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Research Article

Approximations of Numerical Method for Neutral Stochastic Functional Differential Equations with Markovian Switching

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Stochastic systems with Markovian switching have been used in a variety of application areas, including biology, epidemiology, mechanics, economics, and finance. In this paper, we study the Euler-Maruyama (EM) method for neutral stochastic functional differential equations with Markovian switching. The main aim is to show that the numerical solutions will converge to the true solutions. Moreover, we obtain the convergence order of the approximate solutions.

1. Introduction

Stochastic systems with Markovian switching have been successfully used in a variety of application areas, including biology, epidemiology, mechanics, economics, and finance [1]. As well as deterministic neutral functional differential equations and stochastic functional differential equations (SFDEs), most neutral stochastic functional differential equations with Markovian switching (NSFDEsMS) cannot be solved explicitly, so numerical methods become one of the powerful techniques. A number of papers have studied the numerical analysis of the deterministic neutral functional differential equations, for example, [2, 3] and references therein. The numerical solutions of SFDEs and stochastic systems with Markovian switching have also been studied extensively by many authors. Here we mention some of them, for example, [4–16]. moreover, Many well-known theorems in SFDEs are successfully extended to NSFDEs, for example, [17–24] discussed the stability analysis of the true solutions.

However, to the best of our knowledge, little is yet known about the numerical solutions for NSFDEsMS. In this paper we will extend the method developed by [5, 16] to

NSFDEsMS and study strong convergence for the Euler-Maruyama approximations under the local Lipschitz condition, the linear growth condition, and contractive mapping. The three conditions are standard for the existence and uniqueness of the true solutions. Although the method of analysis borrows from [5], the existence of the neutral term and Markovian switching essentially complicates the problem. We develop several new techniques to cope with the difficulties which have risen from the two terms. Moreover, we also generalize the results in [19].

In Section 2, we describe some preliminaries and define the EM method for NSFDEsMS and state our main result that the approximate solutions strongly converge to the true solutions. The proof of the result is rather technical so we present several lemmas in Section 3 and then complete the proof in Section 4. In Section 5, under the global Lipschitz condition, we reveal the order of the convergence of the approximate solutions. Finally, a conclusion is made in Section 6.

2. Preliminaries and EM Scheme

Throughout this paper let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all P-null sets. Let $w(t) = (w_1(t), \dots, w_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^n$. Let $\mathbb{R}_+ = [0, \infty)$, and let $\tau > 0$. Denoted by $C([-\tau, 0], \mathbb{R}^n)$ the family of continuous functions from $[-\tau, 0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-\tau\leq\theta\leq 0}|\varphi(\theta)|$. Let $p > 0$ and $L^p_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ be the family of $\mathcal{F}_0$-measurable $C([-\tau, 0]; \mathbb{R}^n)$-valued random variables ξ such that $\mathbb{E}\|\xi\|^p < \infty$. If $x(t)$ is an $\mathbb{R}^n$-valued stochastic process on $t \in [-\tau, \infty)$, we let $x_t = \{x(t+\theta) : -\tau \leq \theta \leq 0\}$ for $t \geq 0$.

Let $r(t)$, $t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \dots, N\}$ with the generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\{r(t+\Delta) = j \mid r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ij}\Delta + o(\Delta) & \text{if } i = j, \end{cases} \tag{2.1}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{i\neq j}\gamma_{ij}$. We assumethat the Markov chain $r(\cdot)$ is independent of the Brownian motion $w(\cdot)$. It is well known that almost every sample path of $r(\cdot)$ is a right-continuous step function with finite number of simple jumps in any finite subinterval of $\Re_+ = [0, \infty)$.

In this paper, we consider the n-dimensional NSFDEsMS

$$d[x(t) - u(x_t, r(t))] = f(x_t, r(t))dt + g(x_t, r(t))dw(t), \quad t \geq 0, \tag{2.2}$$

with initial data $x_0 = \xi \in L^p_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ and $r(0) = i_0 \in S$, where $f : C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n$, $g : C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^{n\times m}$ and $u : C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n$. As a standing hypothesis we assume that both f and g are sufficiently smooth so that (2.2) has a unique solution. We refer the reader to Mao [10, 12] for the conditions on the existence and uniqueness of the solution $x(t)$. The initial data ξ and i_0 could be random, but the Markov property ensures that it is sufficient to consider only the case when both x_0 and i_0 are constants.

To analyze the Euler-Maruyama (EM) method, we need the following lemma (see [6, 7, 10–12, 16]).

Lemma 2.1. *Given $\Delta > 0$, let $r_k^\Delta = r(k\Delta)$ for $k \geq 0$. Then $\{r_k^\Delta, k = 0, 1, 2, \ldots\}$ is a discrete Markov chain with the one-step transition probability matrix*

$$P(\Delta) = \left(P_{ij}(\Delta)\right)_{N\times N} = e^{\Delta\Gamma}. \tag{2.3}$$

For the completeness, we give the simulation of the Markov chain as follows. Given a stepsize $\Delta > 0$, we compute the one-step transition probability matrix

$$P(\Delta) = \left(P_{ij}(\Delta)\right)_{N\times N} = e^{\Delta\Gamma}. \tag{2.4}$$

Let $r_0^\Delta = i_0$ and generate a random number ξ_1 which is uniformly distributed in $[0, 1]$. Define

$$r_1^\Delta = \begin{cases} i_1, & \text{if } i_1 \in S - \{N\} \text{ such that } \sum_{j=1}^{i_1-1} P_{i_0,j}(\Delta) \leq \xi_1 < \sum_{j=1}^{i_1} P_{i_0,j}(\Delta), \\ N, & \text{if } \sum_{j=1}^{N-1} P_{i_0,j}(\Delta) \leq \xi_1, \end{cases} \tag{2.5}$$

where we set $\sum_{i=1}^{0} P_{i_0,j}(\Delta) = 0$ as usual. Generate independently a new random number ξ_2 which is again uniformly distributed in $[0, 1]$, and then define

$$r_2^\Delta = \begin{cases} i_2, & \text{if } i_2 \in S - \{N\} \text{ such that } \sum_{j=1}^{i_2-1} P_{r_1^\Delta,j}(\Delta) \leq \xi_2 < \sum_{j=1}^{i_2} P_{r_1^\Delta,j}(\Delta), \\ N, & \text{if } \sum_{j=1}^{N-1} P_{r_1^\Delta,j}(\Delta) \leq \xi_2. \end{cases} \tag{2.6}$$

Repeating this procedure a trajectory of $\{r_k^\Delta, k = 0, 1, 2, \ldots\}$ can be generated. This procedure can be carried out independently to obtain more trajectories.

Now we can define the Euler-Maruyama (EM) approximate solution for (2.2) on the finite time interval $[0, T]$. Without loss of any generality, we may assume that T/τ is a rational number; otherwise we may replace T by a larger number. Let the step size $\Delta \in (0, 1)$ be a fraction of τ and T, namely, $\Delta = \tau/N = T/M$ for some integers $N > \tau$ and $M > T$. The explicit discrete EM approximate solution $\overline{y}(k\Delta)$, $k \geq -N$ is defined as follows:

$$\begin{aligned} \overline{y}(k\Delta) &= \xi(k\Delta), \quad -N \leq k \leq 0, \\ \overline{y}((k+1)\Delta) &= \overline{y}(k\Delta) + u\left(\overline{y}_{k\Delta}, r_k^\Delta\right) - u\left(\overline{y}_{(k-1)\Delta}, r_{k-1}^\Delta\right) + f\left(\overline{y}_{k\Delta}, r_k^\Delta\right)\Delta \\ &\quad + g\left(\overline{y}_{k\Delta}, r_k^\Delta\right)\Delta w_k, \quad 0 \leq k \leq M-1, \end{aligned} \tag{2.7}$$

where $\Delta w_k = w((k+1)\Delta) - w(k\Delta)$ and $\overline{y}_{k\Delta} = \{\overline{y}_{k\Delta}(\theta) : -\tau \le \theta \le 0\}$ is a $C([-\tau,0];\mathbb{R}^n)$-valued random variable defined by

$$\begin{aligned}\overline{y}_{k\Delta}(\theta) &= \overline{y}((k+i)\Delta) + \frac{\theta - i\Delta}{\Delta}\left[\overline{y}((k+i+1)\Delta) - \overline{y}((k+i)\Delta)\right] \\ &= \frac{\Delta - (\theta - i\Delta)}{\Delta}\overline{y}((k+i)\Delta) + \frac{\theta - i\Delta}{\Delta}\overline{y}((k+i+1)\Delta),\end{aligned} \tag{2.8}$$

for $i\Delta \le \theta \le (i+1)\Delta$, $i = -N, -N+1, \dots, -1$, where in order for $\overline{y}_{-\Delta}$ to be well defined, we set $\overline{y}(-(N+1)\Delta) = \xi(-N\Delta)$.

That is, $\overline{y}_{k\Delta}(\cdot)$ is the linear interpolation of $\overline{y}((k-N)\Delta), \overline{y}((k-N+1)\Delta), \dots, \overline{y}(k\Delta)$. We hence have

$$\begin{aligned}\left|\overline{y}_{k\Delta}(\theta)\right| &= \frac{\Delta - (\theta - i\Delta)}{\Delta}\left|\overline{y}((k+i)\Delta)\right| + \frac{\theta - i\Delta}{\Delta}\left|\overline{y}((k+i+1)\Delta)\right| \\ &\le \left|\overline{y}((k+i)\Delta)\right| \vee \left|\overline{y}((k+i+1)\Delta)\right|.\end{aligned} \tag{2.9}$$

We therefore obtain

$$\left|\overline{y}_{k\Delta}(\theta)\right| = \max_{-N\le i\le 0}\left|\overline{y}((k+i)\Delta)\right|, \quad \text{for any } k = -1, 0, 1, \dots, M-1. \tag{2.10}$$

It is obvious that $\|\overline{y}_{-\Delta}\| \le \|\overline{y}_0\|$.

In our analysis it will be more convenient to use continuous-time approximations. We hence introduce the $C([-\tau,0];\mathbb{R}^n)$-valued step process

$$\begin{aligned}\overline{y}_t &= \sum_{k=0}^{M-2}\overline{y}_{k\Delta}1_{[k\Delta,(k+1)\Delta)}(t) + \overline{y}_{(M-1)\Delta}1_{[(M-1)\Delta,M\Delta]}(t), \\ \overline{r}(t) &= \sum_{k=0}^{M-1} r_k^{\Delta}1_{[k\Delta,(k+1)\Delta)}(t),\end{aligned} \tag{2.11}$$

and we define the continuous EM approximate solution as follows: let $y(t) = \xi(t)$ for $-\tau \le t \le 0$, while for $t \in [k\Delta, (k+1)\Delta]$, $k = 0, 1, \dots, M-1$,

$$\begin{aligned}y(t) = \xi(0) &+ u\left(\overline{y}_{(k-1)\Delta} + \frac{t - k\Delta}{\Delta}\left(\overline{y}_{k\Delta} - \overline{y}_{(k-1)\Delta}\right), r_k^{\Delta}\right) - u\left(\overline{y}_{-\Delta}, r_0^{\Delta}\right) \\ &+ \int_0^t f\left(\overline{y}_s, \overline{r}(s)\right)ds + \int_0^t g\left(\overline{y}_s, \overline{r}(s)\right)dw(s).\end{aligned} \tag{2.12}$$

Clearly, (2.12) can also be written as

$$y(t) = \overline{y}(k\Delta) + u\left(\overline{y}_{(k-1)\Delta} + \frac{t-k\Delta}{\Delta}\left(\overline{y}_{k\Delta} - \overline{y}_{(k-1)\Delta}\right), r_k^{\Delta}\right) - u\left(\overline{y}_{(k-1)\Delta}, r_{k-1}^{\Delta}\right) + \int_{k\Delta}^{t} f(\overline{y}_s, \overline{r}(s))ds + \int_{k\Delta}^{t} g(\overline{y}_s, \overline{r}(s))dw(s). \tag{2.13}$$

In particular, this shows that $y(k\Delta) = \overline{y}(k\Delta)$, that is, the discrete and continuous EM approximate solutions coincide at the grid points. We know that $y(t)$ is not computable because it requires knowledge of the entire Brownian path, not just its Δ-increments. However, $y(k\Delta) = \overline{y}(k\Delta)$, so the error bound for $y(t)$ will automatically imply the error bound for $\overline{y}(k\Delta)$. It is then obvious that

$$\left\|\overline{y}_{k\Delta}\right\| \leq \left\|y_{k\Delta}\right\|, \quad \forall k = 0, 1, 2, \ldots, M-1. \tag{2.14}$$

Moreover, for any $t \in [0, T]$,

$$\begin{aligned}
\sup_{0\leq t\leq T}\left\|\overline{y}_t\right\| &= \sup_{0\leq k\leq M-1}\left\|\overline{y}_{k\Delta}\right\| \leq \sup_{0\leq k\leq M-1}\left\|y_{k\Delta}\right\| \\
&= \sup_{0\leq k\leq M-1}\sup_{-\tau\leq\theta\leq 0}\left|y(k\Delta+\theta)\right| \\
&\leq \sup_{0\leq t\leq T}\sup_{-\tau\leq\theta\leq 0}\left|y(t+\theta)\right| \\
&\leq \sup_{-\tau\leq s\leq T}\left|y(s)\right|,
\end{aligned} \tag{2.15}$$

and letting $[t/\Delta]$ be the integer part of t/Δ, then

$$\left\|\overline{y}_t\right\| = \left\|\overline{y}_{[t/\Delta]\Delta}\right\| \leq \left\|y_{[t/\Delta]\,\Delta}\right\| \leq \sup_{-\tau\leq s\leq t}\left|y(s)\right|. \tag{2.16}$$

These properties will be used frequently in what follows, without further explanation.

