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*Research Article*

# Explicit Iterative Method for Variational Inequalities on Hadamard Manifolds

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An explicit iterative method for solving the variational inequalities on Hadamard manifold is suggested and analyzed using the auxiliary principle technique. The convergence of this new method requires only the partially relaxed strongly monotonicity, which is a weaker condition than monotonicity. Results can be viewed as refinement and improvement of previously known results.

## 1. Introduction

In recent years, much attention has been given to study the variational inequalities and related problems on the Riemannian manifold and Hadamard manifold. This framework is useful for the development of various fields on nonlinear setting. Several ideas and techniques from the Euclidean space have been extended and generalized to this nonlinear framework. Hadamard manifolds are examples of hyperbolic spaces and geodesics, see [1–7] and the references therein. Nemeth [8], Tang et al. [6], and Colao et al. [2] have considered the variational inequalities and equilibrium problems on Hadamard manifolds. They have studied the existence of a solution of the equilibrium problems under some suitable conditions. Several methods have been developed for solving the variational inequalities and related problems in the linear-normed spaces. The auxiliary principle technique is a powerful tool to suggest and analyze several implicit and explicit iterative methods for solving the equilibrium problems and variational inequalities. This technique is due to Glowinski et al. [9]. M. A. Noor and K. I. Noor [10]; Noor et al. [11] have used the auxiliary principle technique to suggest some iterative methods for solving the variational inequalities and equilibrium problems on Hadamard manifolds. We again use the auxiliary principle technique to suggest and analyze an explicit iterative method for solving the variational inequalities, and this is the main motivation of this paper. We show that the convergence of this new method requires only the partially relaxed strongly monotonicity, which is a weaker condition than monotonicity. This represents the refinement of previously known results for the variational inequalities. We hope that the technique and idea of this paper may stimulate further research in this area.

## 2. Preliminaries

We now recall some fundamental and basic concept that need for a reading of this paper. These results and concepts can be found in the books on Riemannian geometry [2, 3, 5].

Let $M$ be a simply connected $m$-dimensional manifold. Given $x \in M$, the tangent space of $M$ at $x$ is denoted by $T_xM$ and the tangent bundle of $M$ by $TM = \cup_{x\in M} T_xM$, which is naturally a manifold. A vector field $A$ on $M$ is a mapping of $M$ into $TM$ which associates to each point $x \in M$ a vector $A(x) \in T_xM$. We always assume that $M$ can be endowed with a Riemannian metric to become a Riemannian manifold. We denote by $\langle , \cdot, \rangle$ the scalar product on $T_xM$ with the associated norm $\|\cdot\|_x$, where the subscript $x$ will be omitted. Given a piecewise smooth curve $\gamma : [a,b] \to M$ joining $x$ to $y$ (that is, $\gamma(a) = x$ and$\gamma(b) = y$) by using the metric, we can define the length of $\gamma$ as $L(\gamma) = \int_a^b \|\gamma'(t)\| dt$. Then, for any $x, y \in M$, the Riemannian distance $d(x,y)$, which includes the original topology on $M$, is defined by minimizing this length over the set of all such curves joining $x$ to $y$.

Let $\Delta$ be the Levi-Civita connection with $(M, \langle\cdot,\cdot\rangle)$. Let $\gamma$ be a smooth curve in $M$. A vector field $A$ is said to be parallel along $\gamma$ if $\Delta_{\gamma'} A = 0$. If $\gamma'$ itself is parallel along $\gamma$, we say that $\gamma$ is a geodesic, and in this case, $\|\gamma'\|$ is constant. When $\|\gamma'\| = 1$, $\gamma$ is said to be normalized. A geodesic joining $x$ to $y$ in $M$ is said to be minimal if its length equals $d(x,y)$.

A Riemannian manifold is complete, if for any $x \in M$ all geodesics emanating from $x$ are defined for all $t \in R$. By the Hopf-Rinow theorem, we know that if $M$ is complete, then any pair of points in $M$ can be joined by a minimal geodesic. Moreover, $(M,d)$ is a complete metric space, and bounded closed subsets are compact.

Let $M$ be complete. Then the exponential map $\exp_x : T_xM \to M$ at $x$ is defined by $\exp_x v = \gamma_v(1,x)$ for each $v \in T_xM$, where $\gamma(\cdot) = \gamma_v(\cdot, x)$ is the geodesic starting at $x$ with velocity $v$(i.e., $\gamma(0) = x$ and $\gamma'(0) = v$). Then $\exp_x tv = \gamma_v(t,x)$ for each real number $t$.

