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Research Article

Equivalent Lagrangians: Generalization, Transformation Maps, and Applications

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Equivalent Lagrangians are used to find, via transformations, solutions and conservation law of a given differential equation by exploiting the possible existence of an isomorphic algebra of Lie point symmetries and, more particularly, an isomorphic Noether point symmetry algebra. Applications include ordinary differential equations such as the *Kummer equation* and the *combined gravity-inertial-Rossby wave equation* and certain classes of partial differential equations related to multidimensional wave equations.

1. Introduction

The method of equivalent Lagrangians is used to find the solutions of a given differential equation by exploiting the possible existence of an isomorphic algebra of Lie point symmetries and, more particularly, an isomorphic algebra of Noether point symmetries. The underlying idea of the method is to construct a regular point transformation which maps the Lagrangian of a "simpler" differential equation (with known solutions) to the Lagrangian of the differential equation in question. Once determined, this point transformation will then provide a way of mapping the solutions of the simpler differential equation to the solutions of the equation we seek to solve. This transformation can also be used to find conserved quantities for the equation in question, if the conserved quantities for the simpler differential equation are known. In the sections that follow, the method of equivalent Lagrangians is described for scalar second-order ordinary differential equations and for partial differential equations in two independent variables.

Some well-known ordinary differential equations in mathematical physics such as the Kummer equation and the combined gravity-inertial-rossby wave equation are analysed. Also, using the standard Lagrangian and previous knowledge of the (1+1) wave equation, we find some interesting properties of certain classes of partial differential equations like the canonical form of the wave equation, the wave equation with dissipation and the Klein-Gordon equation.

2. Equivalent Lagrangians

Consider an rth-order system of partial differential equations (DEs) of n-independent variables $x = (x^1, x^2, \dots, x^n)$ and m-dependent variables $u = (u^1, u^2, \dots, u^m)$:

$$G^{\mu}\left(x, u, u_{(1)}, \dots, u_{(r)}\right) = 0, \qquad \mu = 1, \dots, \tilde{m}, \tag{2.1}$$

where $u_{(1)}, u_{(2)}, \dots, u_{(r)}$ denote the collections of all first-, second-, $\dots$, rth-order partial derivatives. The total differentiation operator with respect to x^i is given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^{\alpha} \frac{\partial}{\partial u^{\alpha}} + u_{ij}^{\alpha} \frac{\partial}{\partial u_j^{\alpha}} + \dots, \qquad i = 1, \dots, n. \tag{2.2}$$

A current $T = (T^1, \dots, T^n)$ is conserved if it satisfies

$$D_i T^i = 0 \tag{2.3}$$

along the solutions of (2.1).

The Euler-Lagrange (Euler) operator is defined by

$$\frac{\delta}{\delta u^{\alpha}} = \frac{\partial}{\partial u^{\alpha}} + \sum_{s \geq 1} (-1)^s D_{i_1}, \dots, D_{i_s} \frac{\partial}{\partial u^{\alpha}_{i_1 \cdots i_s}}, \qquad \alpha = 1, \dots, m. \tag{2.4}$$

Hence, the Euler-Lagrange (Euler) equations are of the form

$$\frac{\partial L}{\partial u^{\alpha}} = 0, \quad \alpha = 1, \dots, m, \tag{2.5}$$

where L is a *Lagrangian* of some order; the solutions of (2.5) are the optimizers of the functional

$$\int L\left(x, u, u_{(1)}, \dots\right) \, dx. \tag{2.6}$$

A vector field X of the form

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^{\alpha} \frac{\partial}{\partial u^{\alpha}}, \quad \xi^i, \eta^{\alpha} \in \mathcal{A}, \tag{2.7}$$

which leaves (2.6) invariant is known as a Noether symmetry, where $\mathcal{A}$ is the space of differential functions. Equivalently, X is a Noether symmetry of L if there is a vector $B = (B^1, \ldots, B^n) \in \mathcal{A}$ such that

$$X(L) + LD_i\left(\xi^i\right) = D_i\left(B^i\right), \tag{2.8}$$

where X is prolonged to the degree of L (see [1]). If the vector B is identically zero, then X is a strict Noether symmetry, see Ibragimov et al. [2].