For the existence and uniqueness of the solution of (2.2) and the boundedness of the solution's moments, we impose the following hypotheses (e.g., see [11]).

Assumption 2.2. For each integer $j \geq 1$ and $i \in S$, there exists a positive constant C_j such that

$$\left|f(\varphi,i) - f(\psi,i)^2\right| \vee \left|g(\varphi,i) - g(\psi,i)^2\right| \leq C_j\|\varphi-\psi\|^2 \tag{2.17}$$

for $\varphi, \psi \in C([-\tau,0];\mathbb{R}^n)$ with $\|\varphi\| \vee \|\psi\| \leq j$.

Assumption 2.3. There is a constant $K > 0$ such that

$$\left|f(\varphi,i)\right|^2 \vee \left|g(\varphi,i)\right|^2 \leq K\left(1 + \|\varphi\|^2\right), \tag{2.18}$$

for $\varphi \in C([-\tau,0];\mathbb{R}^n)$ and $i \in S$.

Assumption 2.4. There exists a constant $\kappa \in (0,1)$ such that for all $\varphi, \psi \in C([-\tau,0];\mathbb{R}^n)$ and $i \in S$,

$$\left|u(\varphi,i) - u(\psi,i)\right| \leq \kappa\|\varphi - \psi\|, \tag{2.19}$$

for $\varphi, \psi \in C([-\tau,0];\mathbb{R}^n)$ and $u(0,i) = 0$.

We also impose the following condition on the initial data.

Assumption 2.5. $\xi \in L^p_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$ for some $p \geq 2$, and there exists a nondecreasing function $\alpha(\cdot)$ such that

$$\mathbb{E}\left(\sup_{-\tau\leq s\leq t\leq 0}|\xi(t) - \xi(s)|^2\right) \leq \alpha(t-s), \tag{2.20}$$

with the property $\alpha(s) \to 0$ as $s \to 0$.

From Mao and Yuan [11], we may therefore state the following theorem.

Theorem 2.6. *Let $p \geq 2$. If Assumptions 2.3–2.5 are satisfied, then*

$$\mathbb{E}\left(\sup_{-\tau\leq t\leq T}|x(t)|^p\right) \leq H_{\kappa,p,T,K,\xi}, \tag{2.21}$$

for any $T > 0$, where $H_{\kappa,p,T,K,\xi}$ is a constant dependent on κ, p, T, K, ξ.

The primary aim of this paper is to establish the following strong mean square convergence theorem for the EM approximations.

Theorem 2.7. *If Assumptions 2.2–2.5 hold,*

$$\lim_{\Delta\to 0}\mathbb{E}\left(\sup_{0\leq t\leq T}\left|x(t) - y(t)\right|^2\right) = 0. \tag{2.22}$$

The proof of this theorem is very technical, so we present some lemmas in the next section, and then complete the proof in the sequent section.

3. Lemmas

Lemma 3.1. *If Assumptions 2.3–2.5 hold, for any $p \geq 2$, there exists a constant $H(p)$ such that*

$$\mathbb{E}\left(\sup_{-\tau\leq t\leq T}\left|y(t)\right|^p\right) \leq H(p), \tag{3.1}$$

where $H(p)$ is independent of Δ.

Proof. For $t \in [k\Delta, (k+1)\Delta]$, $k = 0, 1, 2, \ldots, M-1$, set $\tilde{y}(t) := y(t) - u(\overline{y}_{(k-1)\Delta} + (t - k\Delta)(\overline{y}_{k\Delta} - \overline{y}_{(k-1)\Delta})/\Delta, r_k^{\Delta})$ and

$$h(t) := \mathbb{E}\left(\sup_{-\tau \le s \le t} |y(s)|^p\right), \qquad \tilde{h}(t) := \mathbb{E}\left(\sup_{0 \le s \le t} |\tilde{y}(s)|^p\right). \tag{3.2}$$

Recall the inequality that for $p \ge 1$ and any $\varepsilon > 0$, $|x+y|^p \le (1+\varepsilon)^{p-1}(|x|^p + \varepsilon^{1-p}|y|^p)$. Then we have, from Assumption 2.4,

$$\begin{aligned} |y(t)|^p &\le (1+\varepsilon)^{p-1}\left(|\tilde{y}(t)|^p + \varepsilon^{1-p}\left|u\left(\overline{y}_{(k-1)\Delta} + \frac{(t-k\Delta)\left(\overline{y}_{k\Delta} - \overline{y}_{(k-1)\Delta}\right)}{\Delta}, r_k^{\Delta}\right)\right|^p\right) \\ &\le (1+\varepsilon)^{p-1}\left(|\tilde{y}(t)|^p + \varepsilon^{1-p}\kappa^p\left\|\overline{y}_{(k-1)\Delta} + \frac{t-k\Delta}{\Delta}\left(\overline{y}_{k\Delta} - \overline{y}_{(k-1)\Delta}\right)\right\|^p\right). \end{aligned} \tag{3.3}$$

By $\|\overline{y}_{-\Delta}\| \le \|\overline{y}_0\|$, noting $k = 0, 1, 2, \ldots, M-1$,

$$\begin{aligned} \left\|\overline{y}_{(k-1)\Delta} + \frac{t-k\Delta}{\Delta}\left(\overline{y}_{k\Delta} - \overline{y}_{(k-1)\Delta}\right)\right\|^p &\le \left|\frac{(k+1)\Delta - t}{\Delta}\left\|\overline{y}_{(k-1)\Delta}\right\| + \frac{t-k\Delta}{\Delta}\|\overline{y}_{k\Delta}\|\right|^p \\ &\le \left[\frac{(k+1)\Delta - t}{\Delta}\left(\sup_{-\tau \le s \le t}|y(s)|\right) + \frac{t-k\Delta}{\Delta}\left(\sup_{-\tau \le s \le t}|y(s)|\right)\right]^p \\ &\le \sup_{-\tau \le s \le t}|y(s)|^p. \end{aligned} \tag{3.4}$$

Consequently, choose $\varepsilon = \kappa/(1-\kappa)$, then

$$|y(t)|^p \le (1-\kappa)^{1-p}|\tilde{y}(t)|^p + \kappa\left(\sup_{-\tau \le s \le t}|y(s)|^p\right). \tag{3.5}$$

Hence,

$$\begin{aligned} h(t) &\le \mathbb{E}\|\xi\|^p + \mathbb{E}\left(\sup_{0 \le s \le t}|y(s)|^p\right) \\ &\le \mathbb{E}\|\xi\|^p + \kappa h(t) + (1-\kappa)^{1-p}\tilde{h}(t), \end{aligned} \tag{3.6}$$

which implies

$$h(t) \le \frac{\mathbb{E}\|\xi\|^p}{1-\kappa} + \frac{\tilde{h}(t)}{(1-\kappa)^p}. \tag{3.7}$$

Since

$$\tilde{y}(t) = \tilde{y}(0) + \int_0^t f(\overline{y}_s, \overline{r}(s))ds + \int_0^t g(\overline{y}_s, \overline{r}(s))dw(s), \tag{3.8}$$

with $\tilde{y}(0) = \overline{y}(0) - u(\overline{y}_{-\Delta})$, by the Hölder inequality, we have

$$|\tilde{y}(t)|^p \leq 3^{p-1}\left[|\tilde{y}(0)|^p + t^{p-1}\int_0^t |f(\overline{y}_s, \overline{r}(s))|^p ds + \left|\int_0^t g(\overline{y}_s, \overline{r}(s))dw(s)\right|^p\right]. \tag{3.9}$$

Hence, for any $t_1 \in [0, T]$,

$$\tilde{h}(t_1) \leq 3^{p-1}\left[\mathbb{E}|\tilde{y}(0)|^p + T^{p-1}\mathbb{E}\int_0^{t_1} |f(\overline{y}_s, \overline{r}(s))|^p ds + \mathbb{E}\left(\sup_{0\leq t\leq t_1}\left|\int_0^t g(\overline{y}_s, \overline{r}(s))dw(s)\right|^p\right)\right]. \tag{3.10}$$

By Assumption 2.4 and the fact $\|\overline{y}_{-\Delta}\| \leq \|\overline{y}_0\|$, we compute that

$$\begin{aligned}
\mathbb{E}|\tilde{y}(0)|^p &= \mathbb{E}\left|\overline{y}(0) - u\left(\overline{y}_{-\Delta}, r_0^{\Delta}\right)\right|^p \\
&\leq \mathbb{E}\left(|\overline{y}(0)| + \kappa\|\overline{y}_{-\Delta}\|\right)^p \\
&\leq \mathbb{E}\left(|\overline{y}(0)| + \kappa\|\overline{y}_0\|\right)^p \\
&\leq \mathbb{E}(|\xi(0)| + \kappa\|\xi\|)^p \\
&\leq (1+\kappa)^p\mathbb{E}\|\xi\|^p.
\end{aligned} \tag{3.11}$$

Assumption 2.3 and the Hölder inequality give

$$\begin{aligned}
\mathbb{E}\int_0^{t_1} |f(\overline{y}_s, \overline{r}(s))|^p ds &\leq \mathbb{E}\int_0^{t_1} K^{p/2}\left(1 + \|\overline{y}_s\|^2\right)^{p/2} ds \\
&\leq K^{p/2}2^{(p-2)/2}\mathbb{E}\int_0^{t_1}(1 + \|\overline{y}_s\|^p)ds \\
&\leq K^{p/2}2^{(p-2)/2}\left[T + \int_0^{t_1}\mathbb{E}\left(\sup_{-\tau\leq t\leq s}|y(t)|^p ds\right)\right].
\end{aligned} \tag{3.12}$$

Applying the Burkholder-Davis-Gundy inequality, the Hölder inequality and Assumption 2.3 yield

$$
\begin{aligned}
\mathbb{E}\left(\sup_{0\le t\le t_1}\left|\int_0^t g(\overline{y}_s,\overline{r}(s))dw(s)\right|^p\right) &\le C_p\mathbb{E}\left(\int_0^{t_1}\left|g(\overline{y}_s,\overline{r}(s))\right|^2 ds\right)^{p/2} \\
&\le C_pT^{(p-2)/2}\mathbb{E}\int_0^{t_1}\left|g(\overline{y}_s,\overline{r}(s))\right|^p ds \\
&\le C_pT^{(p-2)/2}\mathbb{E}\int_0^{t_1}K^{p/2}\left(1+\|\overline{y}_s\|^2\right)^{p/2}ds \\
&\le C_pT^{(p-2)/2}K^{p/2}2^{(p-2)/2}\mathbb{E}\int_0^{t_1}\left(1+\|\overline{y}_s\|^p\right)ds \\
&\le C_pT^{(p-2)/2}K^{p/2}2^{(p-2)/2}\left[T+\int_0^{t_1}\mathbb{E}\left(\sup_{-\tau\le t\le s}|y(t)|^p\right)ds\right],
\end{aligned}
\tag{3.13}
$$

where C_p is a constant dependent only on p. Substituting (3.11), (3.12), and (3.13) into (3.10) gives

$$
\begin{aligned}
\widetilde{h}(t_1) &\le 3^{p-1}\left[(1+\kappa)^p\mathbb{E}\|\xi\|^p+K^{p/2}2^{(p-2)/2}T^p+C_p(2T)^{(p-2)/2}K^{p/2}T\right] \\
&\quad+3^{p-1}\left[K^{p/2}2^{(p-2)/2}T^{p-1}+C_p(2T)^{(p-2)/2}K^{p/2}\right]\int_0^{t_1}\mathbb{E}\left(\sup_{-\tau\le t\le s}|y(t)|^p\right)ds \\
&=: C_1+C_2\int_0^{t_1}h(s)ds.
\end{aligned}
\tag{3.14}
$$

