Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 516897, 19 pages
doi:10.1155/2012/516897

Research Article

Strong Convergence of a Hybrid Iteration Scheme for Equilibrium Problems, Variational Inequality Problems and Common Fixed Point Problems, of Quasi-ϕ-Asymptotically Nonexpansive Mappings in Banach Spaces

Jing Zhao[1, 2]

[1] *College of Science, Civil Aviation University of China, Tianjin 300300, China*
[2] *Tianjin Key Laboratory for Advanced Signal Processing, Civil Aviation University of China, Tianjin 300300, China*

Correspondence should be addressed to Jing Zhao, zhaojing200103@163.com

Received 3 April 2012; Accepted 5 June 2012

Academic Editor: Hong-Kun Xu

We introduce an iterative algorithm for finding a common element of the set of common fixed points of a finite family of closed quasi-ϕ-asymptotically nonexpansive mappings, the set of solutions of an equilibrium problem, and the set of solutions of the variational inequality problem for a γ-inverse strongly monotone mapping in Banach spaces. Then we study the strong convergence of the algorithm. Our results improve and extend the corresponding results announced by many others.

1. Introduction and Preliminary

Let E be a Banach space with the dual E^*. A mapping $A : D(A) \subset E \to E^*$ is said to be monotone if, for each $x, y \in D(A)$, the following inequality holds:

$$\langle Ax - Ay, x - y \rangle \geq 0. \tag{1.1}$$

A is said to be γ-inverse strongly monotone if there exists a positive real number γ such that

$$\langle x - y, Ax - Ay \rangle \geq \gamma \|Ax - Ay\|^2, \quad \forall x, y \in D(A). \tag{1.2}$$

If A is γ-inverse strongly monotone, then it is Lipschitz continuous with constant $1/\gamma$, that is, $\|Ax - Ay\| \leq (1/\gamma)\|x - y\|$, for all $x, y \in D(A)$, and hence uniformly continuous.

Let C be a nonempty closed convex subset of E and $f : C \times C \to \mathbb{R}$ a bifunction, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for f is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0 \tag{1.3}$$

for all $y \in C$. The set of solutions of (1.3) is denoted by $\mathrm{EP}(f)$. Given a mapping $T : C \to E^*$, let $f(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then $\hat{x} \in \mathrm{EP}(f)$ if and only if $\langle T\hat{x}, y - \hat{x}\rangle \geq 0$ for all $y \in C$; that is, $\hat{x}$ is a solution of the variational inequality. Numerous problems in physics, optimization, engineering, and economics reduce to find a solution of (1.3). Some methods have been proposed to solve the equilibrium problem; see, for example, Blum and Oettli [1] and Moudafi [2]. For solving the equilibrium problem, let us assume that f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t \to 0} f(tz + (1 - t)x, y) \leq f(x, y)$;

(A4) for each $x \in C$, the function $y \mapsto f(x, y)$ is convex and lower semicontinuous.

Let E be a Banach space with the dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \left\{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\right\}, \tag{1.4}$$

where $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing. Let $\dim E \geq 2$, and the modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(\tau) = \sup\left\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1; \|y\| = \tau\right\}. \tag{1.5}$$

The space E is said to be smooth if $\rho_E(\tau) > 0$, for all $\tau > 0$ and E is called uniformly smooth if and only if $\lim_{t \to 0^+}(\rho_E(t)/t) = 0$. A Banach space E is said to be strictly convex if $\|x + y\|/2 < 1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \left\|\frac{x + y}{2}\right\| : \|x\| = \|y\| = 1; \epsilon = \|x - y\|\right\}. \tag{1.6}$$

E is called uniformly convex if and only if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0, 2]$. Let $p > 1$, then E is said to be p-uniformly convex if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in (0, 2]$. Observe that every p-uniformly convex space is uniformly convex. We know that if E is uniformly smooth, strictly convex, and reflexive, then the normalized duality mapping J is single-valued, one-to-one, onto and uniformly norm-to-norm continuous on each bounded subset of E. Moreover, if E is a reflexive and strictly convex Banach space with a strictly convex dual, then J^{-1} is single-valued, one-to-one, surjective, and it is the duality mapping from E^* into E and thus $JJ^{-1} = I_{E^*}$ and $J^{-1}J = I_E$ (see, [3]). It is also well known that E is uniformly smooth if and only if E^* is uniformly convex.

Let C be a nonempty closed convex subset of a Banach space E and $T : C \to C$ a mapping. A point $x \in C$ is said to be a fixed point of T provided $Tx = x$. A point $x \in C$ is said to be an asymptotic fixed point of T provided C contains a sequence $\{x_n\}$ which converges weakly to x such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. In this paper, we use $F(T)$ and $\widetilde{F(T)}$ to denote the fixed point set and the asymptotic fixed point set of T and use $\to$ to denote the strong convergence and weak convergence, respectively. Recall that a mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.7}$$

A mapping $T : C \to C$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\}$ of real numbers with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in C, \forall n \geq 1. \tag{1.8}$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [4] in 1972. They proved that if C is a nonempty bounded closed convex subset of a uniformly convex Banach space E, then every asymptotically nonexpansive self-mapping T of C has a fixed point. Further, the set $F(T)$ is closed and convex. Since 1972, a host of authors have studied the weak and strong convergence problems of the iterative algorithms for such a class of mappings (see, e.g., [4–6] and the references therein).

It is well known that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and, consequently, it is not available in more general Banach spaces. In this connection, Alber [7] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Next, we assume that E is a smooth Banach space. Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{1.9}$$

Following Alber [7], the generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the following minimization problem:

$$\phi(\overline{x}, x) = \inf_{y\in C} \phi(y, x). \tag{1.10}$$

It follows from the definition of the function ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2, \quad \forall x, y \in E. \tag{1.11}$$

If E is a Hilbert space, then $\phi(y, x) = \|y - x\|^2$ and $\Pi_C = P_C$ is the metric projection of H onto C.

Remark 1.1 (see [8, 9]). If E is a reflexive, strictly convex, and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$.