It is well known that if the Noether symmetry algebras for two Lagrangians, L and $\overline{L}$, are isomorphic, the Lagrangians can be mapped from one to the other. In light of this, we define the notion of equivalent Lagrangians.

Definition 2.1. Two Lagrangians, $L = L(x, u, u_{(1)}, \ldots, u_{(r)})$ and $\overline{L} = \overline{L}(X, U, U_{(1)}, \ldots, U_{(r)})$, are said to be equivalent if and only if there exists a transformation, $X = X(x, u)$ and $U = U(x, u)$, such that

$$L(x, u, u_{(1)}, \ldots) = \overline{L}(X, U, U_{(1)}, \ldots) J(x, u, u_{(1)}), \tag{2.9}$$

where J is the determinant of the Jacobian matrix, see Kara [3].

For ordinary differential equations in which $u = u(x)$, the definition of equivalence up to gauge is as follows.

Definition 2.2. Two Lagrangians, L and $\overline{L}$, are said to be equivalent up to gauge if and only if there exists a transformation, $X = X(x, u)$ and $U = U(x, u)$, such that

$$L(x, u, u') = \overline{L}(X, U, U') \frac{dU}{dx} + f_x + u' f_u, \tag{2.10}$$

where the gauge function, f, is an arbitrary function of x and u, see Kara and Mahomed [4].

Remark 2.3. The definitions imply that given a variational differential equation with corresponding Lagrangian L, we can find a regular point transformation $X = X(x, u)$ and $U = U(x, u)$ which maps L to another (equivalent) Lagrangian $\overline{L}$. This regular point transformation also maps the solutions of the differential equation associated with $\overline{L}$ to the solutions of the original differential equation.

Also, once we have found the regular point transformation $X = X(x, u)$ and $U = U(x, u)$ mentioned above, it is possible to use this transformation to map the (known) conserved quantities of the differential equation associated with $\overline{L}$ to the conserved quantities of the equation in question.

As an illustration, consider the well-known harmonic oscillator ordinary differential equation

$$y'' + y = 0, \tag{2.11}$$

with Lagrangian

$$L = \frac{1}{2}y'^2 - \frac{1}{2}y^2. \tag{2.12}$$

Using the method of equivalent Lagrangians detailed in the following sections, one can find the regular point transformation $X = X(x, y)$ and $Y = Y(x, y)$ that maps the Lagrangian

$$\overline{L} = \frac{1}{2}Y'^2, \tag{2.13}$$

associated with the free particle differential equation

$$Y'' = 0, \tag{2.14}$$

to the Lagrangian (2.12) associated with the differential equation (2.11). The transformation in question is given by the equations $X = \tan x$ and $Y = y\sec x$. This transformation in turn maps the solutions of (2.14) to the solutions of (2.11). Furthermore, we can use it to find the conserved quantities of (2.11).

Consider, for example, the known conserved quantity

$$\overline{I} = XY' - Y \tag{2.15}$$

of (2.14). Using transformations $X = X(x, y)$ and $Y = Y(x, y)$ above, it follows that a conserved quantity for (2.11) is

$$\begin{aligned} I &= X\left(\frac{dY}{dX}\right) - Y \\ &= y' \sin x + y \sin x \tan x - y\sec x. \end{aligned} \tag{2.16}$$

This is verified by $dI/dx = y'' \sin x + y \sin x = \sin x(y'' + y)$. I is the well-known integral $y' \sin x - y \cos x$.

3. Applications to ODEs

Second-order ordinary differential equations (ODEs) can be divided into equivalence classes based on their Lie symmetries [5]. Two equations belong to the same equivalence class if there exists a diffeomorphism that transforms one of the equations to the other [5]. If a second-order ordinary differential equation admits eight Lie symmetries (the maximum number of Lie symmetries of a scalar second-order ordinary differential equation, by Lie's "Counting Theorem," it belongs to the equivalence class of the equation $Y'' = 0$ [5]. Hence, it can be mapped to this equation by means of a regular point transformation.

Mahomed et al. [5] prove that the maximum dimension of the Noether symmetry algebra for a scalar second-order ordinary differential equation is five and that (2.14) with standard Lagrangian (2.13) attains this maximum. This five-dimensional Noether algebra is unique (see [5]), and so for any scalar second-order ordinary differential equation with Lagrangian, L, generating a five-dimensional Noether algebra, L can be mapped to $\overline{L}$ by means of a regular point transformation $X = X(x, y)$ and $Y = Y(x, y)$ (this transformation evidently also transforms the corresponding Euler-Lagrange equations, for L and $\overline{L}$, respectively, from one to the other [5]).

