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Research Article

Convex Polyhedron Method to Stability of Continuous Systems with Two Additive Time-Varying Delay Components

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This paper is concerned with delay-dependent stability for continuous systems with two additive time-varying delay components. By constructing a new class of Lyapunov functional and using a new convex polyhedron method, a new delay-dependent stability criterion is derived in terms of linear matrix inequalities. The obtained stability criterion is less conservative than some existing ones. Finally, numerical examples are given to illustrate the effectiveness of the proposed method.

1. Introduction

Robust stability of dynamic interval systems covering interval matrices and interval polynomials has attracted considerable attention over last decades. Reference [1] presents some necessary and sufficient conditions for the quadratic stability and stabilization of dynamic interval systems. It is well known that time delay frequently occurs in many industrial and engineering systems, such as manufacturing systems, telecommunication, and economic systems, and is a major cause of instability and poor performance. Over the past decades, much efforts have been invested in the stability analysis of time-delay systems [2–16]. Reference [2] deals with the problem of quadratic stability analysis and quadratic stabilization for uncertain linear discrete time systems with state delay. Reference [3] deals with the quadratic stability and linear state-feedback and output-feedback stabilization of switched delayed linear dynamic systems. However, almost all the reported results mentioned above on time-delay systems are based on the following basic mathematical model:

$$\dot{x}(t) = Ax(t) + A_d x(t - d(t)), \tag{1.1}$$

where $d(t)$ is a time delay in the state $x(t)$, which is often assumed to be either constant or time-varying satisfying certain conditions, for example,

$$0 \leqslant d(t) \leqslant \overline{d} < \infty, \qquad \dot{d}(t) \leqslant \tau < \infty. \tag{1.2}$$

Sometimes in practical situations, however, signals transmitted from one point to another may experience a few segments of networks, which can possibly induce successive delays with different properties due to the variable network transmission conditions. Thus, in recent papers [15, 16], a new model for time-delay systems with multiple additive time-varying delay components has been proposed:

$$\dot{x}(t) = Ax(t) + A_d x\left(t - \sum_{i=1}^{n} d_i(t)\right), \tag{1.3}$$

$$0 \leqslant d_i(t) \leqslant d_i < \infty, \qquad \dot{d}_i(t) \leqslant \tau_i < \infty. \tag{1.4}$$

To make the stability analysis simpler, we proceed by considering the system (1.3) with two additive delay components as follows:

$$\begin{gathered} \dot{x}(t) = Ax(t) + A_d x(t - d_1(t) - d_2(t)), \\ x(t) = \phi(t), \quad t \in [-d, 0]. \end{gathered} \tag{1.5}$$

Here, $x(t) \in \mathbb{R}^n$ is the state vector; $d_1(t)$ and $d_2(t)$ represent the two delay components in the state, and we denote $d(t) = d_1(t) + d_2(t)$; A, A_d are system matrices with appropriate dimensions. It is assumed that

$$\begin{aligned} 0 \leqslant d_1(t) \leqslant d_1 < \infty, \qquad \dot{d}_1(t) \leqslant \tau_1 < \infty, \\ 0 \leqslant d_2(t) \leqslant d_2 < \infty, \qquad \dot{d}_2(t) \leqslant \tau_2 < \infty, \end{aligned} \tag{1.6}$$

and $d = d_1 + d_2$, $\tau = \tau_1 + \tau_2$. $\phi(t)$ is the initial condition on the segment $[-d, 0]$.

The purpose of our paper is to derive new stability conditions under which system (1.5) is asymptotically stable for all delays $d_1(t)$ and $d_2(t)$ satisfying (1.6). One possible approach to check the stability of this system is to simply combine $d_1(t)$ and $d_2(t)$ into one delay $d(t)$ with

$$0 \leqslant d(t) \leqslant d_1 + d_2 < \infty, \qquad \dot{d}(t) \leqslant \tau_1 + \tau_2 < \infty. \tag{1.7}$$

Then, the system (1.5) becomes

$$\dot{x}(t) = Ax(t) + A_d x(t - d(t)),$$
$$x(t) = \phi(t), \quad t \in [-d, 0]. \tag{1.8}$$