Let C be a nonempty, closed, and convex subset of a smooth Banach E and T a mapping from C into itself. The mapping T is said to be relatively nonexpansive if $\widetilde{F(T)} = F(T) \neq \emptyset$, $\phi(p, Tx) \leq \phi(p, x)$, for all $x \in C$, $p \in F(T)$. The mapping T is said to be ϕ-nonexpansive if $\phi(Tx, Ty) \leq \phi(x, y)$, for all $x, y \in C$. The mapping T is said to be quasi-ϕ-nonexpansive if $F(T) \neq \emptyset$, $\phi(p, Tx) \leq \phi(p, x)$, for all $x \in C$, $p \in F(T)$. The mapping T is said to be relatively asymptotically nonexpansive if there exists some real sequence $\{k_n\}$ with $k_n \geq 1$ and $k_n \to 1$ as $n \to \infty$ such that $\widetilde{F(T)} = F(T) \neq \emptyset$, $\phi(p, T^n x) \leq k_n \phi(p, x)$, for all $x \in C$, $p \in F(T)$. The mapping T is said to be ϕ-asymptotically nonexpansive if there exists some real sequence $\{k_n\}$ with $k_n \geq 1$ and $k_n \to 1$ as $n \to \infty$ such that $\phi(T^n x, T^n y) \leq k_n \phi(x, y)$, for all $x, y \in C$. The mapping T is said to be quasi-ϕ-asymptotically nonexpansive if there exists some real sequence $\{k_n\}$ with $k_n \geq 1$ and $k_n \to 1$ as $n \to \infty$ such that $F(T) \neq \emptyset$, $\phi(p, T^n x) \leq k_n \phi(p, x)$, for all $x \in C$, $p \in F(T)$. The mapping T is said to be asymptotically regular on C if, for any bounded subset K of C, $\limsup_{n \to \infty} \{\|T^{n+1}x - T^n x\| : x \in K\} = 0$. The mapping T is said to be closed on C if, for any sequence $\{x_n\}$ such that $\lim_{n \to \infty} x_n = x_0$ and $\lim_{n \to \infty} T x_n = y_0$, $T x_0 = y_0$.

We remark that a ϕ-asymptotically nonexpansive mapping with a nonempty fixed point set $F(T)$ is a quasi-ϕ-asymptotically nonexpansive mapping, but the converse may be not true. The class of quasi-ϕ-nonexpansive mappings and quasi-ϕ-asymptotically nonexpansive mappings is more general than the class of relatively nonexpansive mappings and relatively asymptotically nonexpansive mappings, respectively.

Recently, many authors studied the problem of finding a common element of the set of fixed points of nonexpansive or relatively nonexpansive mappings, the set of solutions of an equilibrium problem, and the set of solutions of variational inequalities in the frame work of Hilbert spaces and Banach spaces, respectively; see, for instance, [10–21] and the references therein.

In 2009, Takahashi and Zembayashi [22] introduced the following iterative process:

$$
\begin{gathered}
x_0 = x \in C, \\
y_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J S x_n), \\
u_n \in C, \quad f(u_n, y) + \frac{1}{r_n} \langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\
H_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
W_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\
x_{n+1} = \Pi_{H_n \cap W_n} x, \quad \forall n \geq 1,
\end{gathered}
\tag{1.12}
$$

where $f : C \times C \to \mathbb{R}$ is a bifunction satisfying (A1)–(A4), J is the normalized duality mapping on E and $S : C \to C$ is a relatively nonexpansive mapping. They proved the sequence $\{x_n\}$ defined by (1.12) converges strongly to a common point of the set of solutions of the equilibrium problem (1.3) and the set of fixed points of S provided the control sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfy appropriate conditions in Banach spaces.

Qin et al. [8] introduced the following iterative scheme on the equilibrium problem (1.3) and a family of quasi-ϕ-nonexpansive mapping:

$$\begin{gathered}
x_0 \in E, \qquad C_1 = C, \qquad x_1 = \Pi_{C_1} x_0, \\
y_n = J^{-1}\left(\alpha_{n,0} J x_n + \Sigma_{i=1}^{N} \alpha_{n,i} J T_i x_n\right), \\
u_n \in C, \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n\rangle \geq 0, \quad \forall y \in C, \\
C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1.
\end{gathered} \tag{1.13}$$

Strong convergence theorems of common elements are established in a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property.

Very recently, for finding a common element of $\cap_{i=1}^{r} F(T_i) \cap \mathrm{EP}(f, B) \cap \mathrm{VI}(A, C)$ Zegeye [23] proposed the following iterative algorithm:

$$\begin{gathered}
x_0 \in C_0 = C, \\
z_n = \Pi_C J^{-1}(J x_n - \lambda_n A x_n), \\
y_n = J^{-1}(\alpha_0 J x_n + \Sigma_{i=1}^{r} \alpha_i J T_i z_n), \\
u_n \in C, \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n\rangle \geq 0, \quad \forall y \in C, \\
C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1,
\end{gathered} \tag{1.14}$$

where $T_i : C \to C$ is closed quasi-ϕ-nonexpansive mapping $(i = 1, \ldots, r)$, $f : C \times C \to \mathbb{R}$ is a bifunction satisfying (A1)–(A4) and A is a γ-inverse strongly monotone mapping of C into E^*. Strong convergence theorems for iterative scheme (1.14) are obtained under some conditions on parameters in 2-uniformly convex and uniformly smooth real Banach space E.

In this paper, inspired and motivated by the works mentioned above, we introduce an iterative process for finding a common element of the set of common fixed points of a finite family of closed quasi-ϕ-asymptotically nonexpansive mappings, the solution set of equilibrium problem, and the solution set of the variational inequality problem for a γ-inverse strongly monotone mapping in Banach spaces. The results presented in this paper improve and generalize the corresponding results announced by many others.

In order to the main results of this paper, we need the following lemmas.

Lemma 1.2 (see [24]). *Let E be a 2-uniformly convex and smooth Banach space. Then, for all $x, y \in E$, one has*

$$\|x - y\| \leq \frac{2}{c^2}\|Jx - Jy\|, \tag{1.15}$$

where J is the normalized duality mapping of E and $1/c(0 < c \leq 1)$ is the 2-uniformly convex constant of E.

Lemma 1.3 (see [7, 25]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y), \quad \forall x \in C,\ y \in E. \tag{1.16}$$

Lemma 1.4 (see [25]). *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty}\phi(x_n, y_n) = 0$, then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

Lemma 1.5 (see [7]). *Let C be a nonempty closed convex subset of a smooth Banach space E, let $x \in E$ and let $z \in C$. Then*

$$z = \Pi_C x \Longleftrightarrow \langle y - z, Jx - Jz\rangle \leq 0, \quad \forall y \in C. \tag{1.17}$$

We denote by $N_C(v)$ the normal cone for $C \subset E$ at a point $v \in C$, that is, $N_C(v) = \{x^* \in E^* : \langle v - y, x^*\rangle \geq 0$, for all $y \in C\}$. We shall use the following lemma.

Lemma 1.6 (see [26]). *Let C be a nonempty closed convex subset of a Banach space E and let A be a monotone and hemicontinuous operator of C into E^* with $C = D(A)$. Let $S \subset E \times E^*$ be an operator defined as follows:*

$$Sv = \begin{cases} Av + N_C(v), & v \in C, \\ \emptyset, & v \notin C. \end{cases} \tag{1.18}$$

Then S is maximal monotone and $S^{-1}(0) = \mathrm{VI}(C, A)$.

