(C_1, C_2) \subseteq S$ or $(C_2, C_1) \subseteq S$, we assume without loss of generality that $(C_1, C_2) \subseteq S$. Now three different cases occur: $V_1 \neq \emptyset = V_2$; $V_1 = \emptyset \neq V_2$; $V_1 \neq \emptyset \neq V_2$.

Consider at first the case when $V_1 \neq \emptyset = V_2$. By Lemma 2.1, we deduce that $\{x \in V(D) : (a,x) \in V(C_1)\}$ induces a strongly connected component of D as well as $\{y \in V(D) : (a,y) \in V(C_2)\}$ in this case. Furthermore, $(\{x \in V(D) : (a,x) \in V(C_1)\}, \{y \in V(D) : (a,y) \in V(C_2)\})$ is a directed cut of D and $(\{(a,x) : (a,x) \in V(C_1)\}, \{(b,y) : (b,y) \in V(C_2)\}) \cup (\{(b,x) : (b,x) \in V(C_1)\}, \{(a,y) : (a,y) \in V(C_2)\}) \subseteq S$. It follows from these observations that

$$\begin{aligned} |S| &\geq \left|\{(a,x) \in V(C_1)\}, \{(b,y) \in V(C_2)\}\right| \\ &\quad + \left|\{(b,x) \in V(C_1)\}, \{(a,y) \in V(C_2)\}\right| \\ &\geq 2\lambda(D). \end{aligned} \tag{4.1}$$

Consider secondly the case when $V_1 = \emptyset \neq V_2$. Let $M = \{x \in V(D) : (a,x) \in V(C_1)\}$ and $N = \{y \in V(D) : (b,y) \in V(C_1)\}$. Then (M, N) is a partition of $V(D)$ and $(C_1, C_2) = \{((a,x),(b,y)) : (x,y) \in A(D[M])\} \cup \{((b,x),(a,y)) : (x,y) \in A(D[N])\}$. Since $D - A(D[M]) - A(D[N])$ is a bipartite subgraph of D, it follows that $|(C_1, C_2)| = |A(D[M])| + |A(D[N])| \geq \beta(D)$. Recalling that $(C_1, C_2) \subseteq S$, we have $|S| \geq |(C_1, C_2)| \geq \beta(D)$.

Consider finally the case when $V_1 \neq \emptyset \neq V_2$. Let

$$\begin{aligned} H &= \{x \in V(D) : (a,x) \in V(C_1),\ x \in V_1\}, \\ Q &= \{x \in V(D) : (a,x) \in V(C_2),\ x \in V_1\}, \\ W &= \{x \in V(D) : (a,x) \in V(C_1),\ x \in V_2\}, \\ Z &= \{x \in V(D) : (a,x) \in V(C_2),\ x \in V_2\}. \end{aligned} \tag{4.2}$$

Then (H, Q, W, Z) is a partition of $V(D)$, refer to (1) of Figure 2. Since $(C_1, C_2) \subseteq S$, the arcs in this set is removed in Figure 2. If $H \neq \emptyset \neq Q$, then the set of arcs from $\{(a,x) : x \in H\} \cup \{(b,x) : x \in H\}$ to C_2 is a directed cut of $\vec{K}_2 \times D$, but it has less arcs than S. This contradiction shows that either $H = \emptyset$ or $Q = \emptyset$. Assume without loss of generality that $Q = \emptyset$. Then $\vec{K}_2 \times D - S$ can be depicted as (2) of Figure 2.

On the one hand, for every arc $(x,y) \in (H, D - H)$, if $y \in W$ then $((a,x),(b,y)) \in S$; if $y \in Z$ then $((b,x),(a,y)) \in S$. On the other hand, for every arc $(x,y) \in A(D[W])$ the

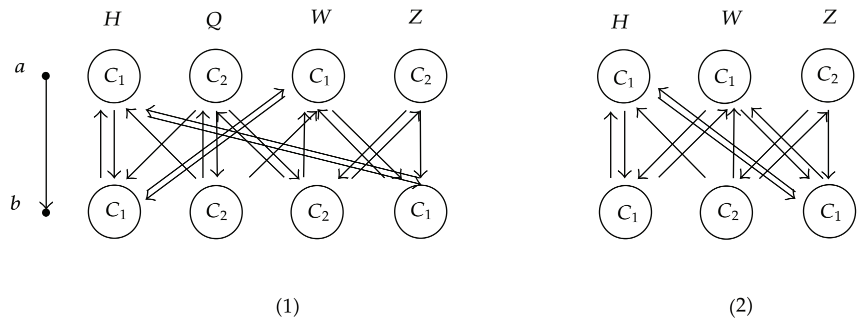


Figure 2: Partition of $V(\vec{K}_2 \times D - S)$.

arc $((a,x),(b,y)) \in S$ and for every arc $(x,y) \in A(D[Z])$ the arc $((b,x),(a,y)) \in S$. It follows from these observations that

$$\begin{aligned} |S| &\geq |[H, D-H]| + (|A(D[W])| + |A(D[Z])|) \\ &\geq \min\{j + \beta_j : \lambda(D) \leq j \leq \delta(D)\}, \end{aligned} \tag{4.3}$$

where $[H, D-H]$ represents the set of arcs with ends in H and $D-H$, respectively. And so, the theorem follows from Claims 1 and 2. $\square$

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References

[1] W. Imrich and S. Klavžar, *Product Graph, Structure and Recognition*, Wiley-Interscience, New York, NY, USA, 2000.
[2] Y. Chao, *Connectivity and fault-diameter of product graph [Ph.D. thesis]*, University of Science and Technology of China, 2007.
[3] P. M. Weichsel, "The Kronecker product of graphs," *Proceedings of the American Mathematical Society*, vol. 13, pp. 47–52, 1962.
[4] R. Guji and E. Vumar, "A note on the connectivity of Kronecker products of graphs," *Applied Mathematics Letters*, vol. 22, no. 9, pp. 1360–1363, 2009.
[5] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, Macmillan Press, London, UK, 1976.