We use the method of equivalent Lagrangians detailed above to find solutions and conserved quantities for two scalar second-order ordinary differential equations, namely, the Kummer equation and the combined gravity-inertial Rossby wave equation.

3.1. The Kummer Equation

The Kummer equation, also called the confluent hypergeometric function, has several applications in theoretical physics. It models the velocity distribution of electrons in a high-frequency gas discharge. Using the solutions of this equation, together with kinetic theory, it is thus possible to predict the high-frequency breakdown electric field for gases (see [6]). The differential equation is given by

$$xy'' + (2k - x)y' - ky = 0, \tag{3.1}$$

where k is an arbitrary constant. By rearranging this equation and multiplying by an integrating factor $kxe - x$, we discover that a Lagrangian for this equation is

$$L = \frac{1}{2}x^2 ke^{-x}\left(y' + \frac{k}{x}y^2\right). \tag{3.2}$$

Equation (3.1) has 8 Lie symmetries. Therefore, it can be mapped, via a point transformation $X = X(x, y)$ and $Y = Y(x, y)$ to equation (2.14), with Lagrangian (2.13), which is known to have five Noether symmetries. It can be shown that the Lagrangian for the Kummer equation (3.1), given by (3.2), also has five Noether symmetries. Therefore, Lagrangians (3.2) and (2.13) are equivalent. Invoking Definition 2.2 and substituting L and $\overline{L}$ into (2.10), we can find the point transformations $X = X(x, y)$ and $Y = Y(x, y)$ that map (3.2) to (2.13), and hence (3.1) to (2.14).

Equation (2.10) gives us

$$\begin{aligned} \frac{1}{2}x^{2k}e^{-x}\left(y'^2 + \frac{k}{x}y^2\right) &= \frac{1}{2}y'^2\frac{dX}{dx} + f_x + y'f_y \\ &= \frac{1}{2}\left(\frac{Y_x + 2Y_xY_yy' + y'^2Y_y^2}{X_x + y'X_y}\right) + f_x + y'f_y. \end{aligned} \tag{3.3}$$

In order to simplify the above equation, we assume that X is a function of x only and is of the form $X = \int a^2 dx$, where a is a function of x. In fact, this assumption is not essential. It turns out that the coefficient of the cubic term in the subsequent separation leads to $X_y = 0$. The above equation becomes

$$\frac{1}{2}x^{2k}e^{-x}\left(y'^2 + \frac{k}{x}y^2\right) = \frac{1}{2}\left(\frac{Y_x^2 + 2Y_xY_yy' + y'^2Y_y^2}{a^2}\right) + f_x + y'f_y. \tag{3.4}$$

Now, since the variables x, y, and y' are all linearly independent, we can separate (3.4) by powers of y', after which we obtain a system of three equations:

$$\frac{1}{2}x^{2k}e^{-x} = \frac{1}{2}\left(\frac{Y_y^2}{a^2}\right), \tag{3.5}$$

$$0 = \frac{1}{2}\left(\frac{2Y_xY_y}{a^2}\right) + f_y, \tag{3.6}$$

$$\frac{1}{2}x^{2k}e^{-x}y^2 = \frac{1}{2}\left(\frac{Y_x^2}{a^2}\right) + f_x. \tag{3.7}$$

From (3.5), we get that

$$Y_y = ax^ke^{-(1/2)x}. \tag{3.8}$$

Integrating with respect to y results in the expression

$$Y = ax^ke^{-(1/2)x}y + b(x), \tag{3.9}$$

for which we assume that $b(x) = 0$. We can differentiate (3.9) partially with respect to x, and substitute expressions for Y_x and Y_y (given above) into (3.6), in order to obtain the expression

$$f_y = -x^{2k-1}e^{-x}y\left(\frac{\dot{a}}{a}x + k - \frac{1}{2}x\right), \tag{3.10}$$

from which we get

$$f = -\frac{1}{2}x^{2k-1}e^{-x}y^2\left(\frac{\dot{a}}{a}x + k - \frac{1}{2}x\right) + c(x), \tag{3.11}$$

where we again assume that $c(x) = 0$.