We make use of the function $V : E \times E^* \to \mathbb{R}$ defined by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2, \tag{1.19}$$

for all $x \in E$ and $x^* \in E^*$ (see [7]). That is, $V(x, x^*) = \phi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E^*$.

Lemma 1.7 (see [7]). *Let E be a reflexive, strictly convex, and smooth Banach space with E^* as its dual. Then,*

$$V(x, x^*) + 2\left\langle J^{-1}x^* - x, y^*\right\rangle \leq V(x, x^* + y^*) \tag{1.20}$$

for all $x \in E$ and x^*, $y^* \in E^*$.

Lemma 1.8 (see [1]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4), *and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C. \tag{1.21}$$

Lemma 1.9 (see [22]). *Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4). *For $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\} \tag{1.22}$$

for all $x \in E$. Then, the following hold:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive, that is, for any $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle; \tag{1.23}$$

(3) $F(T_r) = \mathrm{EP}(f)$;

(4) $\mathrm{EP}(f)$ *is closed and convex;*

(5) $\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x)$, *for all $q \in F(T_r)$.*

Lemma 1.10 (see [8, 23]). *Let E be a uniformly convex Banach space, $s > 0$ a positive number and $B_s(0)$ a closed ball of E. Then there exists a strictly increasing, continuous, and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\left\| \sum_{i=0}^{N} (\alpha_i x_i) \right\|^2 \leq \sum_{i=0}^{N} \alpha_i \|x_i\|^2 - \alpha_k \alpha_l g(\|x_k - x_l\|) \tag{1.24}$$

for any $k, l \in \{0, 1, \ldots, N\}$, for all $x_0, x_1, \ldots, x_N \in B_s(0) = \{x \in E : \|x\| \leq s\}$ and $\alpha_0, \alpha_1, \ldots, \alpha_n \in [0, 1]$ such that $\sum_{i=0}^{N} \alpha_i = 1$.

Lemma 1.11 (see [27]). *Let E be a uniformly convex and uniformly smooth Banach space, C a nonempty, closed, and convex subset of E, and T a closed quasi-ϕ-asymptotically nonexpansive mapping from C into itself. Then $F(T)$ is a closed convex subset of C.*

2. Main Results

Theorem 2.1. *Let C be a nonempty, closed, and convex subset of a 2-uniformly convex and uniformly smooth real Banach space E and $T_i : C \to C$ a closed quasi-ϕ-asymptotically nonexpansive mapping with sequence $\{k_{n,i}\} \subset [1, \infty)$ such that $\lim_{n \to \infty} k_{n,i} = 1$ for each $1 \leq i \leq N$. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). Let A be a γ-inverse strongly monotone mapping of C into E^**

with constant $\gamma > 0$ such that $F = (\bigcap_{i=1}^{N} F(T_i)) \bigcap \mathrm{EP}(f) \bigcap \mathrm{VI}(C, A) \neq \emptyset$ and F is bounded. Assume that T_i is asymptotically regular on C for each $1 \leq i \leq N$ and $\|Ax\| \leq \|Ax - Ap\|$ for all $x \in C$ and $p \in F$. Define a sequence $\{x_n\}$ in C in the following manner:

$$
\begin{gathered}
x_0 \in C_0 = C \text{ chosen arbitrarily}, \\
z_n = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \\
y_n = J^{-1}(\alpha_{n,0} J x_n + \alpha_{n,1} J T_1^n z_n + \cdots + \alpha_{n,N} J T_N^n z_n), \\
u_n \in C, \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\
C_{n+1} = \left\{ z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \sum_{i=1}^{N} \alpha_{n,i}(k_{n,i} - 1) L_n \right\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_0
\end{gathered}
\tag{2.1}
$$

for every $n \geq 0$, where $\{r_n\}$ is a real sequence in $[a, \infty)$ for some $a > 0$, J is the normalized duality mapping on E and $L_n = \sup\{\phi(p, x_n) : p \in F\} < \infty$. Assume that $\{\alpha_{n,0}\}$, $\{\alpha_{n,1}\}, \ldots, \{\alpha_{n,N}\}$ are real sequences in $(0,1)$ such that $\sum_{i=0}^{N} \alpha_{n,i} = 1$ and $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$, for all $i \in \{1, 2, \ldots, N\}$. Let $\{\lambda_n\}$ be a sequence in $[s, t]$ for some $0 < s < t < c^2\gamma/2$, where $1/c$ is the 2-uniformly convex constant of E. Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Proof. We break the proof into nine steps.

Step 1. $\Pi_F x_0$ is well defined for $x_0 \in C$.

By Lemma 1.11 we know that $F(T_i)$ is a closed convex subset of C for every $1 \leq i \leq N$. Hence $F = (\bigcap_{i=1}^{N} F(T_i)) \bigcap \mathrm{EP}(f) \bigcap \mathrm{VI}(C, A) \neq \emptyset$ is a nonempty closed convex subset of C. Consequently, $\Pi_F x_0$ is well defined for $x_0 \in C$.

Step 2. C_n is closed and convex for each $n \geq 0$.

It is obvious that $C_0 = C$ is closed and convex. Suppose that C_n is closed and convex for some integer n. Since the defining inequality in C_{n+1} is equivalent to the inequality:

$$
2\langle z, Jx_n - Ju_n \rangle \leq \|x_n\|^2 - \|u_n\|^2 + \sum_{i=1}^{N} \alpha_{n,i}(k_{n,i} - 1) L_n, \tag{2.2}
$$

we have that C_{n+1} is closed and convex. So C_n is closed and convex for each $n \geq 0$. This in turn shows that $\Pi_{C_{n+1}} x_0$ is well defined.

Step 3. $F \subset C_n$ for all $n \geq 0$.