By using some existing stability conditions, the stability of system (1.8) can be readily checked. As discussed in [15, 16], however, since this approach does not make full use of the information on $d_1(t)$ and $d_2(t)$, it would be inevitably conservative for some situations. Recently, some new delay-dependent stability criteria have been obtained for system (1.5) in [15, 16], by making full use of the information on $d_1(t)$ and $d_2(t)$. However, the stability result is conservative because of overly bounding some integrals appearing in the derivative of the Lyapunov functional. On the one hand, the integral $-\int_{t-d_1}^{t} \dot{x}^T(s) Z_1 \dot{x}(s) ds$ in [15] was enlarged as $-\int_{t-d_1(t)}^{t} \dot{x}^T(s) Z_1 \dot{x}(s) ds$, with $-\int_{t-d_1}^{t-d_1(t)} \dot{x}^T(s) Z_1 \dot{x}(s) ds$ discarded. On the other hand, some integrals were estimated conservatively. Take $-\int_{t-d_1(t)}^{t} \dot{x}^T(s) Z_1 \dot{x}(s) ds$ as an example, by introducing

$$0 = 2\zeta^T S \left[x(t) - x(t - d_1(t)) - \int_{t-d_1(t)}^{t} \dot{x}(s) ds \right] \tag{1.9}$$

with an appropriate vector $\zeta(t)$ and a matrix S, respectively, it was estimated as

$$2\zeta^T(t) S [x(t) - x(t - d_1(t))] + \zeta^T(t) d_1 S Z_1^{-1} S^T \zeta(t) \tag{1.10}$$

with $d_1(t) S Z_1^{-1} S^T$ enlarged as $d_1 S Z_1^{-1} S^T$.

The problem of delay-dependent stability criterion for continuous systems with two additive time-varying delay components has been considered in this paper. By constructing a new class of Lyapunov functional and using a new convex polyhedron method, a new stability criterion is derived in terms of linear matrix inequalities. The obtained stability criterion is less conservative than some existing ones. Finally, numerical examples are given to indicate less conservatism of the stability results.

Definition 1.1. Let $\Phi_1, \Phi_2, \ldots, \Phi_N : \mathcal{R}^m \to \mathcal{R}^n$ be a given finite number of functions such that they have positive values in an open subset D of $\mathcal{R}^m$. Then, a reciprocally convex combination of these functions over D is a function of the form

$$\frac{1}{\alpha_1}\Phi_1 + \frac{1}{\alpha_2}\Phi_2 + \cdots + \frac{1}{\alpha_N}\Phi_N : D \longrightarrow R^n, \tag{1.11}$$

where the real numbers α_i satisfy $\alpha_i > 0$ and $\sum_i \alpha_i = 1$.

The following Lemma 1.2 suggests a lower bound for a reciprocally convex combination of scalar positive functions $\Phi_i = f_i$.

Lemma 1.2 (See [10]). *Let $f_1, f_2, \ldots, f_N : \mathcal{R}^m \to \mathcal{R}$ have positive values in an open subset D of $\mathcal{R}^m$. Then, the reciprocally convex combination of f_i over D satisfies*

$$\min_{\{\alpha_i \mid \alpha_i > 0, \sum_i \alpha_i = 1\}} \sum_i \frac{1}{\alpha_i} f_i(t) = \sum_i f_i(t) + \max_{g_{i,j}(t)} \sum_{i \neq j} g_{i,j}(t) \tag{1.12}$$

subject to

$$\left\{ g_{i,j} : R^m \longrightarrow R,\ g_{j,i}(t) \triangleq g_{i,j}(t), \begin{bmatrix} f_i(t) & g_{i,j}(t) \\ g_{i,j}(t) & f_j(t) \end{bmatrix} \geqslant 0 \right\}. \tag{1.13}$$

In the following, we present a new stability criterion by a convex polyhedron method and Lemma 1.2.