Hence from (3.7), we have

$$
\begin{aligned}
h(t_1) &\le \frac{\mathbb{E}\|\xi\|^p}{1-\kappa}+\frac{1}{(1-\kappa)^p}\left[C_1+C_2\int_0^{t_1}h(s)ds\right] \\
&\le \frac{\mathbb{E}\|\xi\|^p}{1-\kappa}+\frac{C_1}{(1-\kappa)^p}+\frac{C_2}{(1-\kappa)^p}\int_0^{t_1}h(s)ds.
\end{aligned}
\tag{3.15}
$$

By the Gronwall inequality we find that

$$
h(T)\le\left[\frac{\mathbb{E}\|\xi\|^p}{1-\kappa}+\frac{C_1}{(1-\kappa)^p}\right]e^{C_2T/(1-\kappa)^p}.
\tag{3.16}
$$

From the expressions of C_1 and C_2, we know that they are positive constants dependent only on ξ, κ, K, p, and T, but independent of Δ. The proof is complete. □

Lemma 3.2. *If Assumptions 2.3–2.5 hold, then for any integer $l > 1$,*

$$\mathbb{E}\left(\sup_{0\le k\le M-1}\left\|\overline{y}_{k\Delta}-\overline{y}_{(k-1)\Delta}\right\|^2\right)\le c_1'+c_1\alpha(\Delta)+\overline{c}_1(l)\Delta^{(l-1)/l}=:\gamma(\Delta),\tag{3.17}$$

where $c_1 = 1/(1-2\kappa)$, $c_1' = (8\kappa/(1-2\kappa))H(2)$, and $\overline{c}_1(l)$ is a constant dependent on l but independent of Δ.

Proof. For $\theta\in[i\Delta,(i+1)\Delta]$, where $i=-N,-(N+1),\dots,-1$, from (2.7),

$$\begin{aligned}\left|\overline{y}_{k\Delta}-\overline{y}_{(k-1)\Delta}\right| &\le \frac{(i+1)\Delta-\theta}{\Delta}\left|\overline{y}((k+i)\Delta)-\overline{y}((k-1+i)\Delta)\right|\\ &\quad+\frac{\theta-i\Delta}{\Delta}\left|\overline{y}((k+i+1)\Delta)-\overline{y}((k+i)\Delta)\right|\\ &\le \left|\overline{y}((k+i)\Delta)-\overline{y}((k-1+i)\Delta)\right|\vee\left|\overline{y}((k+i+1)\Delta)-\overline{y}((k+i)\Delta)\right|,\end{aligned}\tag{3.18}$$

so

$$\left\|\overline{y}_{k\Delta}-\overline{y}_{(k-1)\Delta}\right\|\le\sup_{-N\le i\le 0}\left|\overline{y}((k+i)\Delta)-\overline{y}((k-1+i)\Delta)\right|.\tag{3.19}$$

We therefore have

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le k\le M-1}\left\|\overline{y}_{k\Delta}-\overline{y}_{(k-1)\Delta}\right\|^2\right)&\le\mathbb{E}\left[\sup_{0\le k\le M-1}\left(\sup_{-N\le i\le 0}\left|\overline{y}((k+i)\Delta)-\overline{y}((k-1+i)\Delta)\right|^2\right)\right]\\ &\le\mathbb{E}\left(\sup_{-N\le k\le M-1}\left|\overline{y}(k\Delta)-\overline{y}((k-1)\Delta)\right|^2\right).\end{aligned}\tag{3.20}$$

When $-N\le k\le 0$, by Assumption 2.5 and $\overline{y}(-(N+1)\Delta)=\xi(-N\Delta)$,

$$\begin{aligned}\mathbb{E}\left(\sup_{-N\le k\le 0}\left|\overline{y}(k\Delta)-\overline{y}((k-1)\Delta)\right|^2\right)&\le\mathbb{E}\left(\sup_{-N\le k\le 0}\left|\xi(k\Delta)-\xi((k-1)\Delta)\right|^2\right)\\ &\le\alpha(\Delta).\end{aligned}\tag{3.21}$$

When $1 \le k \le M-1$, from (2.13), we have

$$\overline{y}(k\Delta) - \overline{y}((k-1)\Delta) = u\Big(\overline{y}_{(k-1)\Delta}, r^{\Delta}_{k-1}\Big) - u\Big(\overline{y}_{(k-2)\Delta}, r^{\Delta}_{k-2}\Big) + f\Big(\overline{y}_{(k-1)\Delta}, r^{\Delta}_{k-1}\Big)\Delta + g\Big(\overline{y}_{(k-1)\Delta}, r^{\Delta}_{k-1}\Big)\Delta w_{k-1}. \tag{3.22}$$

Recall the elementary inequality, for any $x, y > 0$ and $\varepsilon \in (0,1)$, $(x+y)^2 \le x^2/\varepsilon + y^2/(1-\varepsilon)$. Then

$$\begin{aligned}
&\left|\overline{y}(k\Delta) - \overline{y}((k-1)\Delta)\right|^2 \\
&\quad\le \frac{1}{\varepsilon}\left|u\Big(\overline{y}_{(k-1)\Delta}, r^{\Delta}_{k-1}\Big) - u\Big(\overline{y}_{(k-2)\Delta}, r^{\Delta}_{k-1}\Big) + u\Big(\overline{y}_{(k-2)\Delta}, r^{\Delta}_{k-1}\Big) - u\Big(\overline{y}_{(k-2)\Delta}, r^{\Delta}_{k-2}\Big)\right|^2 \\
&\qquad+ \frac{1}{1-\varepsilon}\left|f\Big(\overline{y}_{(k-1)\Delta}, r^{\Delta}_{k-1}\Big)\Delta + g\Big(\overline{y}_{(k-1)\Delta}, r^{\Delta}_{k-1}\Big)\Delta w_{k-1}\right|^2 \\
&\quad\le \frac{2\kappa^2}{\varepsilon}\left\|\overline{y}_{(k-1)\Delta} - \overline{y}_{(k-2)\Delta}\right\|^2 + \frac{8\kappa^2}{\varepsilon}\left\|\overline{y}_{(k-2)\Delta}\right\|^2 \\
&\qquad+ \frac{2}{1-\varepsilon}\left|f\Big(\overline{y}_{(k-1)\Delta}, r^{\Delta}_{k-1}\Big)\right|^2\Delta^2 + \frac{2}{1-\varepsilon}\left|g\Big(\overline{y}_{(k-1)\Delta}, r^{\Delta}_{k-1}\Big)\Delta w_{k-1}\right|^2.
\end{aligned} \tag{3.23}$$

Consequently

$$\begin{aligned}
&\mathbb{E}\left(\sup_{1\le k\le M-1}\left|\overline{y}(k\Delta) - \overline{y}((k-1)\Delta)\right|^2\right) \\
&\quad\le \frac{2\kappa^2}{\varepsilon}\mathbb{E}\left(\sup_{1\le k\le M-1}\left\|\overline{y}_{(k-1)\Delta} - \overline{y}_{(k-2)\Delta}\right\|^2\right) + \frac{8\kappa^2}{\varepsilon}\mathbb{E}\left(\sup_{1\le k\le M-1}\left\|\overline{y}_{(k-2)\Delta}\right\|^2\right) \\
&\qquad+ \frac{2\Delta^2}{1-\varepsilon}\mathbb{E}\left(\sup_{1\le k\le M-1}\left|f\Big(\overline{y}_{(k-1)\Delta}, \overline{r}(t)\Big)\right|^2\right) + \frac{2}{1-\varepsilon}\mathbb{E}\left(\sup_{1\le k\le M-1}\left|g\Big(\overline{y}_{(k-1)\Delta}, \overline{r}(t)\Big)\Delta w_{k-1}\right|^2\right).
\end{aligned} \tag{3.24}$$

We deal with these four terms, separately. By (3.21),

$$\begin{aligned}
&\mathbb{E}\left(\sup_{1\le k\le M-1}\left\|\overline{y}_{(k-1)\Delta} - \overline{y}_{(k-2)\Delta}\right\|^2\right) \\
&\quad\le \mathbb{E}\left(\sup_{1\le k\le M-1}\sup_{-N\le i\le 0}\left|\overline{y}((k+i-1)\Delta) - \overline{y}((k-2+i)\Delta)\right|^2\right) \\
&\quad\le \mathbb{E}\left(\sup_{-N\le k\le M-1}\left|\overline{y}(k\Delta) - \overline{y}((k-1)\Delta)\right|^2\right)
\end{aligned}$$

$$\leq \mathbb{E}\left(\sup_{-N\leq k\leq 0}\left|\overline{y}(k\Delta)-\overline{y}((k-1)\Delta)\right|^2\right)+\mathbb{E}\left(\sup_{1\leq k\leq M-1}\left|\overline{y}(k\Delta)-\overline{y}((k-1)\Delta)\right|^2\right)$$
$$\leq \alpha(\Delta)+\mathbb{E}\left(\sup_{1\leq k\leq M-1}\left|\overline{y}(k\Delta)-\overline{y}((k-1)\Delta)\right|^2\right). \tag{3.25}$$

Noting that $\mathbb{E}[\sup_{-\tau\leq t\leq T}|\overline{y}(t)|^2]\leq \mathbb{E}[\sup_{-\tau\leq t\leq T}|y(t)|^2]\leq H(2)$ (where $H(p)$ has been defined in Lemma 3.1), by Assumption 2.3 and (2.15),

$$\begin{aligned}
&\mathbb{E}\left(\sup_{1\leq k\leq M-1}\left|f\left(\overline{y}_{(k-1)\Delta},r_{k-1}^{\Delta}\right)\right|^2\right)\\
&\quad\leq \mathbb{E}\left(\sup_{1\leq k\leq M-1}K\left(1+\left\|\overline{y}_{(k-1)\Delta}\right\|^2\right)\right)\\
&\quad\leq K+K\mathbb{E}\left(\sup_{1\leq k\leq M-1}\sup_{-N\leq i\leq 0}\left|\overline{y}((k-1+i)\Delta)\right|^2\right)\\
&\quad\leq K+K\mathbb{E}\left(\sup_{-N\leq k\leq M-1}\left|\overline{y}(k\Delta)\right|^2\right)\\
&\quad\leq K+K\mathbb{E}\left(\sup_{-\tau\leq t\leq T}\left|\overline{y}(t)\right|^2\right)\\
&\quad\leq K(1+H(2)).
\end{aligned}\tag{3.26}$$

By the Hölder inequality, for any integer $l>1$,

$$\begin{aligned}
&\mathbb{E}\left(\sup_{1\leq k\leq M-1}\left|g\left(\overline{y}_{(k-1)\Delta},r_{k-1}^{\Delta}\right)\Delta w_{k-1}\right|^2\right)\\
&\quad\leq \mathbb{E}\left(\sup_{1\leq k\leq M-1}\left|g\left(\overline{y}_{(k-1)\Delta},\overline{r}(t)\right)\right|^2\sup_{1\leq k\leq M-1}|\Delta w_{k-1}|^2\right)\\
&\quad\leq \left[\mathbb{E}\left(\sup_{1\leq k\leq M-1}\left|g\left(\overline{y}_{(k-1)\Delta},r_{k-1}^{\Delta}\right)\right|^{2l/(l-1)}\right)\right]^{(l-1)/l}\left[\mathbb{E}\left(\sup_{1\leq k\leq M-1}|\Delta w_{k-1}|^{2l}\right)\right]^{1/l}\\
&\quad\leq \left[\mathbb{E}\left(\sup_{0\leq k\leq M-1}\left(K\left(1+\|\overline{y}_{k\Delta}\|^2\right)\right)^{l/(l-1)}\right)\right]^{(l-1)/l}\left[\mathbb{E}\left(\sum_{k=0}^{M-1}|\Delta w_k|^{2l}\right)\right]^{1/l}\\
&\quad\leq \left[K^{l/(1-l)}\mathbb{E}\left(1+\left(\sup_{0\leq k\leq M-1}\|\overline{y}_{k\Delta}\|^2\right)\right)^{l/(l-1)}\right]^{(l-1)/l}\left[\left(\sum_{k=0}^{M-1}\mathbb{E}|\Delta w_k|^{2l}\right)\right]^{1/l}
\end{aligned}$$

$$\leq \left[2^{1/(l-1)}K^{l/(1-l)}\left(1+\mathbb{E}\left(\sup_{0\leq k\leq M-1}\|\overline{y}_{k\Delta}\|^{2l/(l-1)}\right)\right)\right]^{(l-1)/l}\left[\left(\sum_{k=0}^{M-1}(2l-1)!!\Delta^{l}\right)\right]^{1/l}$$
$$\leq \left[2^{1/(l-1)}K^{l/(1-l)}\left(1+H\left(\frac{2l}{l-1}\right)\right)\right]^{(l-1)/l}\left[(2l-1)!!T\Delta^{l-1}\right]^{1/l}$$
$$\leq D(l)\Delta^{(l-1)/l}, \tag{3.27}$$

where $D(l)$ is a constant dependent on l.