We do this by induction. For $n = 0$, we have $F \subset C = C_0$. Suppose that $F \subset C_n$ for some $n \geq 0$. Let $p \in F \subset C$. Putting $u_n = T_{r_n} y_n$ for all $n \geq 0$, we have that T_{r_n} is quasi-ϕ-nonexpansive from Lemma 1.9. Since T_i is quasi-ϕ-asymptotically nonexpansive, we have

$$\begin{aligned}
\phi(p, u_n) &= \phi(p, T_{r_n} y_n) \leq \phi(p, y_n) \\
&= \phi\left(p, J^{-1}\left(\alpha_{n,0} J x_n + \sum_{i=1}^{N} \alpha_{n,i} J T_i^n z_n\right)\right) \\
&= \|p\|^2 - 2\left\langle p, \alpha_{n,0} J x_n + \sum_{i=1}^{N} \alpha_{n,i} J T_i^n z_n \right\rangle + \left\| \alpha_{n,0} J x_n + \sum_{i=1}^{N} \alpha_{n,i} J T_i^n z_n \right\|^2 \\
&\leq \|p\|^2 - 2\alpha_{n,0} \langle p, J x_n \rangle - 2\sum_{i=1}^{N} \alpha_{n,i} \langle p, J T_i^n z_n \rangle + \alpha_{n,0} \|x_n\|^2 + \sum_{i=1}^{N} \alpha_{n,i} \left\| T_i^n z_n \right\|^2 \\
&= \alpha_{n,0} \phi(p, x_n) + \sum_{i=1}^{N} \alpha_{n,i} \phi(p, T_i^n z_n) \\
&\leq \alpha_{n,0} \phi(p, x_n) + \sum_{i=1}^{N} \alpha_{n,i} k_{n,i} \phi(p, z_n).
\end{aligned} \tag{2.3}$$

Moreover, by Lemmas 1.3 and 1.7, we get that

$$\begin{aligned}
&\phi(p, z_n) \\
&\quad = \phi\left(p, \Pi_C J^{-1}(J x_n - \lambda_n A x_n)\right) \\
&\quad \leq \phi\left(p, J^{-1}(J x_n - \lambda_n A x_n)\right) \\
&\quad = V(p, J x_n - \lambda_n A x_n) \\
&\quad \leq V(p, (J x_n - \lambda_n A x_n) + \lambda_n A x_n) - 2\left\langle J^{-1}(J x_n - \lambda_n A x_n) - p, \lambda_n A x_n \right\rangle \\
&\quad = V(p, J x_n) - 2\lambda_n \left\langle J^{-1}(J x_n - \lambda_n A x_n) - p, A x_n \right\rangle \\
&\quad = \phi(p, x_n) - 2\lambda_n \langle x_n - p, A x_n \rangle - 2\lambda_n \left\langle J^{-1}(J x_n - \lambda_n A x_n) - x_n, A x_n \right\rangle \\
&\quad \leq \phi(p, x_n) - 2\lambda_n \langle x_n - p, A x_n - A p \rangle - 2\lambda_n \langle x_n - p, A p \rangle \\
&\qquad + 2\left\langle J^{-1}(J x_n - \lambda_n A x_n) - x_n, -\lambda_n A x_n \right\rangle.
\end{aligned} \tag{2.4}$$

Thus, since $p \in \mathrm{VI}(C, A)$ and A is γ-inverse strongly monotone, we have from (2.4) that

$$\phi(p, z_n) \leq \phi(p, x_n) - 2\lambda_n \gamma \left\| A x_n - A p \right\|^2 + 2\left\langle J^{-1}(J x_n - \lambda_n A x_n) - x_n, -\lambda_n A x_n \right\rangle. \tag{2.5}$$

Therefore, from (2.5), Lemma 1.2 and the fact that $\lambda_n < c^2\gamma/2$ and $\|Ax\| \leq \|Ax - Ap\|$ for all $x \in C$ and $p \in F$, we have

$$\begin{aligned}\phi(p, z_n) &\leq \phi(p, x_n) - 2\lambda_n\gamma\|Ax_n - Ap\|^2 + \frac{4}{c^2}\lambda_n^2\|Ax_n - Ap\|^2 \\ &= \phi(p, x_n) + 2\lambda_n\left(\frac{2}{c^2}\lambda_n - \gamma\right)\|Ax_n - Ap\|^2 \\ &\leq \phi(p, x_n).\end{aligned} \tag{2.6}$$

Substituting (2.6) into (2.3), we get

$$\phi(p, u_n) \leq \phi(p, x_n) + \sum_{i=1}^{N}\alpha_{n,i}(k_{n,i} - 1)L_n, \tag{2.7}$$

that is, $p \in C_{n+1}$. By induction, $F \subset C_n$ and the iteration algorithm generated by (2.1) is well defined.

Step 4. $\lim_{n\to\infty}\phi(x_n, x_0)$ exists and $\{x_n\}$ is bounded.

Noticing that $x_n = \Pi_{C_n}x_0$ and Lemma 1.3, we have

$$\phi(x_n, x_0) = \phi(\Pi_{C_n}x_0, x_0) \leq \phi(p, x_0) - \phi(p, x_n) \leq \phi(p, x_0) \tag{2.8}$$

for all $p \in F$ and $n \geq 0$. This shows that the sequence $\{\phi(x_n, x_0)\}$ is bounded. From $x_n = \Pi_{C_n}x_0$ and $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we obtain that

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \geq 0, \tag{2.9}$$

which implies that $\{\phi(x_n, x_0)\}$ is nondecreasing. Therefore, the limit of $\{\phi(x_n, x_0)\}$ exists and $\{x_n\}$ is bounded.

Step 5. We have $x_n \to x^* \in C$.

By Lemma 1.3, we have, for any positive integer $m \geq n$, that

$$\phi(x_m, x_n) = \phi(x_m, \Pi_{C_n}x_0) \leq \phi(x_m, x_0) - \phi(\Pi_{C_n}x_0, x_0) = \phi(x_m, x_0) - \phi(x_n, x_0). \tag{2.10}$$

In view of Step 4 we deduce that $\phi(x_m, x_n) \to 0$ as $m, n \to \infty$. It follows from Lemma 1.4 that $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence of C. Since E is a Banach space and C is closed subset of E, there exists a point $x^* \in C$ such that $x_n \to x^*$ $(n \to \infty)$.

Step 6. We have $x^* \in \bigcap_{i=1}^{N} F(T_i)$.

By taking $m = n + 1$ in (2.10), we have

$$\lim_{n\to\infty}\phi(x_{n+1}, x_n) = 0. \tag{2.11}$$

From Lemma 1.4, it follows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{2.12}$$

Noticing that $x_{n+1}\in C_{n+1}$, we obtain

$$\phi(x_{n+1},u_n)\le\phi(x_{n+1},x_n)+\sum_{i=1}^{N}\alpha_{n,i}(k_{n,i}-1)L_n. \tag{2.13}$$