2. Main Results

Theorem 2.1. *System* (1.5) *with delays $d_1(t)$ and $d_2(t)$ satisfying* (1.6) *is asymptotically stable if there exist symmetric positive definite matrices P, Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Z, Z_1, Z_2 and any matrices S_{12}, N, M, L, S, P_1, P_2 with appropriate dimensions, such that the following LMIs hold:*

$$\begin{bmatrix} Z & S_{12} \\ * & Z \end{bmatrix} \geqslant 0, \tag{2.1}$$

$$\overline{E}_{13} = \begin{bmatrix} E & -d_1 N & -d_2 L \\ * & -d_1 Z_1 & 0 \\ * & * & -d_2 Z_2 \end{bmatrix} < 0, \tag{2.2}$$

$$\overline{E}_{14} = \begin{bmatrix} E & -d_1 N & -d_2 S \\ * & -d_1 Z_1 & 0 \\ * & * & -d_2 Z_2 \end{bmatrix} < 0, \tag{2.3}$$

$$\overline{E}_{23} = \begin{bmatrix} E & -d_1 M & -d_2 L \\ * & -d_1 Z_1 & 0 \\ * & * & -d_2 Z_2 \end{bmatrix} < 0, \tag{2.4}$$

$$\overline{E}_{24} = \begin{bmatrix} E & -d_1 M & -d_2 S \\ * & -d_1 Z_1 & 0 \\ * & * & -d_2 Z_2 \end{bmatrix} < 0, \tag{2.5}$$

where

$$E = \begin{bmatrix} E_{11} & E_{12} & S_{12}^T & 0 & 0 & 0 & 0 & E_{18} \\ * & E_{22} & E_{23} & 0 & 0 & 0 & 0 & A_d^T P_2^T \\ * & * & E_{33} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & E_{44} & 0 & 0 & 0 & 0 \\ * & * & * & * & -Q_4 & 0 & 0 & 0 \\ * & * & * & * & * & E_{66} & 0 & 0 \\ * & * & * & * & * & * & -Q_6 & 0 \\ * & * & * & * & * & * & * & E_{88} \end{bmatrix} + \begin{bmatrix} N+L & 0 & 0 & M-N & -M & S-L & -S & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} N+L & 0 & 0 & M-N & -M & S-L & -S & 0 \end{bmatrix}^T,$$
$$E_{11} = Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Q_6 - Z + P_1 A + A^T P_1^T, \qquad E_{12} = -S_{12}^T + Z + P_1 A_d,$$
$$E_{18} = P - P_1 + A^T P_2^T, \qquad E_{22} = -(1-\tau)Q_1 - 2Z + S_{12} + S_{12}^T, \qquad E_{23} = -S_{12}^T + Z,$$
$$E_{33} = -Q_2 - Z, \qquad E_{44} = -(1-\tau_1)Q_3, \qquad E_{66} = -(1-\tau_2)Q_5,$$
$$E_{88} = d^2 Z + d_1 Z_1 + d_2 Z_2 - P_2 - P_2^T. \tag{2.6}$$

Proof. Construct a new Lyapunov functional candidate as

$$\begin{aligned} V(x(t)) =& V_1(x(t)) + V_2(x(t)) + V_3(x(t)) + V_4(x(t)), \\ V_1(x(t)) =& x^T(t) P x(t), \\ V_2(x(t)) =& \int_{t-d(t)}^{t} x^T(s) Q_1 x(s) ds + \int_{t-d}^{t} x^T(s) Q_2 x(s) ds + \int_{t-d_1(t)}^{t} x^T(s) Q_3 x(s) ds \\ &+ \int_{t-d_1}^{t} x^T(s) Q_4 x(s) ds + \int_{t-d_2(t)}^{t} x^T(s) Q_5 x(s) ds + \int_{t-d_2}^{t} x^T(s) Q_6 x(s) ds, \\ V_3(x(t)) =& d \int_{-d}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) Z \dot{x}(s) ds\, d\theta, \\ V_4(x(t)) =& \int_{-d_1}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) Z_1 \dot{x}(s) ds\, d\theta + \int_{-d_2}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) Z_2 \dot{x}(s) ds\, d\theta. \end{aligned} \tag{2.7}$$

□

Remark 2.2. Our paper fully uses the information about $d(t)$, $d_1(t)$, and $d_2(t)$, but [15, 16] only use the information about $d_1(t)$ and $d_2(t)$, when constructing the Lyapunov functional $V(x(t))$. So the Lyapunov functional in our paper is more general than that in [15, 16], and the stability criteria in our paper may be more applicable.