For (3.7), we can substitute our expression for Y_x to obtain

$$xk = \left(\frac{\dot{a}}{a}\right)x^2 + 2\left(\frac{\dot{a}}{a}\right)x\left(k - \frac{1}{2}x\right) + \left(k - \frac{1}{2}x\right)^2 + f_x. \tag{3.12}$$

Making the substitution

$$A = \frac{\dot{a}}{a} \tag{3.13}$$

simplifies the above equation to

$$xk = Ax^2 + 2Ax\left(k - \frac{1}{2}x\right) + \left(k - \frac{1}{2}x\right)^2 + f_x. \tag{3.14}$$

We then differentiate (3.11) partially with respect to x and obtain an expression for f_x, which we can substitute into the above equation. This simplifies to

$$0 = A^2 - \dot{A} - \frac{k}{x}\left(\frac{k}{x} - 1\right) - \frac{1}{4}. \tag{3.15}$$

Integrating the equation $A = \dot{a}/a$ gives us the expression

$$a = e^{\int A dx}, \tag{3.16}$$

where A satisfies (3.15). Hence, we have that

$$\begin{aligned} X &= \int e^{2\int A dx} dx, \\ Y &= e^{\int A dx} x^k e^{-(1/2)x} y. \end{aligned} \tag{3.17}$$

Equation (3.17) defines our regular point transformations $X = X(x, y)$ and $Y = Y(x, y)$, which transform (3.1) to (2.14).

We know that the solution to (2.14) is given by $Y = \alpha X + \beta$, where α and β are arbitrary constants. Therefore, we can substitute expressions (3.17), for X and Y, respectively, to obtain an expression for y which is the solution to the Kummer equation (3.1). As before, the point transformations found above can also be used to find the conserved quantities of the Kummer equation.

3.2. The Combined Gravity-Inertial-Rossby Wave Equation

The combined gravity-inertial-Rossby wave equation is given by

$$y'' + g(x)y = 0, \tag{3.18}$$

where $g(x)$ is an arbitrary function of x. The derivation of this equation is outlined in MCkenzie [7]. Very briefly, the governing equations for the combined gravity-inertial-Rossby waves on a β-plane reduce to a partial differential equation, which, with Fourier plane wave analysis, becomes a second-order ordinary differential equation describing the latitudinal structure of the perturbations. In (3.18), x and y are local Cartesian coordinates and $g(x)$ is the wave number, see MCkenzie [7]. By inspection, we find that

$$L = \frac{1}{2}y'^2 - \frac{1}{2}g(x)y^2 \tag{3.19}$$

is a Lagrangian for (3.18). As for the Kummer equation and its corresponding Lagrangian, it can be shown that (3.18) with Lagrangian (3.19) has an eight-dimensional Lie symmetry algebra and a five-dimensional Noether algebra. Therefore, this equation can be mapped to (2.14) using the method of equivalent Lagrangians. We follow the same procedure as for the Kummer equation in the previous section, with our aim being to find the regular point transformations $X = X(x, y)$ and $Y = Y(x, y)$ that map (3.18) to (2.14).

As before, we begin by substituting expressions for L and $\overline{L}$ (given in (3.19) and (2.13), resp.) into (2.10). This gives the equation

$$\frac{1}{2}y'^2 - \frac{1}{2}g(x)y^2 = \frac{1}{2}\left(\frac{Y_x + 2Y_xY_yy' + y'^2Y_y^2}{X_x + y'X_y}\right) + f_x + y'f_y. \tag{3.20}$$

Again we assume that X is of the form $X = \int a^2 dx$, where $a = a(x)$, which simplifies the above equation to

$$\frac{1}{2}y'^2 - \frac{1}{2}g(x)y^2 = \frac{1}{2}\left(\frac{Y_x^2 + 2Y_xY_yy' + y'^2Y_y^2}{a^2}\right) + f_x + y'f_y. \tag{3.21}$$