From (2.11), $\lim_{n\to\infty}k_{n,i}=1$ for any $1\le i\le N$, and Lemma 1.4, we know

$$\lim_{n\to\infty}\|x_{n+1}-u_n\|=0. \tag{2.14}$$

Notice that

$$\|x_n-u_n\|\le\|x_n-x_{n+1}\|+\|x_{n+1}-u_n\| \tag{2.15}$$

for all $n\ge 0$. It follows from (2.12) and (2.14) that

$$\lim_{n\to\infty}\|x_n-u_n\|=0, \tag{2.16}$$

which implies that $u_n\to x^*$ as $n\to\infty$. Since J is uniformly norm-to-norm continuous on bounded sets, from (2.16), we have

$$\lim_{n\to\infty}\|Jx_n-Ju_n\|=0. \tag{2.17}$$

Let $s=\sup\{\|x_n\|,\|T_1^nx_n\|,\|T_2^nx_n\|,\ldots,\|T_N^nx_n\|:n\in\mathbb{N}\}$. Since E is uniformly smooth Banach space, we know that E^* is a uniformly convex Banach space. Therefore, from Lemma 1.10 we have, for any $p\in F$, that

$$\begin{aligned}
\phi(p,u_n)&=\phi(p,T_{r_n}y_n)\\
&\le\phi(p,y_n)\\
&=\phi\left(p,J^{-1}\left(\alpha_{n,0}Jx_n+\sum_{i=1}^{N}\alpha_{n,i}JT_i^nz_n\right)\right)\\
&=\|p\|^2-2\alpha_{n,0}\langle p,Jx_n\rangle-2\sum_{i=1}^{N}\alpha_{n,i}\langle p,JT_i^nz_n\rangle+\left\|\alpha_{n,0}Jx_n+\sum_{i=1}^{N}\alpha_{n,i}JT_i^nz_n\right\|^2
\end{aligned}$$

$$\begin{aligned}
&\leq \|p\|^2 - 2\alpha_{n,0}\langle p, Jx_n\rangle - 2\sum_{i=1}^{N}\alpha_{n,i}\langle p, JT_i^n z_n\rangle + \alpha_{n,0}\|x_n\|^2 \\
&\quad + \sum_{i=1}^{N}\alpha_{n,i}\|T_i^n z_n\|^2 - \alpha_{n,0}\alpha_{n,1}g(\|Jx_n - JT_1^n z_n\|) \\
&= \alpha_{n,0}\phi(p, x_n) + \sum_{i=1}^{N}\alpha_{n,i}\phi(p, T_i^n z_n) - \alpha_{n,0}\alpha_{n,1}g(\|Jx_n - JT_1^n z_n\|) \\
&\leq \alpha_{n,0}\phi(p, x_n) + \sum_{i=1}^{N}\alpha_{n,i}k_{n,i}\phi(p, z_n) - \alpha_{n,0}\alpha_{n,1}g(\|Jx_n - JT_1^n z_n\|).
\end{aligned} \tag{2.18}$$

Therefore, from (2.6) and (2.18), we have

$$\begin{aligned}
\phi(p, u_n) \leq \phi(p, x_n) &+ \sum_{i=1}^{N}\alpha_{n,i}(k_{n,i} - 1)\phi(p, x_n) - \alpha_{n,0}\alpha_{n,1}g(\|Jx_n - JT_1^n z_n\|) \\
&+ 2\lambda_n\left(\frac{2}{c^2}\lambda_n - \gamma\right)\|Ax_n - Ap\|^2\sum_{i=1}^{N}\alpha_{n,i}k_{n,i}.
\end{aligned} \tag{2.19}$$

It follows from $\lambda_n < c^2\gamma/2$ that

$$\alpha_{n,0}\alpha_{n,1}g(\|Jx_n - JT_1^n z_n\|) \leq \phi(p, x_n) - \phi(p, u_n) + \sum_{i=1}^{N}\alpha_{n,i}(k_{n,i} - 1)\phi(p, x_n). \tag{2.20}$$

On the other hand, we have

$$\begin{aligned}
&|\phi(p, x_n) - \phi(p, u_n)| \\
&\quad = \left|\|x_n\|^2 - \|u_n\|^2 - 2\langle p, Jx_n - Ju_n\rangle\right| \\
&\quad \leq |\|x_n\| - \|u_n\||(\|x_n\| + \|u_n\|) + 2\|Jx_n - Ju_n\|\|p\| \\
&\quad \leq \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|Jx_n - Ju_n\|\|p\|.
\end{aligned} \tag{2.21}$$

It follows from (2.16) and (2.17) that

$$\lim_{n\to\infty}(\phi(p, x_n) - \phi(p, u_n)) = 0. \tag{2.22}$$

Since $\lim_{n\to\infty}k_{n,i} = 1$ and $\liminf_{n\to\infty}\alpha_{n,0}\alpha_{n,1} > 0$, from (2.20) and (2.22) we have

$$\lim_{n\to\infty}g(\|Jx_n - JT_1^n z_n\|) = 0. \tag{2.23}$$

Therefore, from the property of g, we obtain

$$\lim_{n\to\infty} \left\| Jx_n - JT_1^n z_n \right\| = 0. \tag{2.24}$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \left\| x_n - T_1^n z_n \right\| = 0, \tag{2.25}$$

and hence $T_1^n z_n \to x^*$ as $n \to \infty$. Since $\|T_1^{n+1} z_n - x^*\| \le \|T_1^{n+1} z_n - T_1^n z_n\| + \|T_1^n z_n - x^*\|$, it follows from the asymptotic regularity of T_1 that

$$\lim_{n\to\infty} \left\| T_1^{n+1} z_n - x^* \right\| = 0. \tag{2.26}$$

That is, $T_1(T_1^n z_n) \to x^*$ as $n \to \infty$. From the closedness of T_1, we get $T_1 x^* = x^*$. Similarly, one can obtain that $T_i x^* = x^*$ for $i = 2, \ldots, N$. So, $x^* \in \bigcap_{i=1}^N F(T_i)$.

Moreover, from (2.19) we have that

$$\begin{aligned}
&2\lambda_n \left(\gamma - \frac{2}{c^2}\lambda_n \right) \left\| Ax_n - Ap \right\|^2 (1 - \alpha_{n,0}) \\
&\quad\le 2\lambda_n \left(\gamma - \frac{2}{c^2}\lambda_n \right) \left\| Ax_n - Ap \right\|^2 \sum_{i=1}^N \alpha_{n,i} k_{n,i} \\
&\quad\le \phi(p, x_n) - \phi(p, u_n) + \sum_{i=1}^N \alpha_{n,i}(k_{n,i} - 1)\phi(p, x_n),
\end{aligned} \tag{2.27}$$

which implies that

$$\lim_{n\to\infty} \left\| Ax_n - Ap \right\| = 0. \tag{2.28}$$

Now, Lemmas 1.3 and 1.7 imply that

$$\begin{aligned}
\phi(x_n, z_n) &= \phi\Big(x_n, \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)\Big) \\
&\leq \phi\Big(x_n, J^{-1}(Jx_n - \lambda_n Ax_n)\Big) \\
&= V(x_n, Jx_n - \lambda_n Ax_n) \\
&\leq V(x_n, (Jx_n - \lambda_n Ax_n) + \lambda_n Ax_n) - 2\Big\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, \lambda_n Ax_n \Big\rangle \\
&= \phi(x_n, x_n) + 2\Big\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, -\lambda_n Ax_n \Big\rangle \\
&= 2\Big\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, -\lambda_n Ax_n \Big\rangle \\
&\leq 2\Big\| J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}Jx_n \Big\| \cdot \|\lambda_n Ax_n\|.
\end{aligned} \tag{2.29}$$

In view of Lemma 1.2 and the fact that $\|Ax\| \leq \|Ax - Ap\|$ for all $x \in C$, $p \in F$, we have

$$\phi(x_n, z_n) \leq \frac{4}{c^2}\lambda_n^2 \|Ax_n - Ap\|^2 \leq \frac{4}{c^2} t^2 \|Ax_n - Ap\|^2. \tag{2.30}$$

From (2.28) and Lemma 1.4 we get

$$\lim_{n \to \infty} \|x_n - z_n\| = 0, \tag{2.31}$$

and hence $z_n \to x^*$ as $n \to \infty$.