The time derivative of $V(x(t))$ along the trajectory of system (1.5) is given by

$$\dot{V}_1(x(t)) = 2x^T(t)P\dot{x}(t), \tag{2.8}$$

$$\begin{aligned}\dot{V}_2(x(t)) = {}& x^T(t)(Q_1+Q_2+Q_3+Q_4+Q_5+Q_6)x(t) - (1-\tau)x^T(t-d(t))Q_1x(t-d(t)) \\ & - x^T(t-d)Q_2x(t-d) - (1-\tau_1)x^T(t-d_1(t))Q_3x(t-d_1(t)) \\ & - x^T(t-d_1)Q_4x(t-d_1) - (1-\tau_2)x^T(t-d_2(t))Q_5x(t-d_2(t)) \\ & - x^T(t-d_2)Q_6x(t-d_2),\end{aligned} \tag{2.9}$$

$$\dot{V}_3(x(t)) = d^2\dot{x}^T(t)Z\dot{x}(t) - d\int_{t-d}^{t-d(t)}\dot{x}^T(s)Z\dot{x}(s)ds - d\int_{t-d(t)}^{t}\dot{x}^T(s)Z\dot{x}(s)ds, \tag{2.10}$$

$$\begin{aligned}\dot{V}_4(x(t)) = {}& \dot{x}^T(t)(d_1Z_1+d_2Z_2)\dot{x}(t) - \int_{t-d_1}^{t-d_1(t)}\dot{x}^T(s)Z_1\dot{x}(s)ds - \int_{t-d_1(t)}^{t}\dot{x}^T(s)Z_1\dot{x}(s)ds \\ & - \int_{t-d_2}^{t-d_2(t)}\dot{x}^T(s)Z_2\dot{x}(s)ds - \int_{t-d_2(t)}^{t}\dot{x}^T(s)Z_2\dot{x}(s)ds.\end{aligned} \tag{2.11}$$

The $\dot{V}_3(x(t))$ is upper bounded by

$$\begin{aligned}\dot{V}_3(x(t)) \leqslant {}& d^2\dot{x}^T(t)Z\dot{x}(t) - \frac{d}{d-d(t)}\zeta^T(t)(e_2-e_3)Z(e_2-e_3)^T\zeta(t) \\ & - \frac{d}{d(t)}\zeta^T(t)(e_1-e_2)Z(e_1-e_2)^T\zeta(t)\end{aligned} \tag{2.12}$$

$$\leqslant d^2\dot{x}^T(t)Z\dot{x}(t) - \zeta^T(t)\begin{bmatrix} e_2^T - e_3^T \\ e_1^T - e_2^T \end{bmatrix}^T \begin{bmatrix} Z & S_{12} \\ S_{12}^T & Z \end{bmatrix} \begin{bmatrix} e_2^T - e_3^T \\ e_1^T - e_2^T \end{bmatrix}\zeta(t), \tag{2.13}$$

where the inequality in (2.12) comes from the Jensen inequality lemma, and that of (2.13) from Lemma 1.2 as

$$-\zeta^T(t)\begin{bmatrix} \sqrt{\dfrac{\beta}{\alpha}}(e_2-e_3)^T \\ -\sqrt{\dfrac{\alpha}{\beta}}(e_1-e_2)^T \end{bmatrix}^T \begin{bmatrix} Z & S_{12} \\ S_{12}^T & Z \end{bmatrix} \begin{bmatrix} \sqrt{\dfrac{\beta}{\alpha}}(e_2-e_3)^T \\ -\sqrt{\dfrac{\alpha}{\beta}}(e_1-e_2)^T \end{bmatrix}\zeta(t) \leqslant 0, \tag{2.14}$$

where

$$\zeta^T(t) = \begin{bmatrix} x^T(t) & x^T(t-d(t)) & x^T(t-d) & x^T(t-d_1(t)) & x^T(t-d_1) & x^T(t-d_2(t)) \\ x^T(t-d_2) & \dot{x}^T(t) \end{bmatrix},$$
$$e_1 = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T, \quad e_2 = \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T, \quad e_3 = \begin{pmatrix} 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T, \tag{2.15}$$

$\alpha = (d - d(t))/d$, $\beta = d(t)/d$. Note that when $d(t) = d$ or $d(t) = 0$, one can obtain $\zeta^T(t)(e_2 - e_3) = 0$ or $\zeta^T(t)(e_1 - e_2) = 0$, respectively. So the relation (2.13) also holds.