Substituting (3.25), (3.26), and (3.27) into (3.24), choosing $\varepsilon=\kappa$, and noting $\Delta\in(0,1)$, we have

$$\begin{aligned}&\mathbb{E}\left(\sup_{1\leq k\leq M-1}\left|\overline{y}(k\Delta)-\overline{y}((k-1)\Delta)\right|^{2}\right)\\&\quad\leq \frac{2\kappa}{1-2\kappa}\alpha(\Delta)+\frac{8\kappa}{1-2\kappa}H(2)+\frac{2K(1+H(2))+2D(l)}{(1-2\kappa)^{2}}\Delta^{(l-1)/l}.\end{aligned} \tag{3.28}$$

Combining (3.21) with (3.28), from (3.20), we have

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\leq k\leq M-1}\left\|\overline{y}_{k\Delta}-\overline{y}_{(k-1)\Delta}\right\|^{2}\right)\\&\quad\leq \mathbb{E}\left(\sup_{-N\leq k\leq M-1}\left|\overline{y}(k\Delta)-\overline{y}((k-1)\Delta)\right|^{2}\right)\\&\quad\leq \mathbb{E}\left(\sup_{-N\leq k\leq 0}\left|\overline{y}(k\Delta)-\overline{y}((k-1)\Delta)\right|^{2}\right)+\mathbb{E}\left(\sup_{1\leq k\leq M-1}\left|\overline{y}(k\Delta)-\overline{y}((k-1)\Delta)\right|^{2}\right)\\&\quad\leq \frac{1}{1-2\kappa}\alpha(\Delta)+\frac{8\kappa}{1-2\kappa}H(2)+\frac{2K(1+H(2))+2D(l)}{(1-2\kappa)^{2}}\Delta^{(l-1)/l},\end{aligned} \tag{3.29}$$

as required. □

Lemma 3.3. *If Assumptions 2.3–2.5 hold,*

$$\mathbb{E}\left(\sup_{0\leq s\leq T}\left\|y_s-\overline{y}_s\right\|^{2}\right)\leq c_2'+c_2\alpha(2\Delta)+\overline{c}_2(l)\Delta^{(l-1)/l}=:\beta(\Delta), \tag{3.30}$$

where c_2, c_2' are a constant independent of l and Δ, $\overline{c}_2(l)$ is a constant dependent on l but independent of Δ.

Proof. Fix any $s \in [0,T]$ and $\theta \in [-\tau, 0]$. Let $k_s \in \{0,1,2,\ldots,M-1\}$, $k_\theta \in \{-N,-N+1,\ldots,-1\}$, and $k_{s\theta} \in \{-N,-N+1,\ldots,M-1\}$ be the integers for which $s \in [k_s\Delta,(k_s+1)\Delta]$, $\theta \in [k_\theta\Delta,(k_\theta+1)\Delta]$, and $s+\theta \in [k_{s\theta}\Delta,(k_{s\theta}+1)\Delta]$, respectively. Clearly,

$$\begin{gathered} 0 \le s+\theta-(k_s+k_\theta) \le 2\Delta, \\ k_{s\theta}-(k_s+k_\theta) \in \{0,1,2\}. \end{gathered} \tag{3.31}$$

From (2.7),

$$\begin{aligned} \overline{y}_s &= \overline{y}_{k_s\Delta}(\theta) \\ &= \overline{y}((k_s+k_\theta)\Delta) + \frac{\theta-k_\theta\Delta}{\Delta}\left(\overline{y}((k_s+k_\theta+1)\Delta) - \overline{y}(k_s+k_\theta)\Delta\right), \end{aligned} \tag{3.32}$$

which yields

$$\begin{aligned} \left|y_s-\overline{y}_s\right| &= \left|y(s+\theta)-\overline{y}_{k_s\Delta}(\theta)\right| \\ &\le \left|y(s+\theta)-\overline{y}((k_s+k_\theta)\Delta)\right| + \frac{\theta-k_\theta\Delta}{\Delta}\left|\overline{y}((k_s+k_\theta+1)\Delta)-\overline{y}((k_s+k_\theta)\Delta)\right| \\ &\le \left|y(s+\theta)-\overline{y}((k_s+k_\theta)\Delta)\right| + \left|\overline{y}((k_s+k_\theta+1)\Delta)-\overline{y}((k_s+k_\theta)\Delta)\right|, \end{aligned} \tag{3.33}$$

so by (3.20) and Lemma 3.2, noting $\overline{y}(M\Delta) = \overline{y}((M-1)\Delta)$ from (2.15),

$$\begin{aligned} \mathbb{E}\left(\sup_{0\le s\le T}\left\|y_s-\overline{y}_s\right\|^2\right) &\le 2\mathbb{E}\left[\sup_{0\le s\le T}\left(\sup_{-\tau\le\theta\le 0}\left|y(s+\theta)-\overline{y}((k_s+k_\theta)\Delta)\right|^2\right)\right] \\ &\quad + 2\mathbb{E}\left[\sup_{0\le k_s\le M-1}\left(\sup_{-N\le k_\theta\le 0}\left|\overline{y}((k_s+k_\theta+1)\Delta)-\overline{y}((k_s+k_\theta)\Delta)\right|^2\right)\right] \\ &\le 2\mathbb{E}\left(\sup_{-\tau\le\theta\le 0, 0\le s\le T}\left|y(s+\theta)-\overline{y}((k_s+k_\theta)\Delta)\right|^2\right) + 2\gamma(\Delta). \end{aligned} \tag{3.34}$$

Therefore, it is a key to compute $\mathbb{E}(\sup_{-\tau\le\theta\le 0,0\le s\le T}|y(s+\theta)-\overline{y}((k_s+k_\theta)\Delta)|^2)$. We discuss the following four possible cases.

Case 1 ($k_s+k_\theta \ge 0$). We again divide this case into three possible subcases according to $k_{s\theta}-(k_s+k_\theta) \in \{0,1,2\}$.

Subcase 1 ($k_{s\theta}-(k_s+k_\theta)=0$). From (2.13),

$$y(s+\theta)-\overline{y}((k_s+k_\theta)\Delta)=u\left(\overline{y}_{(k_{s\theta}-1)\ \Delta}+\frac{s+\theta-k_{s\theta}\Delta}{\Delta}\left(\overline{y}_{k_{s\theta}\Delta}-\overline{y}_{(k_{s\theta}-1)\Delta}\right),r^{\Delta}_{k_{s\theta}}\right)-u\left(\overline{y}_{(k_{s\theta}-1)\Delta},r^{\Delta}_{k_{s\theta}-1}\right)+\int_{k_{s\theta}\Delta}^{s+\theta}f\left(\overline{y}_v,\overline{r}(v)\right)dv+\int_{k_{s\theta}\Delta}^{s+\theta}g\left(\overline{y}_v,\overline{r}(v)\right)dw(v), \tag{3.35}$$

which yields

$$\begin{aligned}\mathbb{E}&\left(\sup_{-\tau\le\theta\le0,0\le s\le T,k_s+k_\theta\ge0}\left|y(s+\theta)-\overline{y}((k_s+k_\theta)\Delta)\right|^2\right)\\&\le3\mathbb{E}\left(\sup_{-\tau\le\theta\le0,0\le s\le T,k_{s\theta}\ge0}\left|u\left(\overline{y}_{(k_{s\theta}-1)\Delta}+\frac{s+\theta-k_{s\theta}\Delta}{\Delta}\left(\overline{y}_{k_{s\theta}\Delta}-\overline{y}_{(k_{s\theta}-1)\Delta},r^{\Delta}_{k_{s\theta}}\right)\right)-u\left(\overline{y}_{(k_{s\theta}-1)\Delta},r^{\Delta}_{k_{s\theta}-1}\right)\right|^2\right)+3\mathbb{E}\left(\sup_{-\tau\le\theta\le0,0\le s\le T,k_{s\theta}\ge0}\left|\int_{k_{s\theta}\Delta}^{s+\theta}f\left(\overline{y}_r,\overline{r}(r)\right)dr\right|^2\right)\\&\quad+3\mathbb{E}\left(\sup_{-\tau\le\theta\le0,0\le s\le T,k_{s\theta}\ge0}\left|\int_{k_{s\theta}\Delta}^{s+\theta}g\left(\overline{y}_r,\overline{r}(r)\right)dw(r)\right|^2\right).\end{aligned} \tag{3.36}$$

From Assumption 2.4, (3.24), (3.26), and Lemma 3.2, we have

$$\begin{aligned}\mathbb{E}&\left(\sup_{-\tau\le\theta\le0,0\le s\le T,k_{s\theta}\ge0}\left|u\left(\overline{y}_{(k_{s\theta}-1)\Delta}+\frac{s+\theta-k_{s\theta}\Delta}{\Delta}\left(\overline{y}_{k_{s\theta}\Delta}-\overline{y}_{(k_{s\theta}-1)\Delta},r^{\Delta}_{k_{s\theta}}\right)\right)-u\left(\overline{y}_{(k_{s\theta}-1)\Delta},r^{\Delta}_{k_{s\theta}-1}\right)\right|^2\right)\\&\le2\kappa^2\mathbb{E}\left(\sup_{-\tau\le\theta\le0,0\le s\le T,k_{s\theta}\ge0}\left\|\frac{s+\theta-k_{s\theta}\Delta}{\Delta}\left(\overline{y}_{k_{s\theta}\Delta}-\overline{y}_{(k_{s\theta}-1)\Delta}\right)\right\|^2\right)+8\kappa^2H(2)\\&\le2\kappa^2\mathbb{E}\left(\sup_{-\tau\le\theta\le0,0\le s\le T,k_{s\theta}\ge0}\left\|\left(\overline{y}_{k_{s\theta}\Delta}-\overline{y}_{(k_{s\theta}-1)\Delta}\right)\right\|^2\right)+8\kappa^2H(2)\\&\le2\kappa^2\mathbb{E}\left(\sup_{0\le k_{s\theta}\le M-1}\left\|\left(\overline{y}_{k_{s\theta}\Delta}-\overline{y}_{(k_{s\theta}-1)\Delta}\right)\right\|^2\right)+8\kappa^2H(2)\\&\le2\kappa^2\gamma(\Delta)+8\kappa^2H(2).\end{aligned} \tag{3.37}$$