A complete simply connected Riemannian manifold of nonpositive sectional curvature is called a *Hadamard manifold*. Throughout the remainder of this paper, we always assume that $M$ is an $m$-manifold Hadamard manifold.

We also recall the following well-known results, which are essential for our work.

**Lemma 2.1** (see [5]). *Let $x \in M$. Then $\exp_x : T_xM \to M$ is a diffeomorphism, and for any two points $x, y \in M$, there exists a unique normalized geodesic joining $x$ to $y$, $\gamma_{x,y}$, which is minimal.*

So from now on, when referring to the geodesic joining two points, we mean the unique minimal normalized one. Lemma 2.1 says that $M$ is diffeomorphic to the Euclidean space $R^m$. Thus, $M$ has the same topology and differential structure as $R^m$. It is also known that Hadamard manifolds and euclidean spaces have similar geometrical properties. Recall that a geodesic triangle $\triangle\ (x_1, x_2, x_3)$ of a Riemannian manifold is a set consisting of three points $x_1$, $x_2$, and $x_3$ and three minimal geodesics joining these points.

**Lemma 2.2** ([2, 3, 5] (comparison theorem for triangles)). *Let $\triangle\ (x_1, x_2, x_3)$ be a geodesic triangle. Denote, for each $i = 1,2,3 \pmod 3$, by $\gamma_i : [0, l_i] \to M$ the geodesic joining $x_i$ to $x_{i+1}$,* *and* $\alpha_i ;= L(\gamma_i'(0), -\gamma_l'(i-1)(li-1))$, *the angle between the vectors* $\gamma_i'(0)$ *and* $-\gamma_{i-1}'(l_{i-1})$, *and* $l_i ;= L(\gamma_i)$. *Then*

$$\alpha_1 + \alpha_2 + \alpha_3 \leq \pi, \tag{2.1}$$

$$l_l^2 + l_{i+1}^2 - 2L_i l_{i+1} \cos \alpha_{i+1} \leq l_{i-1}^2. \tag{2.2}$$

*In terms of the distance and the exponential map, the inequality* (2.2) *can be rewritten as*

$$d^2(x_i, x_{i+1}) + d^2(x_{i+1}, x_{i+2}) - 2\left\langle \exp_{x_{i+1}}^{-1} x_i, \exp_{x_{i+1}}^{-1} x_{i+2} \right\rangle \leq d^2(x_{i-1}, x_i), \tag{2.3}$$

*since*

$$\left\langle \exp_{x_{i+1}}^{-1} x_i, \exp_{x_{i+1}}^{-1} x_{i+2} \right\rangle = d(x_i, x_{i+1}) d(x_{i+1}, x_{i+2}) \cos \alpha_{i+1}. \tag{2.4}$$

**Lemma 2.3** (see [5]). *Let* $\triangle(x, y, z)$ *be a geodesic triangle in a Hadamard manifold* $M$. *Then there exist* $x', y', z' \in R^2$ *such that*

$$d(x, y) = \|x' - y'\|, \quad d(y, z) = \|y' - z'\|, \quad d(z, x) = \|z' - x'\|. \tag{2.5}$$

*The triangle* $\triangle(x', y', z')$ *is called the comparison triangle of the geodesic triangle* $\triangle(x, y, z)$, *which is unique up to isometry of* $M$.

From the law of cosines in inequality (2.3), one has the following inequality, which is a general characteristic of the spaces with nonpositive curvature [5]:

$$\left\langle \exp_x^{-1} y, \exp_x^{-1} z \right\rangle + \left\langle \exp_y^{-1} x, \exp_y^{-1} z \right\rangle \geq d^2(x, y). \tag{2.6}$$

From the properties of the exponential map, one has the following known result.

**Lemma 2.4** (see [5]). *Let* $x_0 \in M$ *and* $\{x_n\} \subset M$ *such that* $x_n \to x_0$. *Then the following assertions hold.*

(i) *For any* $y \in M$,

$$\exp_{x_n}^{-1} y \longrightarrow \exp_{x_o}^{-1} y, \qquad \exp_y^{-1} x_n \longrightarrow \exp_y^{-1} x_o. \tag{2.7}$$

(ii) *If* $\{v_n\}$ *is a sequence such that* $v_n \in T_{x_n} M$ *and* $v_n \to v_0$, *then* $v_0 \in T_{x_0} M$.