By the Jensen inequality lemma, it is easy to obtain

$$\begin{aligned}
-\int_{t-d_1(t)}^{t} \dot{x}^T(s) Z_1 \dot{x}(s)ds &\leqslant -d_1(t) U_1^T Z_1 U_1, \\
-\int_{t-d_1}^{t-d_1(t)} \dot{x}^T(s) Z_1 \dot{x}(s)ds &\leqslant -(d_1 - d_1(t)) U_2^T Z_1 U_2, \\
-\int_{t-d_2(t)}^{t} \dot{x}^T(s) Z_2 \dot{x}(s)ds &\leqslant -d_2(t) U_3^T Z_2 U_3, \\
-\int_{t-d_2}^{t-d_2(t)} \dot{x}^T(s) Z_2 \dot{x}(s)ds &\leqslant -(d_2 - d_2(t)) U_4^T Z_2 U_4,
\end{aligned} \tag{2.16}$$

where

$$\begin{aligned}
U_1 &= \frac{1}{d_1(t)} \int_{t-d_1(t)}^{t} \dot{x}(s)ds, \qquad U_2 = \frac{1}{d_1 - d_1(t)} \int_{t-d_1}^{t-d_1(t)} \dot{x}(s)ds, \\
U_3 &= \frac{1}{d_2(t)} \int_{t-d_2(t)}^{t} \dot{x}(s)ds, \qquad U_4 = \frac{1}{d_2 - d_2(t)} \int_{t-d_2}^{t-d_2(t)} \dot{x}(s)ds,
\end{aligned} \tag{2.17}$$

$$\begin{aligned}
&\lim_{d_1(t)\to 0} \frac{1}{d_1(t)} \int_{t-d_1(t)}^{t} \dot{x}(s)ds = \dot{x}(t), \\
&\lim_{d_1(t)\to d_1} \frac{1}{d_1 - d_1(t)} \int_{t-d_1}^{t-d_1(t)} \dot{x}(s)ds = \dot{x}(t-d_1), \\
&\lim_{d_2(t)\to 0} \frac{1}{d_2(t)} \int_{t-d_2(t)}^{t} \dot{x}(s)ds = \dot{x}(t), \\
&\lim_{d_2(t)\to d_2} \frac{1}{d_2 - d_2(t)} \int_{t-d_2}^{t-d_2(t)} \dot{x}(s)ds = \dot{x}(t-d_2).
\end{aligned} \tag{2.18}$$

From the Leibniz-Newton formula, the following equations are true for any matrices N, M, L, S, P_1, P_2 with appropriate dimensions

$$\begin{aligned}
&2\zeta^T(t)N[x(t) - x(t-d_1(t)) - d_1(t)U_1] = 0,\\
&2\zeta^T(t)M[x(t-d_1(t)) - x(t-d_1) - (d_1 - d_1(t))U_2] = 0,\\
&2\zeta^T(t)L[x(t) - x(t-d_2(t)) - d_2(t)U_3] = 0,\\
&2\zeta^T(t)S[x(t-d_2(t)) - x(t-d_2) - (d_2 - d_2(t))U_4] = 0,\\
&2\left[x^T(t)P_1 + \dot{x}^T(t)P_2\right][-\dot{x}(t) + Ax(t) + A_d x(t-d(t))] = 0.
\end{aligned} \tag{2.19}$$