By the Hölder inequality, Assumption 2.3, and Lemma 3.1,

$$\begin{aligned}
&\mathbb{E}\left(\sup_{-\tau\le\theta\le 0,0\le s\le T,k_{s\theta}\ge 0}\left|\int_{k_{s\theta}\Delta}^{s+\theta} f(\overline{y}_r,\overline{r}(r))dr\right|^2\right)\\
&\quad\le \Delta\mathbb{E}\left(\sup_{-\tau\le\theta\le 0,0\le s\le T,k_{s\theta}\ge 0}\int_{k_{s\theta}\Delta}^{s+\theta}\left|f(\overline{y}_r,\overline{r}(r))\right|^2 dr\right)\\
&\quad\le K\Delta\mathbb{E}\left(\sup_{-\tau\le\theta\le 0,0\le s\le T,k_{s\theta}\ge 0}\int_{k_{s\theta}\Delta}^{s+\theta}\left(1+\|\overline{y}_r\|^2\right)dr\right)\\
&\quad\le K\Delta\mathbb{E}\left[\int_0^T\left(1+\sup_{0\le r\le T}\|\overline{y}_r\|^2\right)dr\right]\\
&\quad\le K\Delta\int_0^T\left[1+\mathbb{E}\left(\sup_{-\tau\le t\le T} y(t)|^2\right)\right]dr\\
&\quad\le K\Delta\int_0^T[1+H(2)]dr\\
&\quad\le KT[1+H(2)]\Delta.
\end{aligned}\tag{3.38}$$

Setting $v = s+\theta$ and $k_v = k_{s\theta}$ and applying the Hölder inequality yield

$$\begin{aligned}
&\mathbb{E}\left(\sup_{-\tau\le\theta\le 0,0\le s\le T,k_{s\theta}\ge 0}\left|\int_{k_{s\theta}\Delta}^{s+\theta} g(\overline{y}_r,\overline{r}(r))dw(r)\right|^2\right)\\
&\quad= \mathbb{E}\left(\sup_{0\le v\le T,0\le k_v\le M-1}\left|g\left(\overline{y}_{k_v\Delta},\overline{r}(v)\right)(w(v)-w(k_v\Delta))\right|^2\right)\\
&\quad\le \left[\mathbb{E}\left(\sup_{0\le v\le T,0\le k_v\le M-1}\left|g\left(\overline{y}_{k_v\Delta},\overline{r}(v)\right)\right|^{2l/(l-1)}\right)\right]^{(l-1)/l}\\
&\qquad\times\left[\mathbb{E}\left(\sup_{0\le v\le T,0\le k_v\le M-1}|w(v)-w(k_v\Delta)|^{2l}\right)\right]^{1/l}.
\end{aligned}\tag{3.39}$$

The Doob martingale inequality gives

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le v\le T,0\le k_v\le M-1}|w(v)-w(k_v\Delta)|^{2l}\right)\\
&\quad= \mathbb{E}\left(\sup_{0\le k_v\le M-1}\left(\sup_{k_v\Delta\le v\le(k_v+1)\Delta}|w(v)-w(k_v\Delta)|^{2l}\right)\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \mathbb{E}\left(\sum_{k_v=0}^{M-1}\left(\sup_{k_v\Delta\leq v\leq (k_v+1)\Delta}|w(v)-w(k_v\Delta)|^{2l}\right)\right)\\
&=\sum_{k_v=0}^{M-1}\mathbb{E}\left(\sup_{k_v\Delta\leq v\leq (k_v+1)\Delta}|w(v)-w(k_v\Delta)|^{2l}\right)\\
&\leq \left(\frac{2l}{2l-1}\right)^{2l}\sum_{k_v=0}^{M-1}\mathbb{E}|w((k_v+1)\Delta)-w(k_v\Delta)|^{2l}.
\end{aligned} \tag{3.40}$$

By (3.27), we therefore have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{-\tau\leq\theta\leq 0,0\leq s\leq T,k_{s\theta}\geq 0}\left|\int_{k_{s\theta}\Delta}^{s+\theta}g(\overline{y}_r,\overline{r}(r))dw(r)\right|^2\right)\\
&\leq\left(\frac{2l}{2l-1}\right)^2\left[\mathbb{E}\left(\sup_{0\leq k_v\leq M-1}\left|g\left(\overline{y}_{k_v\Delta},\overline{r}(v)\right)\right|^{2l/(l-1)}\right)\right]^{(l-1)/l}\\
&\quad\times\left[\sum_{k_v=0}^{M-1}\mathbb{E}|w((k_v+1)\Delta)-w(k_v\Delta)|^{2l}\right]^{1/l}\\
&\leq\left(\frac{2l}{2l-1}\right)^2 D(l)\Delta^{(l-1)/l}.
\end{aligned} \tag{3.41}$$

Substituting (3.37), (3.38), and (3.41) into (3.36) and noting $\Delta\in(0,1)$ give

$$\begin{aligned}
&\mathbb{E}\left(\sup_{-\tau\leq\theta\leq 0,0\leq s\leq T,k_s+k_\theta\geq 0}\left|y(s+\theta)-\overline{y}((k_s+k_\theta)\Delta)\right|^2\right)\\
&\leq 6\kappa^2\gamma(\Delta)+24\kappa^2H(2)+3\left(KT(1+H(2))+\left(\frac{2l}{2l-1}\right)^2D(l)\right)\Delta^{(l-1)/l}\\
&=: 6\kappa^2\gamma(\Delta)+24\kappa^2H(2)+c_1(l)\Delta^{(l-1)/l}.
\end{aligned} \tag{3.42}$$

*Subcase*2 ($k_{s\theta}-(k_s+k_\theta)=1$). From (2.13),

$$\begin{aligned}
y(s+\theta)-\overline{y}((k_s+k_\theta)\Delta) &= y(k_{s\theta}\Delta)-\overline{y}((k_s+k_\theta)\Delta)+y(s+\theta)-\overline{y}(k_{s\theta}\Delta)\\
&\leq u\left(\overline{y}_{(k_s+k_\theta)\Delta},r^{\Delta}_{k_s+k_\theta}\right)-u\left(\overline{y}_{(k_s+k_\theta-1)\Delta},r^{\Delta}_{k_s+k_\theta-1}\right)+f\left(\overline{y}_{(k_s+k_\theta)\Delta},r^{\Delta}_{k_s+k_\theta}\right)\Delta\\
&\quad+g\left(\overline{y}_{(k_s+k_\theta)\Delta},r^{\Delta}_{k_s+k_\theta}\right)\Delta w_{k_s+k_\theta}+y(s+\theta)-\overline{y}(k_{s\theta}\Delta),
\end{aligned} \tag{3.43}$$

so we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{-\tau\le\theta\le 0,0\le s\le T,k_s+k_\theta\ge 0}\left|y(s+\theta)-\overline{y}((k_s+k_\theta)\Delta)\right|^2\right)\\
&\quad\le 4\left\{\mathbb{E}\left(\sup_{-\tau\le\theta\le 0,0\le s\le T,k_s+k_\theta\ge 0}\left|u\left(\overline{y}_{(k_s+k_\theta)\Delta},r^{\Delta}_{k_s+k_\theta}\right)-u\left(\overline{y}_{(k_s+k_\theta-1)\Delta},r^{\Delta}_{k_s+k_\theta-1}\right)\right|^2\right)\right.\\
&\qquad+\mathbb{E}\left(\sup_{-\tau\le\theta\le 0,0\le s\le T,k_s+k_\theta\ge 0}\left|f\left(\overline{y}_{(k_s+k_\theta)\Delta},r^{\Delta}_{k_s+k_\theta}\right)\Delta\right|^2\right)\\
&\qquad+\mathbb{E}\left(\sup_{-\tau\le\theta\le 0,0\le s\le T,k_s+k_\theta\ge 0}\left|g\left(\overline{y}_{(k_s+k_\theta)\Delta},r^{\Delta}_{k_s+k_\theta}\right)\Delta w_{k_s+k_\theta}\right|^2\right)\\
&\qquad\left.+\mathbb{E}\left(\sup_{-\tau\le\theta\le 0,0\le s\le T,k_s+k_\theta\ge 0}\left|y(s+\theta)-\overline{y}(k_{s\theta}\Delta)\right|^2\right)\right\}.
\end{aligned}\tag{3.44}$$

Since

$$\mathbb{E}\left(\sup_{-\tau\le\theta\le 0,0\le s\le T,k_s+k_\theta\ge 0}\left|u\left(\overline{y}_{(k_s+k_\theta)\Delta},r^{\Delta}_{k_s+k_\theta}\right)-u\left(\overline{y}_{(k_s+k_\theta-1)\Delta},r^{\Delta}_{k_s+k_\theta-1}\right)\right|^2\right)\le 2\kappa^2\gamma(\Delta)+8\kappa^2H(2),\tag{3.45}$$

from (3.26), (3.27), and the Subcase 1, noting $\Delta\in(0,1)$, we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{-\tau\le\theta\le 0,0\le s\le T,k_s+k_\theta\ge 0}\left|y(s+\theta)-\overline{y}((k_s+k_\theta)\Delta)\right|^2\right)\\
&\quad\le 4\left\{2\kappa^2\gamma(\Delta)+8\kappa^2H(2)+K[1+H(2)]\Delta^2+D(l)\Delta^{(l-1)/l}\right.\\
&\qquad\left.+3\left(2\kappa^2\gamma(\Delta)+8\kappa^2H(2)\right)(\Delta)+c_1(l)\Delta^{(l-1)/l}\right\}\\
&\quad=: 32\kappa^2\gamma(\Delta)+128\kappa^2H(2)+c_2(l)\Delta^{(l-1)/l}.
\end{aligned}\tag{3.46}$$

Subcase 3 ($k_{s\theta}-(k_s+k_\theta)=2$). From (2.13), we have

$$y(s+\theta)-\overline{y}((k_s+k_\theta)\Delta)=y(s+\theta)-y((k_{s\theta}-1)\Delta)+y((k_{s\theta}-1)\Delta)-\overline{y}((k_s+k_\theta)\Delta),\tag{3.47}$$

so from the Subcase 2, we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{-\tau\le\theta\le 0, 0\le s\le T, k_s+k_\theta\ge 0}\left|y(s+\theta)-\overline{y}((k_s+k_\theta)\Delta)\right|^2\right)\\
&\quad\le 2\mathbb{E}\left(\sup_{-\tau\le\theta\le 0, 0\le s\le T, k_s+k_\theta\ge 0}\left|y(s+\theta)-\overline{y}((k_{s\theta}-1)\Delta)\right|^2\right)\\
&\qquad+2\mathbb{E}\left(\sup_{-\tau\le\theta\le 0, 0\le s\le T, k_s+k_\theta\ge 0}\left|y((k_{s\theta}-1)\Delta)-\overline{y}((k_s+k_\theta)\Delta)\right|^2\right)\\
&\quad\le 2\left[32\kappa^2\gamma(\Delta)+128\kappa^2H(2)+c_2(l)\Delta^{(l-1)/l}\right]\\
&\qquad+2\left[2\kappa^2\gamma(\Delta)+8\kappa^2H(2)+K[1+H(2)]\Delta^2+D(l)\Delta^{(l-1)/l}\right]\\
&\quad=: 68\kappa^2\gamma(\Delta)+272\kappa^2H(2)+c_3(l)\Delta^{(l-1)/l}.
\end{aligned}\tag{3.48}$$

From these three subcases, we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{-\tau\le\theta\le 0, 0\le s\le T, k_s+k_\theta\ge 0}\left|y(s+\theta)-\overline{y}((k_s+k_\theta)\Delta)\right|^2\right)\\
&\quad\le 106\kappa^2\gamma(\Delta)+424\kappa^2H(2)+[c_1(l)+c_2(l)+c_3(l)]\Delta^{(l-1)/l}.
\end{aligned}\tag{3.49}$$

Case 2 ($k_s+k_\theta=-1$ and $0\le s+\theta\le\Delta$). In this case, applying Assumption 2.5 and case 1, we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{-\tau\le\theta\le 0, 0\le s\le T}\left|y(s+\theta)-\overline{y}((k_s+k_\theta)\Delta)\right|^2\right)\\
&\quad\le 2\mathbb{E}\left(\sup_{-\tau\le\theta\le 0, 0\le s\le T}\left|y(s+\theta)-\overline{y}(0)\right|^2\right)+2E\left|y(0)-y(-\Delta)\right|^2\\
&\quad\le 212\kappa^2\gamma(\Delta)+848\kappa^2H(2)+2[c_1(l)+c_2(l)+c_3(l)]\Delta^{(l-1)/l}+2\alpha(\Delta).
\end{aligned}\tag{3.50}$$

Case 3 ($k_s+k_\theta=-1$ and $-\Delta\le s+\theta\le 0$). In this case, using Assumption 2.5,

$$\mathbb{E}\left(\sup_{-\tau\le\theta\le 0, 0\le s\le T}\left|y(s+\theta)-\overline{y}((k_s+k_\theta)\Delta)\right|^2\right)\le\alpha(\Delta).\tag{3.51}$$

Case 4 ($k_s + k_\theta \le -2$). In this case, $s+\theta \le 0$, so using Assumption 2.5,

$$\mathbb{E}\left(\sup_{-\tau\le\theta\le 0, 0\le s\le T}\left|y(s+\theta)-\overline{y}((k_s+k_\theta)\Delta)\right|^2\right)\le \alpha(2\Delta). \tag{3.52}$$

Substituting these four cases into (3.34) and noting the expression of $\gamma(\Delta)$, there exist c_2, c_2', and $\overline{c}_2(l)$ such that

$$\mathbb{E}\left(\sup_{0\le s\le T}\left\|y_s-\overline{y}_s\right\|^2\right)\le c_2\alpha(2\Delta)+c_2'+\overline{c}_2(l)\Delta^{(l-1)/l}, \tag{3.53}$$

This proof is complete. □

Remark 3.4. It should be pointed out that much simpler proofs of Lemmas 3.2 and 3.3 can be obtained by choosing $l=2$ if we only want to prove Theorem 2.7. However, here we choose $l>1$ to control the stochastic terms $\beta(\Delta)$ and $\gamma(\Delta)$ by $\Delta^{(l-1)/l}$ in Section 3, which will be used to show the order of the strong convergence.