Separating by powers of y', we obtain the following system of three equations:

$$\frac{1}{2} = \frac{1}{2}\frac{Y_y^2}{a^2}, \tag{3.22}$$

$$0 = \frac{Y_xY_y}{a^2} + f_y, \tag{3.23}$$

$$-\frac{1}{2}g(x)y^2 = \frac{1}{2}\frac{Y_x^2}{a^2} + f_x. \tag{3.24}$$

From (3.22), we deduce that

$$Y = ay + b(x), \tag{3.25}$$

where we can assume that $b(x) = 0$. Substituting expressions for Y_y and Y_x into (3.23), and then integrating with respect to y, we have that

$$f = -\frac{1}{2}\frac{\dot{a}}{a}y^2 + c(x), \tag{3.26}$$

for which we again assume that $c(x) = 0$. Finally, after substituting expressions for Y_x and f_x into (3.24), and making the substitution $A = \dot{a}/a$, we obtain the equation

$$-\frac{1}{2}g(x)y^2 = \frac{1}{2}A^2y^2 - \frac{1}{2}\dot{A}y^2, \tag{3.27}$$

which simplifies to

$$0 = \dot{A} - A^2 - g(x). \tag{3.28}$$

Thus, as before, $a = e^{\int A dx}$, where A satisfies (3.28). Hence, we have that our regular point transformations $X = X(x, y)$ and $Y = Y(x, y)$, which transform (3.18) to (2.14), are given by

$$X = \int e^{2\int A dx}\, dx, \qquad Y = e^{\int A dx} y. \tag{3.29}$$

4. Applications to PDEs

We now study the application of the method to some classes of partial differential equations (PDEs) in two independent variables. We first demonstrate that given a Lagrangian, L, and a known transformation, one can construct an equivalent Lagrangian $\overline{L}$. Following this, we turn our attention to the construction of a standard form for the Lagrangian equivalent to the usual Lagrangian of the standard wave equation. This will enable us to apply the method to partial differential equations whose Lagrangians are known to be equivalent to that of the standard wave equation. In this latter situation, the aim of the method is to construct a transformation that maps one Lagrangian, L, to its equivalent $\overline{L}$.

4.1. Illustrative Example 1

In the first example, we use a given Lagrangian $\overline{L}$ and a given transformation, $X = X(x, t, u)$, $T = T(x, t, u)$, and $U = U(x, t, u)$, in order to construct an equivalent Lagrangian L.

Consider the (1+1) wave equation with unit wave speed,

$$U_{TT} - U_{XX} = 0. \tag{4.1}$$

Equation (4.1) is known to have the Lagrangian

$$\overline{L} = \frac{1}{2}\left(U_T^2 - U_X^2\right). \tag{4.2}$$

Suppose we are given the transformation

$$X = t + x, \qquad T = t - x, \qquad U = u, \tag{4.3}$$

which is the standard transformation to canonical form, see Kara [3]. By making the correct substitutions into (2.9), we can calculate L.

Firstly, the determinant of the Jacobean matrix, J, is given by

$$J = \begin{vmatrix} \dfrac{dX}{dx} & \dfrac{dX}{dt} \\ \dfrac{dT}{dx} & \dfrac{dT}{dt} \end{vmatrix}, \tag{4.4}$$

for two independent variables x and t, see Kara [3].

The Lagrangian L is a function of the variables x, t and u, where $u = u(x,t)$. Hence using our canonical transformation above, we have that $J = 2$.

It follows that

$$L = \frac{1}{2}\left(U_T^2 - U_X^2\right) \cdot 2. \tag{4.5}$$

In order to find U_T and U_X, we note

$$\frac{dU}{dt} = \frac{\partial T}{\partial t}\frac{\partial U}{\partial T} + \frac{\partial X}{\partial t}\frac{\partial U}{\partial X}, \qquad \frac{dU}{dx} = \frac{\partial T}{\partial x}\frac{\partial U}{\partial T} + \frac{\partial X}{\partial x}\frac{\partial U}{\partial X}. \tag{4.6}$$

Using our canonical transformation, $X = X(x,t,u)$, $T = T(x,t,u)$, and $U = U(x,t,u)$, we have the equations $u_t = U_T + U_X$ and $u_x = -U_T + U_X$. Solving these simultaneously, we get that

$$U_T = \frac{1}{2}(u_t - u_x), \qquad U_X = \frac{1}{2}(u_t + u_x). \tag{4.7}$$

These, expressions into (4.5) yield

$$L = \frac{1}{4}\left(u_t^2 - 2u_x u_t + u_x^2\right) - \frac{1}{4}\left(u_t^2 + 2u_x u_t + u_x^2\right) = -u_x u_t. \tag{4.8}$$

Hence, L is equivalent to $\overline{L}$ in the sense of *Definition 2.1*. The Euler-Lagrange equation associated with L is

$$u_{xt} = 0, \tag{4.9}$$

which is the canonical form of the wave equation given in (4.1).