(iii) *Given the sequences* $\{u_n\}$ *and* $\{v_n\}$ *satisfying* $u_n, v_n \in T_{x_n} M$, *if* $u_n \to u_0$ *and* $v_n \to v_0$, *with* $u_0, v_0 \in T_{x_0} M$, *then*

$$\langle u_n, v_n \rangle \longrightarrow \langle u_0, v_0 \rangle. \tag{2.8}$$

A subset $K \subseteq M$ is said to be convex if for any two points $x, y \in K$, the geodesic joining $x$ and $y$ is contained in $K, K$ that is, if $\gamma : [a,b] \to M$ is a geodesic such, that $x = \gamma(a)$ and $y = \gamma(b)$, then $\gamma((1-t)a + tb) \in K,$ for all $t \in [0,1]$. From now on, $K \subseteq M$ will denote a nonempty, closed, and convex set, unless explicitly stated otherwise.

A real-valued function $f$ defined on $K$ is said to convex if for any geodesic $\gamma$ of $M$, the composition function $f \circ \gamma : R \to R$ is convex, that is,

$$(f \circ \gamma)(ta + (1-t)b) \leq t(f \circ \gamma)(a) + (1-t)(f \circ \gamma)(b), \quad \forall a, b \in R,\ t \in [0,1]. \tag{2.9}$$

The subdifferential of a function $f : M \to R$ is the set-valued mapping $\partial f : M \to 2^{TM}$ defined as

$$\partial f(x) = \left\{ u \in T_x M : \left\langle u, \exp_x^{-1} y \right\rangle \leq f(y) - f(x), \forall y \in M \right\}, \quad \forall x \in M, \tag{2.10}$$

and its elements are called subgradients. The subdifferential $\partial f(x)$ at a point $x \in M$ is a closed and convex (possibly empty) Let $D(\partial f)$ denote the domain of $\partial f$ defined by

$$D(\partial f) = \{ x \in M : \partial f(x) \neq \emptyset \}. \tag{2.11}$$

The existence of subgradients for convex functions is guaranteed by the following proposition, see [7].

**Lemma 2.5** (see [5, 7]). *Let $M$ be a Hadamard manifold, and let $f : M \to R$ be convex. Then, for any $x \in M$, the subdifferential $\partial f(x)$ of $f$ at $x$ is nonempty. That is, $D(\partial f) = M$.*

*For a given single-valued vector field $T : M \to TM$, one considers the problem of finding $u \in K$ such that*

$$\left\langle Tu, \exp_u^{-1} v \right\rangle \geq 0, \quad \forall v \in K, \tag{2.12}$$

*which is called the variational inequality. This problem was considered by Nemeth [8], Colao et al. [2], Tang et al. [6], and M. A. Noor and K. I. Noor [10]. They proved the existence of a solution of Problem* (2.12) *using the KKM maps. In the linear setting, variational inequalities have been studied extensively, see [8–10, 12–26] and the references therein.*

*Definition 2.6.* An operator $T$ is said to be partially relaxed strongly monotonicity if and only if there exists a constant $\alpha > 0$ such that

$$\left\langle Tu, \exp_v^{-1} z \right\rangle + \left\langle Tv, \exp_z^{-1} v \right\rangle \leq \alpha d^2(z,u), \quad \forall u, v, z \in M. \tag{2.13}$$

We note that if $z = u$, then partially relaxed strongly monotonicity reduces to monotonicity, but the converse is not true.

## 3. Main Results

We now use the auxiliary principle technique of Glowinski et al. [9] to suggest and analyze an explicit iterative method for solving the variational inequality (2.12) on the Hadamard manifold.

For a given $u \in K$ satisfying (2.12), consider the problem of finding $w \in K$ such that

$$\left\langle \rho T u + \left(\exp_u^{-1} w\right), \exp_w^{-1} v \right\rangle \geq 0, \quad \forall v \in K, \tag{3.1}$$

which is called the auxiliary variational inequality on Hadamard manifolds. We note that if $w = u$, then $w$ is a solution of the variational inequality (2.12). This observation enables as to suggest and analyze the following proximal point method for solving the variational inequality (2.12).

*Algorithm 3.1.* For a given $u_0$, compute the approximate solution by the iterative scheme

$$\left\langle \rho T u_n + \left(\exp_{u_n}^{-1} u_{n+1}\right), \exp_{u_{n+1}}^{-1} v \right\rangle \geq 0, \quad \forall v \in K. \tag{3.2}$$

Algorithm 3.1 is called the explicit iterative method for solving the variational inequality on the Hadamard manifold.

If $M = R^n$, then Algorithm 3.1 collapses to the following.