Lemma 3.5. *If Assumption 2.4 holds,*

$$\mathbb{E}\left(\sup_{0\le t\le T}\left|u(\overline{y}_t,\overline{r}(t))-u(\overline{y}_t,r(t))\right|^2\right)\le 8\kappa^2H(2)L\Delta := \zeta\Delta \tag{3.54}$$

where L is a positive constant independent of Δ.

Proof. Let $n=[T/\Delta]$, the integer part of T/Δ, and 1_G be the indication function of the set G. Then, by Assumption 2.4 and Lemma 3.1, we obtain

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le T}\left|u(\overline{y}_t,\overline{r}(t))-u(\overline{y}_t,r(t))\right|^2\right)\\
&\quad\le \max_{0\le k\le n}\mathbb{E}\left(\sup_{s\in[t_k,t_{k+1})}\left|u(\overline{y}_s,\overline{r}(s))-u(\overline{y}_s,r(s))\right|^2\right)\\
&\quad\le 2\max_{0\le k\le n}\mathbb{E}\left(\sup_{s\in[t_k,t_{k+1})}\left|u(\overline{y}_s,\overline{r}(s))-u(\overline{y}_s,r(s))\right|^2 1_{\{r(s)\neq r(t_k)\}}\right)\\
&\quad\le 4\max_{0\le k\le n}\mathbb{E}\left(\sup_{s\in[t_k,t_{k+1})}\left(\left|u(\overline{y}_s,\overline{r}(s))\right|^2+\left|u(\overline{y}_s,r(s))\right|^2\right)1_{\{r(s)\neq r(t_k)\}}\right)\\
&\quad\le 8\kappa^2\max_{0\le k\le n}\left(\mathbb{E}\left(\sup_{0\le t\le T}\left\|\overline{y}_t\right\|^2\right)\right)\mathbb{E}\left(1_{\{r(s)\neq r(t_k)\}}\right)
\end{aligned}$$

$$\leq 8\kappa^2 H(2)\mathbb{E}\left(1_{\{r(s)\neq r(t_k)\}}\right)$$
$$= 8\kappa^2 H(2)\mathbb{E}\left[\mathbb{E}\left(1_{\{r(s)\neq r(t_k)\}} \mid (t_k)\right)\right], \tag{3.55}$$

where in the last step we use the fact that $\overline{y}_{t_k}$ and $1_{\{r(s)\neq r(t_k)\}}$ are conditionally independent with respect to the $\sigma-$ algebra generated by $r(t_k)$. But, by the Markov property,

$$\begin{aligned}\mathbb{E}\left(1_{\{r(s)\neq r(t_k)\}} \mid r(t_k)\right) &= \sum_{i\in S} 1_{\{r(t_k)=i\}} P(r(s)\neq i \mid r(t_k)=i)\\ &= \sum_{i\in S} 1_{\{r(t_k)=i\}} \sum_{j\neq i}\left(\gamma_{ij}(s-t_k)+o(s-t_k)\right)\\ &\leq \sum_{i\in S} 1_{\{r(t_k)=i\}}\left(\max_{1\leq i\leq N}(-\gamma_{ij})\Delta+o(\Delta)\right)\\ &\leq L\Delta,\end{aligned} \tag{3.56}$$

where L is a positive constant independent of Δ. Then

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left|u(\overline{y}_t,\overline{r}(t))-u(\overline{y}_t,r(t))\right|^2\right)\leq 8\kappa^2 H(2)L\Delta. \tag{3.57}$$

This proof is complete. □

Lemma 3.6. *If Assumption 2.3 holds, there is a constant C, which is independent of Δ such that*

$$\mathbb{E}\int_0^T \left|f(\overline{y}_s,r(s))-f(\overline{y}_s,\overline{r}(s))\right|^2 ds\leq C\Delta, \tag{3.58}$$

$$\mathbb{E}\int_0^T \left|g(\overline{y}_s,r(s))-g(\overline{y}_s,\overline{r}(s))\right|^2 ds\leq C\Delta. \tag{3.59}$$

Proof. Let $n=[T/\Delta]$, the integer part of T/Δ. Then

$$\mathbb{E}\int_0^T \left|f(\overline{y}_s,r(s))-f(\overline{y}_s,\overline{r}(s))\right|^2 ds=\sum_{k=0}^{n}\mathbb{E}\int_{t_k}^{t_{k+1}}\left|f\left(\overline{y}_{t_k},r(s)\right)-f\left(\overline{y}_{t_k},r(t_k)\right)\right|^2 ds, \tag{3.60}$$

with t_{n+1} being T. Let 1_G be the indication function of the set G. Moreover, in what follows, C is a generic positive constant independent of Δ, whose values may vary from line to line. With these notations we derive, using Assumption 2.3, that

$$\begin{aligned}
&\mathbb{E}\int_{t_k}^{t_{k+1}} \left|f\left(\overline{y}_{t_k}, r(s)\right) - f\left(\overline{y}_{t_k}, r(t_k)\right)\right|^2 ds \\
&\quad\le 2\mathbb{E}\int_{t_k}^{t_{k+1}} \left[|f(\overline{y}_{t_k}, r(s))|^2 + \left|f\left(\overline{y}_{t_k}, r(t_k)\right)\right|^2\right] 1_{\{r(s)\neq r(t_k)\}} ds \\
&\quad\le C\mathbb{E}\int_{t_k}^{t_{k+1}} \left(1 + \left\|\overline{y}_{t_k}\right\|^2\right) 1_{\{r(s)\neq r(t_k)\}} ds \\
&\quad\le C\int_{t_k}^{t_{k+1}} \mathbb{E}\left[\mathbb{E}\left[\left(1 + \left\|\overline{y}_{t_k}\right\|^2\right) 1_{\{r(s)\neq r(t_k)\}} \mid r(t_k)\right]\right] ds \\
&\quad\le C\int_{t_k}^{t_{k+1}} \mathbb{E}\left[\mathbb{E}\left[\left(1 + \left\|\overline{y}_{t_k}\right\|^2\right) \mid r(t_k)\right] \mathbb{E}\left[1_{\{r(s)\neq r(t_k)\}} \mid r(t_k)\right]\right] ds,
\end{aligned} \tag{3.61}$$

where in the last step we use the fact that $\overline{y}_{t_k}$ and $1_{\{r(s)\neq r(t_k)\}}$ are conditionally independent with respect to the $\sigma-$ algebra generated by $r(t_k)$. But, by the Markov property,

$$\begin{aligned}
\mathbb{E}\left[1_{\{r(s)\neq r(t_k)\}} \mid r(t_k)\right] &= \sum_{i\in S} 1_{\{r(t_k)=i\}} P(r(s)\neq i \mid r(t_k) = i) \\
&= \sum_{i\in S} 1_{\{r(t_k)=i\}} \sum_{j\neq i} (\gamma_{ij}(s - t_k) + o(s - t_k)) \\
&\le \sum_{i\in S} 1_{\{r(t_k)=i\}} \left(\max_{1\le i\le N}(-\gamma_{ij})\Delta + o(\Delta)\right) \\
&\le C\Delta.
\end{aligned} \tag{3.62}$$

So, by Lemma 3.1,

$$\begin{aligned}
\mathbb{E}\int_{t_k}^{t_{k+1}} \left|f\left(\overline{y}_{t_k}, r(s)\right) - f\left(\overline{y}_{t_k}, r(t_k)\right)\right|^2 ds &\le C\Delta \int_{t_k}^{t_{k+1}} \left(1 + \mathbb{E}\left\|\overline{y}_{t_k}\right\|^2\right) ds \\
&\le C\Delta^2.
\end{aligned} \tag{3.63}$$

Therefore

$$\mathbb{E}\int_0^T \left|f(\overline{y}_s, r(s)) - f(\overline{y}_s, \overline{r}(s))\right|^2 ds \le C\Delta. \tag{3.64}$$

Similarly, we can show (3.59). □

4. Convergence of the EM Methods

In this section we will use the lemmas above to prove Theorem 2.7. From Lemma 3.1 and Theorem 2.6, there exists a positive constant $\widetilde{H}$ such that

$$\mathbb{E}\left(\sup_{-\tau\le t\le T}|x(t)|^p\right)\vee\mathbb{E}\left(\sup_{-\tau\le t\le T}|y(t)|^p\right)\le\widetilde{H}. \tag{4.1}$$

Let j be a sufficient large integer. Define the stopping times

$$u_j := \inf\{t\ge 0 : \|x_t\|\ge j\},\qquad v_j := \inf\{t\ge 0 : \|y_t\|\ge j\},\qquad \rho_j := u_j\wedge v_j, \tag{4.2}$$

where we set $\inf\emptyset=\infty$ as usual. Let

$$e(t) := x(t)-y(t). \tag{4.3}$$

Obviously,

$$\mathbb{E}\left(\sup_{0\le t\le T}|e(t)|^2\right)=\mathbb{E}\left(\sup_{0\le t\le T}|e(t)|^2 1_{\{u_j>T,v_j>T\}}\right)+\mathbb{E}\left(\sup_{0\le t\le T}|e(t)|^2 1_{\{u_j\le T\text{ or }v_j\le T\}}\right). \tag{4.4}$$

Recall the following elementary inequality:

$$a^{\gamma}b^{1-\gamma}\le\gamma a+(1-\gamma)b,\quad\forall a,b>0,\ \gamma\in[0,1]. \tag{4.5}$$

We thus have, for any $\delta>0$,

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le t\le T}|e(t)|^2 1_{\{u_j\le T\text{or }v_j\le T\}}\right)&\le\mathbb{E}\left[\left(\delta\sup_{\{0\le t\le T\}}|e(t)|^p\right)^{2/p}\left(\delta^{-2/(p-2)}1_{\{u_j\le T\text{ or }v_j\le T\}}\right)^{(p-2)/p}\right]\\&\le\frac{2\delta}{p}\mathbb{E}\left(\sup_{0\le t\le T}|e(t)|^p\right)+\frac{p-2}{p\delta^{2/(p-2)}}\mathbb{P}(u_j\le T\text{or }v_j\le T).\end{aligned} \tag{4.6}$$

Hence

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le t\le T}|e(t)|^2\right)\le{}&\mathbb{E}\left(\sup_{0\le t\le T}|e(t)|^2 1_{\{\rho_j>T\}}\right)+\frac{2\delta}{p}\mathbb{E}\left(\sup_{0\le t\le T}|e(t)|^p\right)\\&+\frac{p-2}{p\delta^{2/(p-2)}}\mathbb{P}(u_j\le T\text{ or }v_j\le T).\end{aligned} \tag{4.7}$$

Now,

$$\begin{aligned}
\mathbb{P}(u_j \le T) &\le \mathbb{E}\left(1_{\{u_j\le T\}}\frac{\|x_t\|^p}{j^p}\right)\\
&\le \frac{1}{j^p}\mathbb{E}\left(\sup_{-\tau\le t\le T}|x(t)|^p\right)\\
&\le \frac{\widetilde{H}}{j^p}.
\end{aligned} \tag{4.8}$$