Step 7. We have $x^* \in \mathrm{VI}(C, A)$.

Let $S \subset E \times E^*$ be an operator as follows:

$$Sv = \begin{cases} Av + N_C(v), & v \in C, \\ \emptyset, & v \notin C. \end{cases} \tag{2.32}$$

By Lemma 1.6, S is maximal monotone and $S^{-1}(0) = \mathrm{VI}(C, A)$. Let $(v, w) \in G(S)$. Since $w \in Sv = Av + N_C(v)$, we have $w - Av \in N_C(v)$. It follows from $z_n \in C$ that

$$\langle v - z_n, w - Av \rangle \geq 0. \tag{2.33}$$

On the other hand, from $z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$ and Lemma 1.5 we obtain that

$$\langle v - z_n, Jz_n - (Jx_n - \lambda_n Ax_n) \rangle \geq 0, \tag{2.34}$$

and hence

$$\left\langle v - z_n, \frac{Jx_n - Jz_n}{\lambda_n} - Ax_n \right\rangle \le 0. \tag{2.35}$$

Then, from (2.33) and (2.35), we have

$$\begin{aligned}
\langle v - z_n, w \rangle &\ge \langle v - z_n, Av \rangle \\
&\ge \langle v - z_n, Av \rangle + \left\langle v - z_n, \frac{Jx_n - Jz_n}{\lambda_n} - Ax_n \right\rangle \\
&= \left\langle v - z_n, Av - Ax_n + \frac{Jx_n - Jz_n}{\lambda_n} \right\rangle \\
&= \langle v - z_n, Av - Az_n \rangle + \langle v - z_n, Az_n - Ax_n \rangle + \left\langle v - z_n, \frac{Jx_n - Jz_n}{\lambda_n} \right\rangle \\
&\ge -\|v - z_n\| \cdot \|Az_n - Ax_n\| - \|v - z_n\| \cdot \frac{\|Jx_n - Jz_n\|}{s}.
\end{aligned} \tag{2.36}$$

Hence we have $\langle v - x^*, w \rangle \ge 0$ as $n \to \infty$, since the uniform continuity of J and A imply that the right side of (2.36) goes to 0 as $n \to \infty$. Thus, since S is maximal monotone, we have $x^* \in S^{-1}(0)$ and hence $x^* \in \mathrm{VI}(C, A)$.

Step 8. We have $x^* \in \mathrm{EP}(f) = F(T_r)$.

Let $p \in F$. From $u_n = T_{r_n} y_n$, (2.3), (2.6) and Lemma 1.9 we obtain that

$$\begin{aligned}
\phi(u_n, y_n) &= \phi(T_{r_n} y_n, y_n) \\
&\le \phi(p, y_n) - \phi(p, T_{r_n} y_n) \\
&\le \phi(p, x_n) + \sum_{i=1}^{N} \alpha_{n,i}(k_{n,i} - 1)\phi(p, x_n) - \phi(p, u_n).
\end{aligned} \tag{2.37}$$

It follows from (2.22) and $k_{n,i} \to 1$ that $\phi(u_n, y_n) \to 0$ as $n \to \infty$. Now, by Lemma 1.4 we have that $\|u_n - y_n\| \to 0$ as $n \to \infty$. Consequently, we obtain that $\|Ju_n - Jy_n\| \to 0$ and $y_n \to x^*$ from $u_n \to x^*$ as $n \to \infty$. From the assumption $r_n > a$, we get

$$\lim_{n \to \infty} \frac{\|Ju_n - Jy_n\|}{r_n} = 0. \tag{2.38}$$

Noting that $u_n = T_{r_n} y_n$, we obtain

$$f(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \ge 0, \quad \forall y \in C. \tag{2.39}$$

From (A2), we have

$$\left\langle y - u_n, \frac{Ju_n - Jy_n}{r_n} \right\rangle \geq -f(u_n, y) \geq f(y, u_n), \quad \forall y \in C. \tag{2.40}$$

Letting $n \to \infty$, we have from $u_n \to x^*$, (2.38) and (A4) that $f(y, x^*) \leq 0$ (for all $y \in C$). For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)x^*$. Since $y \in C$ and $x^* \in C$, we have $y_t \in C$ and hence $f(y_t, x^*) \leq 0$. Now, from (A1) and (A4) we have

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, x^*) \leq tf(y_t, y) \tag{2.41}$$

and hence $f(y_t, y) \geq 0$. Letting $t \to 0$, from (A3), we have $f(x^*, y) \geq 0$. This implies that $x^* \in \mathrm{EP}(f)$. Therefore, in view of Steps 6, 7, and 8 we have $x^* \in F$.

Step 9. We have $x^* = \Pi_F x_0$.

From $x_n = \Pi_{C_n} x_0$, we get

$$\langle x_n - z, Jx_0 - Jx_n \rangle \geq 0, \quad \forall z \in C_n. \tag{2.42}$$

Since $F \subset C_n$ for all $n \geq 1$, we arrive at

$$\langle x_n - p, Jx_0 - Jx_n \rangle \geq 0, \quad \forall p \in F. \tag{2.43}$$

Letting $n \to \infty$, we have

$$\langle x^* - p, Jx_0 - Jx^* \rangle \geq 0, \quad \forall p \in F, \tag{2.44}$$

and hence $x^* = \Pi_F x_0$ by Lemma 1.5. This completes the proof.

□

Strong convergence theorem for approximating a common element of the set of solutions of the equilibrium problem and the set of fixed points of a finite family of closed quasi-ϕ-asymptotically nonexpansive mappings in Banach spaces may not require that E be 2-uniformly convex. In fact, we have the following theorem.