*Algorithm 3.2.* For a given $u_0 \in K$, find the approximate solution $u_{n+1}$ by the iterative scheme

$$\langle \rho T u_n + u_{n+1} - u_n, v - u_{n+1} \rangle \geq 0, \quad \forall v \in K, \tag{3.3}$$

which is known as the explicit method for solving the variational inequalities. For the convergence analysis of Algorithm 3.2, see [13, 14].

We now consider the convergence analysis of Algorithm 3.1, and this is the main motivation of our next result.

**Theorem 3.3.** *Let $T$ be a partially relaxed strongly monotone vector field with a constant $\alpha > 0$. Let $u_{n+1}$ be the approximate solution of the variational inequality* (2.12) *obtained from Algorithm 3.1. Then*

$$d^2(u_{n+1}, u) \leq d^2(u_n, u) - (1 - 2\alpha\rho) d^2(u_{n+1}, u_n), \tag{3.4}$$

*where $u \in M$ is the solution of the variational inequality* (2.12).

*Proof.* Let $u \in K$ be a solution of the variational inequality (2.12). Then

$$\left\langle \rho T(u), \exp_u^{-1} v \right\rangle \geq 0, \quad \forall v \in \mathrm{K}. \tag{3.5}$$

Taking $v = u_{n+1}$ in (3.5), we have

$$\left\langle \rho T(u), \exp_u^{-1} u_{n+1} \right\rangle \geq 0. \tag{3.6}$$

Taking $v = u$ in (3.2), we have

$$\left\langle \rho T u_n + \left(\exp_{u_n}^{-1} u_{n+1}\right), \exp_{u_{n+1}}^{-1} u \right\rangle \geq 0. \tag{3.7}$$

From (3.6) and (3.7), we have

$$\begin{aligned} \left\langle \exp_{u_{n+1}}^{-1} u_n, \exp_{u_{n+1}}^{-1} u \right\rangle &\geq -\rho \left\{ \left\langle T(u), \exp_u^{-1} v \right\rangle + \left\langle T u_n, \exp_{u_{n+1}}^{-1} u \right\rangle \right\}. \\ &\geq -\alpha \rho d^2(u_{n+1}, u_n). \end{aligned} \tag{3.8}$$

For the geodesic triangle $\triangle\,(u_n, u_{n+1}, u)$, the inequality (3.2) can be written as

$$d^2(u_{n+1}, u) + d^2(u_{n+1}, u_n) - 2\left\langle \exp_{u_{n+1}}^{-1} u_n, \exp_{u_{n+1}}^{-1} u \right\rangle \leq d^2(u_n, u). \tag{3.9}$$

Thus, from (3.8) and (3.9), we obtained the inequality (3.4), the required result. □

**Theorem 3.4.** *Let $u \in K$ be a solution of* (2.12)*, and let $u_{n+1}$ be the approximate solution obtained from Algorithm 3.1. If $\rho < 1/2\alpha$, then $\lim_{n\to\infty} u_{n+1} = u$*

*Proof*. Let $\hat{u} \in K$ be a solution of (2.12). Then, from (3.4), it follows that the sequence $\{u_n\}$ is bounded and

$$\sum_{n=0}^{\infty} (1 - 2\alpha\rho) d^2(u_{n+1}, u_n) \leq d^2(u_0, u). \tag{3.10}$$

It follows that

$$\lim_{n\to\infty} d(u_{n+1}, \mathrm{u}_n) = 0. \tag{3.11}$$

Let $\hat{u}$ be a cluster point of $\{u_n\}$. Then there exits a subsequence $\{u_{n_i}\}$ such that $\{u_{u_i}\}$ converges to $\hat{u}$. Replacing $u_{n+1}$ by $u_{n_i}$ in (3.2), taking the limit, and using (3.10), we have

$$\left\langle T\hat{u}, \exp_{\hat{u}}^{-1} v \right\rangle \geq 0, \quad \forall v \in K. \tag{3.12}$$

This shows that $\hat{u} \in K$ solves (2.12) and

$$d^2(u_{n+1}, \hat{u}) \leq d^2(u_n, \hat{u}) \tag{3.13}$$

which implies that the sequence $\{u_n\}$ has an unique cluster point and $\lim_{n\to\infty} u_n = \hat{u}$ is a solution of (2.12), the required result. □

## 4. Conclusion

We have used the auxiliary principle technique to suggest and analyze an explicit iterative method for solving the mixed quasivariational inequalities on Hadamard manifolds. Some special cases are also discussed. Convergence analysis of the new proximal point method is proved under weaker conditions. Results obtained in this paper may stimulate further research in this area. The implementation of the new method and its comparison with other methods is an open problem. The ideas and techniques of this paper may be extended for other related optimization problems.

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