Similarly,

$$\mathbb{P}(v_j \le T) \le \frac{\widetilde{H}}{j^p}. \tag{4.9}$$

Thus

$$\begin{aligned}
\mathbb{P}(v_j \le T \text{ or } u_j \le T) &\le \mathbb{P}(v_j \le T) + \mathbb{P}(u_j \le T)\\
&\le \frac{2\widetilde{H}}{j^p}.
\end{aligned} \tag{4.10}$$

We also have

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\le t\le T}|e(t)|^p\right) &\le 2^{p-1}\mathbb{E}\left(\sup_{0\le t\le T}\left(|x_t|^p + |y_t|^p\right)\right)\\
&\le 2^p\widetilde{H}.
\end{aligned} \tag{4.11}$$

Moreover,

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\le t\le T}|e(t)|^2 1_{\{\rho_j>T\}}\right) &= \mathbb{E}\left(\sup_{0\le t\le T}\left|e(t\wedge\rho_j)\right|^2 1_{\{\rho_j>T\}}\right)\\
&\le \mathbb{E}\left(\sup_{0\le t\le T}\left|e(t\wedge\rho_j)\right|^2\right).
\end{aligned} \tag{4.12}$$

Using these bounds in (4.7) yields

$$\mathbb{E}\left(\sup_{0\le t\le T}|e(t)|^2\right) \le \mathbb{E}\left(\sup_{0\le t\le T}\left|e(t\wedge\rho_j)\right|^2\right) + \frac{2^{p+1}\delta\widetilde{H}}{p} + \frac{2(p-2)\widetilde{H}}{p\delta^{2/(p-2)}\, j^p}. \tag{4.13}$$

Setting $v := t \wedge \rho_j$ and for any $\varepsilon \in (0,1)$, by the Hölder inequality, when $v \in [k\Delta, (k+1)\Delta]$, for $k = 0,1,2,\ldots,M-1$,

$$\begin{aligned}
|e(v)|^2 &= \left|x(v)-y(v)\right|^2 \\
&\leq \left|u(x_v, r(v)) - u\left(\overline{y}_{(k-1)\Delta} + \frac{(v-k\Delta)}{\Delta}\left(\overline{y}_{k\Delta} - \overline{y}_{(k-1)\Delta}\right), r_k^{\Delta}\right)\right. \\
&\quad \left. + \int_0^v \left[f(x_s, r(s)) - f\left(\overline{y}_s, \overline{r}(s)\right)\right] ds + \int_0^v \left[g(x_s, r(s)) - g\left(\overline{y}_s, \overline{r}(s)\right)\right] dw(s)\right|^2 \\
&\leq \frac{1}{\varepsilon}\left|u(x_v, r(v)) - u\left(\overline{y}_{(k-1)\Delta} + \frac{(v-k\Delta)}{\Delta}\left(\overline{y}_{k\Delta} - \overline{y}_{(k-1)\Delta}\right), r_k^{\Delta}\right)\right|^2 \\
&\quad + \frac{2}{1-\varepsilon}\left[T\int_0^v \left[f(x_s, r(s)) - f\left(\overline{y}_s, \overline{r}(s)\right)\right]^2 ds + \left|\int_0^v \left[g(x_s, r(s)) - g\left(\overline{y}_s, \overline{r}(s)\right)\right] dw(s)\right|^2\right].
\end{aligned} \tag{4.14}$$

Assumption 2.4 yields

$$\begin{aligned}
&\left|u(x_v, r(v)) - u\left(\overline{y}_{(k-1)\Delta} + \frac{v-k\Delta}{\Delta}\left(\overline{y}_{k\Delta} - \overline{y}_{(k-1)\Delta}\right), r_k^{\Delta}\right)\right|^2 \\
&\quad \leq 2\kappa^2 \left\|x_v - \overline{y}_{(k-1)\Delta} - \frac{v-k\Delta}{\Delta}\left(\overline{y}_{k\Delta} - \overline{y}_{(k-1)\Delta}\right)\right\|^2 \\
&\qquad + 2\left|u\left(\overline{y}_{(k-1)\Delta} + \frac{v-k\Delta}{\Delta}\left(\overline{y}_{k\Delta} - \overline{y}_{(k-1)\Delta}\right), r(v)\right)\right. \\
&\qquad\qquad \left. - u\left(\overline{y}_{(k-1)\Delta} + \frac{v-k\Delta}{\Delta}\left(\overline{y}_{k\Delta} - \overline{y}_{(k-1)\Delta}\right), r_k^{\Delta}\right)\right|^2 \\
&\quad \leq 2\kappa^2 \left\| |x_v - y_v| + |y_v - \overline{y}_v| + \left|\overline{y}_{k\Delta} - \overline{y}_{(k-1)\Delta}\right| - \frac{v-k\Delta}{\Delta}\left|\overline{y}_{k\Delta} - \overline{y}_{(k-1)\Delta}\right|\right\|^2 \\
&\qquad + 2\left|u\left(\overline{y}_{(k-1)\Delta} + \frac{v-k\Delta}{\Delta}\left(\overline{y}_{k\Delta} - \overline{y}_{(k-1)\Delta}\right), r(v)\right)\right. \\
&\qquad\qquad \left. - u\left(\overline{y}_{(k-1)\Delta} + \frac{v-k\Delta}{\Delta}\left(\overline{y}_{k\Delta} - \overline{y}_{(k-1)\Delta}\right), r_k^{\Delta}\right)\right|^2 \\
&\quad \leq \frac{2\kappa^2}{\varepsilon}\|x_v - y_v\|^2 + \frac{4\kappa^2}{1-\varepsilon}\left(\|y_v - \overline{y}_v\|^2 + \left\|\overline{y}_{k\Delta} - \overline{y}_{(k-1)\Delta}\right\|^2\right) \\
&\qquad + 2\left|u\left(\overline{y}_{(k-1)\Delta} + \frac{v-k\Delta}{\Delta}\left(\overline{y}_{k\Delta} - \overline{y}_{(k-1)\Delta}\right), r(v)\right)\right. \\
&\qquad\qquad \left. - u(\overline{y}_{(k-1)\Delta} + \frac{v-k\Delta}{\Delta}(\overline{y}_{k\Delta} - \overline{y}_{(k-1)\Delta}), r_k^{\Delta})\right|^2.
\end{aligned} \tag{4.15}$$

Then, we have

$$
\begin{aligned}
|e(v)|^2 \le{}& \frac{2\kappa^2}{\varepsilon^2}\left\|x_v - y_v\right\|^2 + \frac{4\kappa^2}{\varepsilon(1-\varepsilon)}\left(\left\|y_v - \overline{y}_v\right\|^2 + \left\|\overline{y}_{k\Delta} - \overline{y}_{(k-1)\Delta}\right\|^2\right) \\
&+ 2\left|u\left(\overline{y}_{(k-1)\Delta} + \frac{v-k\Delta}{\Delta}\left(\overline{y}_{k\Delta} - \overline{y}_{(k-1)\Delta}\right), r(v)\right)\right. \\
&\left.\quad - u\left(\overline{y}_{(k-1)\Delta} + \frac{v-k\Delta}{\Delta}\left(\overline{y}_{k\Delta} - \overline{y}_{(k-1)\Delta}\right), r_k^{\Delta}\right)\right|^2 \\
&+ \frac{2}{1-\varepsilon}\left[T\int_0^v \left[f(x_s, r(s)) - f(\overline{y}_s, \overline{r}(s))\right]^2 ds + \left|\int_0^v \left[g(x_s, r(s)) - g(\overline{y}_s, \overline{r}(s))\right] dw(s)\right|^2\right].
\end{aligned}
\tag{4.16}
$$

Hence, for any $t_1 \in [0, T]$, by Lemmas 3.2–3.5,

$$
\begin{aligned}
\mathbb{E}\left[\sup_{0\le t\le t_1}\left|e(t\wedge\rho_j)\right|^2\right] \le{}& \frac{2\kappa^2}{\varepsilon^2}\mathbb{E}\left(\sup_{0\le t\le t_1}\left\|x_{t\wedge\rho_j} - y_{t\wedge\rho_j}\right\|^2\right) \\
&+ \frac{4\kappa^2}{\varepsilon(1-\varepsilon)}\left[\mathbb{E}\left(\sup_{0\le t\le t_1}\left\|y_{t\wedge\rho_j} - \overline{y}_{t\wedge\rho_j}\right\|^2\right)\right. \\
&\left.\quad + \mathbb{E}\left(\sup_{0\le k\le M-1}\left\|\overline{y}_{k\Delta\wedge\rho_j} - \overline{y}_{(k-1)\Delta\wedge\rho_j}\right\|^2\right)\right] \\
&+ 2\mathbb{E}\left[\sup_{0\le k\le M-1}\left|u\left(\overline{y}_{(k-1)\Delta} + \frac{k\Delta\wedge\rho_j - k\Delta}{\Delta}\left(\overline{y}_{k\Delta} - \overline{y}_{(k-1)\Delta}\right), r(k\Delta\wedge\rho_j)\right)\right.\right. \\
&\left.\left.\quad - u\left(\overline{y}_{(k-1)\Delta} + \frac{k\Delta\wedge\rho_j - k\Delta}{\Delta}\left(\overline{y}_{k\Delta} - \overline{y}_{(k-1)\Delta}\right), \overline{r}(k\Delta\wedge\rho_j)\right)\right|^2\right] \\
&+ \frac{2T}{1-\varepsilon}\mathbb{E}\int_0^{t_1\wedge\rho_j}\left[f(x_s, r(s)) - f(\overline{y}_s), \overline{r}(s)\right]^2 ds \\
&+ \frac{2}{1-\varepsilon}\mathbb{E}\left[\sup_{0\le t\le t_1}\left|\int_0^{t\wedge\rho_j}\left[g(x_s, r(s)) - g(\overline{y}_s, \overline{r}(s))\right] dw(s)\right|^2\right] \\
\le{}& \frac{2\kappa^2}{\varepsilon^2}\mathbb{E}\left(\sup_{0\le t\le t_1}\left\|x_{t\wedge\rho_j} - y_{t\wedge\rho_j}\right\|^2\right) + \frac{4\kappa^2}{\varepsilon(1-\varepsilon)}\left(\gamma(\Delta) + \beta(\Delta)\right) + 2\zeta\Delta \\
&+ \frac{2T}{1-\varepsilon}\mathbb{E}\int_0^{t_1\wedge\rho_j}\left[f(x_s, r(s)) - f(\overline{y}_s, \overline{r}(s))\right]^2 ds \\
&+ \frac{2}{1-\varepsilon}\mathbb{E}\left[\sup_{0\le t\le t_1}\left|\int_0^{t\wedge\rho_j}\left[g(x_s, r(s)) - g(\overline{y}_s, \overline{r}(s))\right] dw(s)\right|^2\right].
\end{aligned}
\tag{4.17}
$$

Since $x(t) = y(t) = \xi(t)$ when $t \in [-\tau, 0]$, we have

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\le t\le t_1}\left\|x_{t\wedge\rho_j} - y_{t\wedge\rho_j}\right\|^2\right) &\le \mathbb{E}\left(\sup_{-\tau\le\theta\le 0}\sup_{0\le t\le t_1}\left|x(t\wedge\rho_j+\theta) - y(t\wedge\rho_j+\theta)\right|^2\right)\\
&\le \mathbb{E}\left(\sup_{-\tau\le t\le t_1}\left|x(t\wedge\rho_j) - y(t\wedge\rho_j)\right|^2\right)\\
&= \mathbb{E}\left(\sup_{0\le t\le t_1}\left|x(t\wedge\rho_j) - y(t\wedge\rho_j)\right|^2\right).
\end{aligned}\tag{4.18}$$