Hence, according to (2.8)–(2.19), we can obtain

$$\dot{V}(x(t)) \leqslant \xi^T(t)\overline{E}\xi(t), \tag{2.20}$$

where

$$\begin{gathered}
\xi^T(t) = \left[\zeta^T(t) \;\; U_1^T \;\; U_2^T \;\; U_3^T \;\; U_4^T\right],\\
\overline{E} = \begin{bmatrix}
E & -d_1(t)N & -(d_1 - d_1(t))M & -d_2(t)L & -(d_2 - d_2(t))S\\
* & -d_1(t)Z_1 & 0 & 0 & 0\\
* & * & -(d_1 - d_1(t))Z_1 & 0 & 0\\
* & * & * & -d_2(t)Z_2 & 0\\
* & * & * & * & -(d_2 - d_2(t))Z_2
\end{bmatrix}.
\end{gathered} \tag{2.21}$$

If $\overline{E} < 0$, then there exists a scalar $\varepsilon > 0$, such that

$$\dot{V}(x(t)) \leqslant \xi^T(t)\overline{E}\xi(t) \leqslant -\varepsilon\xi^T(t)\xi(t) \leqslant -\varepsilon x^T(t)x(t) < 0, \quad \forall x(t) \neq 0. \tag{2.22}$$

The $\overline{E} < 0$ leads for $d_1(t) \to d_1$ to $E_1 < 0$ and leads for $d_1(t) \to 0$ to $E_2 < 0$, where

$$E_1 = \begin{bmatrix}
E & -d_1 N & -d_2(t)L & -(d_2 - d_2(t))S\\
* & -d_1 Z_1 & 0 & 0\\
* & * & -d_2(t)Z_2 & 0\\
* & * & * & -(d_2 - d_2(t))Z_2
\end{bmatrix} < 0, \tag{2.23}$$

$$E_2 = \begin{bmatrix}
E & -d_1 M & -d_2(t)L & -(d_2 - d_2(t))S\\
* & -d_1 Z_1 & 0 & 0\\
* & * & -d_2(t)Z_2 & 0\\
* & * & * & -(d_2 - d_2(t))Z_2
\end{bmatrix} < 0. \tag{2.24}$$

It is easy to see that E_1 results from $\overline{E}_{|d_1(t)=d_1}$, where we deleted the zero row and the zero column. Define

$$\begin{aligned} \xi_1^T(t) &= \begin{bmatrix} \zeta^T(t) & U_1^T & U_3^T & U_4^T \end{bmatrix}, \\ \xi_2^T(t) &= \begin{bmatrix} \zeta^T(t) & U_2^T & U_3^T & U_4^T \end{bmatrix}, \end{aligned} \tag{2.25}$$

The LMI (2.23) and (2.24) imply (2.22) because

$$\frac{d_1(t)}{d_1}\zeta_1^T(t)E_1\zeta_1(t) + \frac{d_1 - d_1(t)}{d_1}\zeta_2^T(t)E_2\zeta_2(t) = \xi^T(t)\overline{E}\xi(t) \leqslant -\varepsilon x^T(t)x(t) \tag{2.26}$$

and $\overline{E}$ is convex in $d_1(t) \in [0, d_1]$.

LMI (2.23) leads for $d_2(t) \to d_2$ to LMI (2.2) and for $d_2(t) \to 0$ to LMI (2.3). It is easy to see that $\overline{E}_{13}$ results from $E_{1|d_2(t)=d_2}$, where we deleted the zero row and the zero column. The LMI (2.2) and (2.3) imply (2.23) because

$$\frac{d_2(t)}{d_2}\zeta_{13}^T(t)\overline{E}_{13}\zeta_{13}(t) + \frac{d_2 - d_2(t)}{d_2}\zeta_{14}^T(t)\overline{E}_{14}\zeta_{14}(t) = \xi_1^T(t)E_1\xi_1(t) < 0 \tag{2.27}$$

and E_1 is convex in $d_2(t) \in [0, d_2]$, where

$$\begin{aligned} \xi_{13}^T(t) &= \begin{bmatrix} \zeta^T(t) & U_1^T & U_3^T \end{bmatrix}, \\ \xi_{14}^T(t) &= \begin{bmatrix} \zeta^T(t) & U_1^T & U_4^T \end{bmatrix} \end{aligned} \tag{2.28}$$

$\overline{E}_{13}$ and $\overline{E}_{14}$ are defined in Theorem 2.1.