Theorem 2.2. *Let C be a nonempty, closed, and convex subset of a uniformly convex and uniformly smooth real Banach space E and $T_i : C \to C$ a closed quasi-ϕ-asymptotically nonexpansive mapping with sequence $\{k_{n,i}\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_{n,i} = 1$ for each $1 \leq i \leq N$. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4) *such that $F = (\bigcap_{i=1}^N F(T_i)) \bigcap \mathrm{EP}(f) \neq \emptyset$ and F is bounded.*

Assume that T_i is asymptotically regular on C for each $1 \le i \le N$. Define a sequence $\{x_n\}$ in C in the following manner:

$$\begin{gathered}
x_0 \in C_0 = C \text{ chosen arbitrarily}, \\
y_n = J^{-1}\left(\alpha_{n,0} J x_n + \alpha_{n,1} J T_1^n x_n + \cdots + \alpha_{n,N} J T_N^n x_n\right), \\
u_n \in C, \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \ge 0, \quad \forall y \in C, \\
C_{n+1} = \left\{ z \in C_n : \phi(z, u_n) \le \phi(z, x_n) + \sum_{i=1}^{N} \alpha_{n,i}(k_{n,i} - 1) L_n \right\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_0
\end{gathered} \tag{2.45}$$

for every $n \ge 0$, where $\{r_n\}$ is a real sequence in $[a, \infty)$ for some $a > 0$, J is the normalized duality mapping on E and $L_n = \sup\{\phi(p, x_n) : p \in F\} < \infty$. Assume that $\{\alpha_{n,0}\}$, $\{\alpha_{n,1}\}, \ldots, \{\alpha_{n,N}\}$ are real sequences in $(0,1)$ such that $\sum_{i=0}^{N} \alpha_{n,i} = 1$ and $\liminf_{n \to \infty} \alpha_{n,0}\alpha_{n,i} > 0$, for all $i \in \{1, 2, \ldots, N\}$. Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Proof. Put $A \equiv 0$ in Theorem 2.1. We have $z_n = x_n$. Thus, the method of proof of Theorem 2.1 gives the required assertion without the requirement that E is 2-uniformly convex. □

As some corollaries of Theorems 2.1 and 2.2, we have the following results immediately.

Corollary 2.3. *Let C be a nonempty, closed, and convex subset of a Hilbert space H and $T_i : C \to C$ a closed quasi-ϕ-asymptotically nonexpansive mapping with sequence $\{k_{n,i}\} \subset [1, \infty)$ such that $\lim_{n \to \infty} k_{n,i} = 1$ for each $1 \le i \le N$. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4). *Let A be a γ-inverse strongly monotone mapping of C into H with constant $\gamma > 0$ such that $F = (\bigcap_{i=1}^{N} F(T_i)) \bigcap \mathrm{EP}(f) \bigcap \mathrm{VI}(C, A) \ne \emptyset$ and F is bounded. Assume that T_i is asymptotically regular on C for each $1 \le i \le N$ and $\|Ax\| \le \|Ax - Ap\|$ for all $x \in C$ and $p \in F$. Define a sequence $\{x_n\}$ in C in the following manner:*

$$\begin{gathered}
x_0 \in C_0 = C \text{ chosen arbitrarily}, \\
z_n = P_C(x_n - \lambda_n A x_n), \\
y_n = \alpha_{n,0} x_n + \alpha_{n,1} T_1^n z_n + \cdots + \alpha_{n,N} T_N^n z_n, \\
u_n \in C, \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - y_n \rangle \ge 0, \quad \forall y \in C, \\
C_{n+1} = \left\{ z \in C_n : \|z - u_n\|^2 \le \|z - x_n\|^2 + \sum_{i=1}^{N} \alpha_{n,i}(k_{n,i} - 1) L_n \right\}, \\
x_{n+1} = P_{C_{n+1}} x_0
\end{gathered} \tag{2.46}$$

for every $n \ge 0$, where $\{r_n\}$ is a real sequence in $[a, \infty)$ for some $a > 0$ and $L_n = \sup\{\|x_n - p\|^2 : p \in F\} < \infty$. Assume that $\{\alpha_{n,0}\}$, $\{\alpha_{n,1}\}, \ldots, \{\alpha_{n,N}\}$ are real sequences in $(0,1)$ such that $\sum_{i=0}^{N} \alpha_{n,i} = 1$

and $\liminf_{n\to\infty}\alpha_{n,0}\alpha_{n,i} > 0$*, for all* $i \in \{1,2,\ldots,N\}$*. Let* $\{\lambda_n\}$ *be a sequence in* $[s,t]$ *for some* $0 < s < t < \gamma/2$*. Then the sequence* $\{x_n\}$ *converges strongly to* $P_F x_0$*.*

Corollary 2.4. *Let* C *be a nonempty, closed, and convex subset of a Hilbert space* H *and* $T_i : C \to C$ *a closed quasi-*ϕ*-asymptotically nonexpansive mapping with sequence* $\{k_{n,i}\} \subset [1,\infty)$ *such that* $\lim_{n\to\infty}k_{n,i} = 1$ *for each* $1 \le i \le N$*. Let* f *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* (A1)–(A4) *such that* $F = (\bigcap_{i=1}^{N} F(T_i)) \bigcap \mathrm{EP}(f) \neq \emptyset$ *and* F *is bounded. Assume that* T_i *is asymptotically regular on* C *for each* $1 \le i \le N$*. Define a sequence* $\{x_n\}$ *in* C *in the following manner:*

$$\begin{gathered}
x_0 \in C_0 = C \text{ chosen arbitrarily}, \\
y_n = \alpha_{n,0}x_n + \alpha_{n,1}T_1^n x_n + \cdots + \alpha_{n,N}T_N^n x_n, \\
u_n \in C, \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \ge 0, \quad \forall y \in C, \\
C_{n+1} = \left\{ z \in C_n : \|z - u_n\|^2 \le \|z - x_n\|^2 + \sum_{i=1}^{N}\alpha_{n,i}(k_{n,i} - 1)L_n \right\}, \\
x_{n+1} = P_{C_{n+1}}x_0
\end{gathered} \tag{2.47}$$

for every $n \ge 0$*, where* $\{r_n\}$ *is a real sequence in* $[a,\infty)$ *for some* $a > 0$ *and* $L_n = \sup\{\|x_n - p\|^2 : p \in F\} < \infty$*. Assume that* $\{\alpha_{n,0}\}$*,* $\{\alpha_{n,1}\}$*, …,* $\{\alpha_{n,N}\}$ *are real sequences in* $(0,1)$ *such that* $\sum_{i=0}^{N}\alpha_{n,i} = 1$ *and* $\liminf_{n\to\infty}\alpha_{n,0}\alpha_{n,i} > 0$*, for all* $i \in \{1,2,\ldots,N\}$*. Let* $\{\lambda_n\}$ *be a sequence in* $[s,t]$ *for some* $0 < s < t < \gamma/2$*. Then the sequence* $\{x_n\}$ *converges strongly to* $P_F x_0$*.*

Remark 2.5. Theorems 2.1 and 2.2 extend the main results of [23] from quasi-ϕ-nonexpansive mappings to more general quasi-ϕ-asymptotically nonexpansive mappings.