By Assumption 2.2, Lemma 3.3, and Lemma 3.6, we may compute

$$\begin{aligned}
&\mathbb{E}\int_0^{t_1\wedge\rho_j}\left|f(x_s, r(s)) - f(\overline{y}_s, \overline{r}(s))\right|^2 ds\\
&\quad\le \mathbb{E}\int_0^{t_1\wedge\rho_j}\left|f(x_s, r(s)) - f(\overline{y}_s, r(s)) + f(\overline{y}_s, r(s)) - f(\overline{y}_s, \overline{r}(s))\right|^2 ds\\
&\quad\le 2C_j\mathbb{E}\int_0^{t_1\wedge\rho_j}\left\|x_s - y_s + y_s - \overline{y}_s\right\|^2 ds + 2C\Delta\\
&\quad\le 4C_j\mathbb{E}\int_0^{t_1\wedge\rho_j}\left\|x_s - y_s\right\|^2 ds + 4C_j\mathbb{E}\int_0^{t_1\wedge\rho_j}\left\|y_s - \overline{y}_s\right\|^2 ds + 2C\Delta\\
&\quad\le 4C_j\mathbb{E}\int_0^{t_1\wedge\rho_j}\sup_{-\tau\le\theta\le 0}\left|x(s+\theta) - y(s+\theta)\right|^2 ds + 4C_jT\beta(\Delta) + 2C\Delta\\
&\quad\le 4C_j\mathbb{E}\int_0^{t_1}\sup_{-\tau\le\theta\le 0}\left|x(s\wedge\rho_j+\theta) - y(s\wedge\rho_j+\theta)\right|^2 ds\\
&\qquad + 4C_jT\beta(\Delta) + 2C\Delta\\
&\quad\le 4C_j\mathbb{E}\int_0^{t_1}\sup_{-\tau\le r\le s}\left|x(r\wedge\rho_j) - y(r\wedge\rho_j)\right|^2 ds + 4C_jT\beta(\Delta) + 2C\Delta\\
&\quad= 4C_j\int_0^{t_1}\mathbb{E}\sup_{0\le r\le s}\left|x(r\wedge\rho_j) - y(r\wedge\rho_j)\right|^2 ds + 4C_jT\beta(\Delta) + 2C\Delta.
\end{aligned}\tag{4.19}$$

By the Doob martingale inequality, Lemma 3.3, Lemma 3.6, and Assumption 2.2, we compute

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le t_1}\left|\int_0^{t\wedge\rho_j}\left[g(x_s, r(s)) - g(\overline{y}_s, \overline{r}(s))\right]dw(s)\right|^2\right]\\
&\quad= \mathbb{E}\left[\sup_{0\le t\le t_1}\left|\int_0^{t\wedge\rho_j}\left[g(x_s, r(s)) - g(\overline{y}_s, r(s)) + g(\overline{y}_s, r(s)) - g(\overline{y}_s, \overline{r}(s))\right]dw(s)\right|^2\right]
\end{aligned}$$

$$\leq 8C_j\mathbb{E}\int_0^{t_1\wedge\rho_j}\|x_s-\overline{y}_s\|^2ds+8C\Delta$$

$$\leq 16C_j\int_0^{t_1}\mathbb{E}\left(\sup_{0\leq r\leq s}|x(r\wedge\rho_j)-y(r\wedge\rho_j)|^2\right)ds+16C_jT\beta(\Delta)+8C\Delta. \tag{4.20}$$

Therefore, (4.17) can be written as

$$\begin{aligned}\left(1-\frac{2\kappa^2}{\varepsilon^2}\right)\mathbb{E}\left[\sup_{0\leq t\leq t_1}|e(t\wedge\rho_j)|^2\right]&\leq\frac{4\kappa^2}{\varepsilon(1-\varepsilon)}[\beta(\Delta)+\gamma(\Delta)]+\frac{8C_jT(T+4)}{1-\varepsilon}\beta(\Delta)+\frac{4C\Delta(T+8)}{1-\varepsilon}\\&\quad+2\zeta\Delta+\frac{8C_j(T+4)}{1-\varepsilon}\int_0^{t_1}\mathbb{E}\left(\sup_{0\leq v\leq s}|e(v\wedge\rho_j)|^2\right)ds.\end{aligned} \tag{4.21}$$

Choosing $\varepsilon=(1+\sqrt{2}\kappa)/2$ and noting $\kappa\in(0,1)$, we have

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\leq t\leq t_1}|e(t\wedge\rho_j)|^2\right)\\&\quad\leq\frac{16\kappa^2\left(1+\sqrt{2}\kappa\right)}{\left(1-\sqrt{2}\kappa\right)^2\left(1+3\sqrt{2}\kappa\right)}[\beta(\Delta)+\gamma(\Delta)]+\frac{16C_jT\left(1+\sqrt{2}\kappa\right)^2(T+4)}{\left(1-\sqrt{2}\kappa\right)^2\left(1+3\sqrt{2}\kappa\right)}\beta(\Delta)\\&\qquad+\frac{8C\Delta(T+8)\left(1+\sqrt{2}\kappa\right)^2}{\left(1-\sqrt{2}\kappa\right)^2\left(1+3\sqrt{2}\kappa\right)}+\frac{2\left(1+\sqrt{2}\kappa\right)^2}{\left(1-\sqrt{2}\kappa\right)^2\left(1+3\sqrt{2}\kappa\right)}\zeta\Delta\\&\qquad+\frac{16C_j\left(1+\sqrt{2}\kappa\right)^2(T+4)}{\left(1-\sqrt{2}\kappa\right)^2\left(1+3\sqrt{2}\kappa\right)}\int_0^{t_1}\mathbb{E}\left(\sup_{0\leq s\leq v}|e(s\wedge\rho_j)|^2\right)dv\\&\quad\leq\frac{16}{\left(1-\sqrt{2}\kappa\right)^2}[\beta(\Delta)+\gamma(\Delta)]+\frac{16C_jT(T+4)}{\left(1-\sqrt{2}\kappa\right)^2}\beta(\Delta)+\frac{8C\Delta(T+8)}{\left(1-\sqrt{2}\kappa\right)^2}\\&\qquad+\frac{2}{\left(1-\sqrt{2}\kappa\right)}\zeta\Delta+\frac{16C_j(T+4)}{\left(1-\sqrt{2}\kappa\right)^2}\int_0^{t_1}\mathbb{E}\left(\sup_{0\leq s\leq v}|e(s\wedge\rho_j)|^2\right)dv.\end{aligned} \tag{4.22}$$

By the Gronwall inequality, we have

$$\mathbb{E}\left[\sup_{0\le t\le t_1}\left|e(t\wedge\rho_j)\right|^2\right] \le \left[\frac{16}{(1-\sqrt{2}\kappa)^2}(\beta(\Delta)+\gamma(\Delta)) + \frac{16C_jT(T+4)}{\left(1-\sqrt{2}\kappa\right)^2}\beta(\Delta) + \frac{8C\Delta(T+8)}{\left(1-\sqrt{2}\kappa\right)^2} + \frac{2\zeta\Delta}{\left(1-\sqrt{2}\kappa\right)}\right] \times e^{(16/(1-\sqrt{2}\kappa)^2)C_jT(T+4)}. \tag{4.23}$$

By (4.13),

$$\mathbb{E}\left(\sup_{0\le t\le T}|e(t)|^2\right) \le \left[\frac{16}{\left(1-\sqrt{2}\kappa\right)^2}(\beta(\Delta)+\gamma(\Delta)) + \frac{16C_jT(T+4)}{\left(1-\sqrt{2}\kappa\right)^2}\beta(\Delta) + \frac{8C\Delta(T+8)}{\left(1-\sqrt{2}\kappa\right)^2} + \frac{2\zeta\Delta}{\left(1-\sqrt{2}\kappa\right)}\right] \times e^{(16/(1-\sqrt{2}\kappa)^2)C_jT(T+4)} + \frac{2^{p+1}\delta\widetilde{H}}{p} + \frac{2(p-2)\widetilde{H}}{p\delta^{2/(p-2)}j^p}. \tag{4.24}$$

Given any $\epsilon > 0$, we can now choose δ sufficient small such that $2^{p+1}\delta\widetilde{H}/p \le \epsilon/3$, then choose j sufficient large such that

$$\frac{2(p-2)\widetilde{H}}{p\delta^{2/(p-2)}j^p} < \frac{\epsilon}{3}, \tag{4.25}$$

and finally choose Δ so small such that

$$\left[\frac{16}{\left(1-\sqrt{2}\kappa\right)^2}(\beta(\Delta)+\gamma(\Delta)) + \frac{16C_jT(T+4)}{\left(1-\sqrt{2}\kappa\right)^2}\beta(\Delta) + \frac{8C\Delta(T+8)}{\left(1-\sqrt{2}\kappa\right)^2} + \frac{2\zeta\Delta}{\left(1-\sqrt{2}\kappa\right)}\right] \times e^{(16/(1-\sqrt{2}\kappa)^2)C_jT(T+4)} < \frac{\epsilon}{3} \tag{4.26}$$

and thus $\mathbb{E}(\sup_{0\le t\le T}|e(t)|^2) \le \epsilon$ as required.

Remark 4.1. Obviously, according to Theorem 2.7, for neutral stochastic delay differential equations with Markovian switching [19], we can easily obtain that the numerical solutions converge to the true solutions in mean square under Assumptions 2.2–2.4.

5. Convergence Order of the EM Method

To reveal the convergence order of the EM method, we need the following assumptions.

Assumption 5.1. There exists a constant Q such that for all φ, $\psi \in C([-\tau,0];\mathbb{R}^n)$, $i \in S$, and $t \in [0,T]$,

$$\left|f(\varphi,i) - f(\psi,i)\right|^2 \vee \left|g(\varphi,i) - g(\psi,i)\right|^2 \leq Q\|\varphi - \psi\|^2. \tag{5.1}$$

It is easy to see from the global Lipschitz condition that, for any $\varphi \in C([-\tau,0];\mathbb{R}^n)$,

$$\left|f(\varphi,i)\right|^2 \vee \left|g(\psi,i)\right|^2 \leq 2\left(\left|f(0,i)\right|^2 \vee \left|g(0,i)\right|^2\right) + Q\|\varphi\|^2. \tag{5.2}$$

In other words, the global Lipschitz condition implies linear growth condition with the growth coefficient

$$K = 2\left[\left|f(0,i)\right|^2 \vee \left|g(0,i)\right|^2 \vee Q\right]. \tag{5.3}$$

Assumption 5.2. $\xi \in L^p_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$ for some $p \geq 2$, and there exists a positive constant λ such that

$$\mathbb{E}\left(\sup_{-\tau\leq s\leq t\leq 0} |\xi(t) - \xi(s)|^2\right) \leq \lambda(t-s). \tag{5.4}$$

We can state another theorem, which reveals the order of the convergence.

Theorem 5.3. *If Assumptions 5.1, 5.2, and 2.4 hold, for any positive constant $l > 1$,*

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left|x(t) - y(t)\right|^2\right) \leq O\left(\Delta^{1-1/l}\right). \tag{5.5}$$

Proof. Since $\alpha(\Delta)$ may be replaced by $\lambda\Delta$, from Lemmas 3.2 and 3.3, there exist constants $\widetilde{c}_1(l)$ and $\widetilde{c}_2(l)$ such that $\beta(\Delta) \leq \widetilde{c}_1(l)\Delta^{(l-1)/l}$ and $\gamma(\Delta) \leq \widetilde{c}_2(l)\Delta^{(l-1)/l}$. Here we do not need to define the stopping times u_j and v_j, and we may repeat the proof in Section 4 and directly compute

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\leq t\leq T}|e(t)|^2\right) \leq& \left[\frac{16}{\left(1-\sqrt{2}\kappa\right)^2}\left(\widetilde{c}_1(l)+\widetilde{c}_2(l)\right) + \frac{16QT(T+4)}{\left(1-\sqrt{2}\kappa\right)^2}\widetilde{c}_1(l) + \frac{8C(T+8)}{\left(1-\sqrt{2}\kappa\right)^2} + \frac{2\zeta}{\left(1-\sqrt{2}\kappa\right)}\right] \\
&\times e^{(16/(1-\sqrt{2}\kappa)^2)QT(T+4)}\Delta^{(l-1)/l} \\
\leq& O\left(\Delta^{(l-1)/l}\right).
\end{aligned} \tag{5.6}$$

The proof is complete. □

6. Conclusion

The EM method for neutral stochastic functional differential equations with Markovian switching is studied. The results show that the numerical solution converges to the true solution under the local Lipschitz condition. In addition, the results also show that the order of convergence of the numerical method is close to 1, although the order of the strong convergence in mean square for the EM scheme applied to both SDEs and SFDEs is one [6, 7, 11] under the global Lipschitz condition. Hence, we can control the numerical solution's error; this method may value some path-dependent options more quickly and simply [25].

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