Similarly, the LMI (2.4) and (2.5) imply (2.24) because

$$\frac{d_2(t)}{d_2}\zeta_{23}^T(t)\overline{E}_{23}\zeta_{23}(t) + \frac{d_2 - d_2(t)}{d_2}\zeta_{24}^T(t)\overline{E}_{24}\zeta_{24}(t) = \xi_2^T(t)E_2\xi_2(t) < 0 \tag{2.29}$$

and E_2 is convex in $d_2(t) \in [0, d_2]$, where

$$\begin{aligned} \xi_{23}^T(t) &= \begin{bmatrix} \zeta^T(t) & U_2^T & U_3^T \end{bmatrix}, \\ \xi_{24}^T(t) &= \begin{bmatrix} \zeta^T(t) & U_2^T & U_4^T \end{bmatrix}. \end{aligned} \tag{2.30}$$

$\overline{E}_{23}$ and $\overline{E}_{24}$ are defined in Theorem 2.1. According to the above analysis, one can conclude that the system (1.5) with delays $d_1(t)$ and $d_2(t)$ satisfying (1.6) is asymptotically stable if the LMIs (2.1)–(2.5) hold.

From the proof of Theorem 2.1, one can obtain that $\overline{E}$ is negative definite in the rectangle $0 \leqslant d_1(t) \leqslant d_1$, $0 \leqslant d_2(t) \leqslant d_2$, only if it is negative definite at all vertices. We call this method as the convex polyhedron method.

Remark 2.3. To avoid the emergence of the reciprocally convex combination in (2.12), similar to [9], the integral terms in (2.10) can be upper bounded by

$$\begin{aligned}-d\int_{t-d}^{t}\dot{x}^T(s)Z\dot{x}(s)ds \leqslant -&[x(t-\tau(t))-x(t-d)]^T Z[x(t-\tau(t))-x(t-d)]\\ &-[x(t)-x(t-\tau(t))]^T Z[x(t)-z(t-\tau(t))]\\ &-(1-\gamma)[x(t-\tau(t))-x(t-d)]^T Z[x(t-\tau(t))-x(t-d)]\\ &-\gamma[x(t)-x(t-\tau(t))]^T Z[x(t)-x(t-\tau(t))]\end{aligned} \tag{2.31}$$

which results in a convex combination on γ. However, Theorem 2.1 directly handles the inversely weighted convex combination of quadratic terms of integral quantities by utilizing the result of Lemma 1.2, which achieves performance behavior identical to the approaches based on the integral inequality lemma but with much less decision variables, comparable to those based on the Jensen inequality lemma.

Remark 2.4. Compared to some existing ones, the estimation of $\dot{V}(x(t))$ in the proof of Theorem 2.1 is less conservative due to the convex polyhedron method is employed. More specifically, $-\int_{t-d_1}^{t-d_1(t)}\dot{x}^T(s)Z_1\dot{x}(s)ds$ is retained, while $-\int_{t-d_1}^{t}\dot{x}^T(s)Z_1\dot{x}(s)ds$ is divided into $-\int_{t-d_1(t)}^{t}\dot{x}^T(s)Z_1\dot{x}(s)ds$ and $-\int_{t-d_1}^{t-d_1(t)}\dot{x}^T(s)Z_1\dot{x}(s)ds$. When the two integrals together with others are handled by using free weighting matrix method, instead of enlarging some term $d_1(t)SZ_1^{-1}S^T$ as $d_1SZ_1^{-1}S^T$. The convex polyhedron method is employed to verify the negative definiteness of $\dot{V}(x(t))$. Therefore, Theorem 2.1 is expected to be less conservative than some results in the literature.