Acknowledgments

The research was supported by Fundamental Research Funds for the Central Universities (Program no. ZXH2012K001), supported in part by the Foundation of Tianjin Key Lab for Advanced Signal Processing and it was also supported by the Science Research Foundation Program in Civil Aviation University of China (2012KYM04).

References

[1] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *The Mathematics Student*, vol. 63, no. 1–4, pp. 123–145, 1994.
[2] A. Moudafi, "Second-order differential proximal methods for equilibrium problems," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 4, no. 1, article 18, 2003.
[3] W. Takahashi, *Nonlinear Functional Analysis*, Kindikagaku, Tokyo, Japan, 1988.
[4] K. Goebel and W. A. Kirk, "A fixed point theorem for asymptotically nonexpansive mappings," *Proceedings of the American Mathematical Society*, vol. 35, pp. 171–174, 1972.
[5] J. Schu, "Iterative construction of fixed points of asymptotically nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 158, no. 2, pp. 407–413, 1991.
[6] H. Y. Zhou, Y. J. Cho, and S. M. Kang, "A new iterative algorithm for approximating common fixed points for asymptotically nonexpansive mappings," *Fixed Point Theory and Applications*, vol. 2007, Article ID 64874, 10 pages, 2007.

[7] Y. I. Alber, “Metric and generalized projection operators in Banach spaces: properties and applications,” in *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, vol. 178, pp. 15–50, Marcel Dekker, New York, NY, USA, 1996.

[8] X. Qin, S. Y. Cho, and S. M. Kang, “Strong convergence of shrinking projection methods for quasi-ϕ-nonexpansive mappings and equilibrium problems,” *Journal of Computational and Applied Mathematics*, vol. 234, no. 3, pp. 750–760, 2010.

[9] H. Zhou, G. Gao, and B. Tan, “Convergence theorems of a modified hybrid algorithm for a family of quasi-ϕ-asymptotically nonexpansive mappings,” *Journal of Applied Mathematics and Computing*, vol. 32, no. 2, pp. 453–464, 2010.

[10] F. Cianciaruso, G. Marino, L. Muglia, and Y. Yao, “A hybrid projection algorithm for finding solutions of mixed equilibrium problem and variational inequality problem,” *Fixed Point Theory and Applications*, vol. 2010, Article ID 383740, 19 pages, 2010.

[11] F. Cianciaruso, G. Marino, L. Muglia, and Y. Yao, “On a two-step algorithm for hierarchical fixed point problems and variational inequalities,” *Journal of Inequalities and Applications*, vol. 2009, Article ID 208692, 13 pages, 2009.

[12] S. Saewan and P. Kumam, “A modified hybrid projection method for solving generalized mixed equilibrium problems and fixed point problems in Banach spaces,” *Computers and Mathematics with Applications*, vol. 62, no. 4, pp. 1723–1735, 2011.

[13] S. Saewan and P. Kumam, “Convergence theorems for mixed equilibrium problems, variational inequality problem and uniformly quasi-ϕ-asymptotically nonexpansive mappings,” *Applied Mathematics and Computation*, vol. 218, no. 7, pp. 3522–3538, 2011.

[14] S. Saewan, P. Kumam, and K. Wattanawitoon, “Convergence theorem based on a new hybrid projection method for finding a common solution of generalized equilibrium and variational inequality problems in Banach spaces,” *Abstract and Applied Analysis*, vol. 2010, Article ID 734126, 25 pages, 2010.

[15] Y. Su, D. Wang, and M. Shang, “Strong convergence of monotone hybrid algorithm for hemi-relatively nonexpansive mappings,” *Fixed Point Theory and Applications*, vol. 2008, Article ID 284613, 8 pages, 2008.

[16] A. Tada and W. Takahashi, “Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem,” *Journal of Optimization Theory and Applications*, vol. 133, no. 3, pp. 359–370, 2007.

[17] S. Takahashi and W. Takahashi, “Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces,” *Journal of Mathematical Analysis and Applications*, vol. 331, no. 1, pp. 506–515, 2007.

[18] Y. Yao, R. Chen, and H.-K. Xu, “Schemes for finding minimum-norm solutions of variational inequalities,” *Nonlinear Analysis. Theory, Methods and Applications A*, vol. 72, no. 7-8, pp. 3447–3456, 2010.

[19] Y. Yao, Y. J. Cho, and R. Chen, “An iterative algorithm for solving fixed point problems, variational inequality problems and mixed equilibrium problems,” *Nonlinear Analysis. Theory, Methods and Applications A*, vol. 71, no. 7-8, pp. 3363–3373, 2009.

[20] Y. Yao, M. A. Noor, K. I. Noor, Y.-C. Liou, and H. Yaqoob, “Modified extragradient methods for a system of variational inequalities in Banach spaces,” *Acta Applicandae Mathematicae*, vol. 110, no. 3, pp. 1211–1224, 2010.

[21] Y. Yao, Y.-C. Liou, and Y.-J. Wu, “An extragradient method for mixed equilibrium problems and fixed point problems,” *Fixed Point Theory and Applications*, vol. 2009, Article ID 632819, 15 pages, 2009.

[22] W. Takahashi and K. Zembayashi, “Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces,” *Nonlinear Analysis. Theory, Methods and Applications A*, vol. 70, no. 1, pp. 45–57, 2009.

[23] H. Zegeye, “A hybrid iteration scheme for equilibrium problems, variational inequality problems and common fixed point problems in Banach spaces,” *Nonlinear Analysis. Theory, Methods and Applications A*, vol. 72, no. 3-4, pp. 2136–2146, 2010.

[24] H. K. Xu, “Inequalities in Banach spaces with applications,” *Nonlinear Analysis. Theory, Methods and Applications A*, vol. 16, no. 12, pp. 1127–1138, 1991.

[25] S. Kamimura and W. Takahashi, “Strong convergence of a proximal-type algorithm in a Banach space,” *SIAM Journal on Optimization*, vol. 13, no. 3, pp. 938–945, 2002.

[26] R. T. Rockafellar, “On the maximality of sums of nonlinear monotone operators,” *Transactions of the American Mathematical Society*, vol. 149, pp. 75–88, 1970.

[27] Y. J. Cho, X. Qin, and S. M. Kang, “Strong convergence of the modified Halpern-type iterative algorithms in Banach spaces,” *Analele stiintifice ale Universitatii Ovidius Constanta*, vol. 17, no. 1, pp. 51–68, 2009.