Remark 2.5. The case in which only two additive time-varying delay components appear in the state has been considered, and the idea in this paper can be easily extended to the system (1.3) with multiple additive delay components satisfying (1.4). Choose the Lyapunov functional as

$$\begin{aligned}V(x(t)) &= V_1(x(t))+V_2(x(t))+V_3(x(t))+V_4(x(t)),\\ V_1(x(t)) &= x^T(t)Px(t),\\ V_2(x(t)) &= \int_{t-d(t)}^{t}x^T(s)Q_1x(s)ds+\int_{t-d}^{t}x^T(s)Q_2x(s)ds+\sum_{i=1}^{n}\int_{t-d_i(t)}^{t}x^T(s)Q_{3i}x(s)ds\\ &\quad+\sum_{i=1}^{n}\int_{t-d_i}^{t}x^T(s)Q_{4i}x(s)ds,\\ V_3(x(t)) &= d\int_{-d}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)Z\dot{x}(s)ds\,d\theta,\\ V_4(x(t)) &= \sum_{i=1}^{n}\int_{-d_i}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)Z_i\dot{x}(s)ds\,d\theta.\end{aligned} \tag{2.32}$$

Table 1: Calculated delay bounds for different cases.

Stability conditions	Delay bound d_2 for given d_1			Delay bound d_1 for given d_2		
	$d_1 = 1$	$d_1 = 1.2$	$d_1 = 1.5$	$d_2 = 0.3$	$d_2 = 0.4$	$d_2 = 0.5$
[6, 12, 14]	0.180	0.080	Infeasible	0.880	0.780	0.680
[15]	0.415	0.376	0.248	1.324	1.039	0.806
[16]	0.512	0.406	0.283	1.453	1.214	1.021
Theorem 2.1	0.873	0.673	0.373	1.573	1.473	1.373

Then, the corresponding stability result can be easily derived similar to the proof of Theorem 2.1. The result is omitted due to complicated notation.

Remark 2.6. The stability condition presented in Theorem 2.1 is for the nominal system. However, it is easy to further extend Theorem 2.1 to uncertain systems, where the system matrices A and A_d contain parameter uncertainties either in norm-bounded or polytopic uncertain forms. The reason why we consider the simplest case is to make our idea more lucid and to avoid complicated notations.

3. Illustrative Example

Example 3.1. Consider system (1.5) with the following parameters:

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad \text{assuming } \dot{d}_1(t) \leqslant 0.1,\ \dot{d}_2(t) \leqslant 0.8. \tag{3.1}$$

Our purpose is to calculate the upper bound d_1 of delay $d_1(t)$, or d_2 of delay $d_2(t)$, when the other is known, below which the system is asymptotically stable. By combining the two delay components together, some existing stability results can be applied to this system. The calculation results obtained by Theorem 2.1, in this paper, Theorem 1 in [6, 12, 15, 16], [14, Theorem 2] for different cases are listed in Table 1. It can be seen from the Table 1 that Theorem 2.1, in this paper, yields the least conservative stability test than other results.

Example 3.2. Consider system (1.5) with the following parameters:

$$A = \begin{bmatrix} 0.0 & 1.0 \\ -1.0 & -2.0 \end{bmatrix}, \qquad A_d = \begin{bmatrix} 0.0 & 0.0 \\ -1.0 & 1.0 \end{bmatrix}. \tag{3.2}$$

We assume condition 1: $\dot{d}_1(t) \leqslant 0.2$, $\dot{d}_2(t) \leqslant 0.5$; condition 2: $\dot{d}_1(t) \leqslant 0.2$, $\dot{d}_2(t) \leqslant 0.3$, and under the two cases above, respectively. Table 2 lists the corresponding upper bounds of d_2 for given d_1. This numerical illustrates the effectiveness of the derived results.

Table 2: Allowable upper bound of d_2 for various d_1.

	d_1	0.3	0.5	0.7	0.9
Condition 1	d_2	0.767	0.567	0.367	0.067
Condition 2	d_2	0.968	0.768	0.568	0.368

4. Conclusions

This paper has investigated the stability problem for continuous systems with two additive time-varying delay components. By constructing a new class of Lyapunov functional and using a new convex polyhedron method, a new delay-dependent stability criterion is derived in terms of linear matrix inequalities. The obtained stability criterion is less conservative than some existing ones. Finally, numerical examples are given to illustrate the effectiveness of the proposed method.

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