4.2. Illustrative Example 2

In the previous example, we made use of a canonical transformation in order to find a Lagrangian equivalent to $\overline{L}$. In this example, however, transformed variables are concluded as a consequence of the underlying symmetry structure from which an equivalent Lagrangian is constructed.

It can be verified that

$$\overline{G}(X,T,U) = \frac{\partial}{\partial T} \tag{4.10}$$

is a Noether point symmetry generator for the Lagrangian $\overline{L}$ given by (4.2). Suppose we wish to map $\overline{G}$ to the dilation symmetry generator

$$G(x,t,u) = x\frac{\partial}{\partial x} + \frac{1}{2}u\frac{\partial}{\partial u}. \tag{4.11}$$

Once this mapping is found, it can be used in formula (2.9) to determine L. The formula for change of variables is given by

$$\overline{G}(X,T,U) = G(X)\frac{\partial}{\partial X} + G(T)\frac{\partial}{\partial T} + G(U)\frac{\partial}{\partial U}. \tag{4.12}$$

Substituting the relevant values into (4.12), we obtain the three equations:

$$G(X) = 0, \qquad G(T) = 1, \qquad G(U) = 0. \tag{4.13}$$

We solve these equations using the method of invariants, for which we get that

$$X = \mathcal{F}\left(t, \frac{u^2}{x}\right), \qquad T = \ln x + \mathcal{G}\left(t, \frac{u^2}{x}\right), \qquad U = \mathcal{H}\left(t, \frac{u^2}{x}\right), \tag{4.14}$$

where $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ are arbitrary functions. As an illustration, we choose

$$X = t, \qquad T = \ln x, \qquad U = \frac{u^2}{x}. \tag{4.15}$$

This gives us our transformation. From (4.4), $J = -1/x$. Thus, $U_X = 2uu_t/x$ and $U_T = 2uu_x - (u^2/x)$ so that, by (2.9), we get

$$L = \frac{1}{2}\left(U_T^2 - U_X^2\right).\left(-\frac{1}{x}\right) = \frac{-1}{2x^3}\left(u^2\left(4u_x^2x^2 - 4uu_xx + u^2 - 4u_t^2\right)\right). \tag{4.16}$$

4.3. *Equivalent Lagrangian for the Wave Equation in (1 + 1) Dimension*

We now find an expression for the form of a Lagrangian, L, which is equivalent to the usual Lagrangian of the wave equation, $\overline{L}$. Once we have this form, given any L equivalent to $\overline{L}$, we can find the transformation that maps L to $\overline{L}$, and hence the solutions and conserved quantities of the differential equation associated with L to those of the standard wave equation.

Since

$$U_T = \frac{(dU/dt)X_x - X_t(dU/dx)}{T_t X_x - X_t T_x}, \qquad U_X = \frac{T_t(dU/dx) - T_x(dU/dt)}{T_t X_x - X_t T_x}, \tag{4.17}$$

we get

$$\begin{aligned} &\frac{1}{2}\left(U_T^2 - U_X^2\right) \\ &\quad = \frac{1}{2}\left(\frac{\left(dU^2/dt\right)(X_x^2 - T_x^2) + \left(dU^2/dx\right)(X_t^2 - T_t^2) + 2(dU/dx)(dU/dt)(T_t T_x - X_t X_x)}{T_t^2 X_x^2 - 2T_t T_x X_t X_x + X_t^2 T_x^2}\right). \end{aligned} \tag{4.18}$$

Here, $J = X_x(T_t + u_t T_u) - X_t(T_x + u_x T_u) + X_u(u_x T_t - u_t T_x)$ so that

$$L = \frac{1}{2}\left(U_T^2 - U_X^2\right) J[X_x(T_t + u_t T_u) - X_t(T_x + u_x T_u) + X_u(u_x T_t - u_t T_x)]. \tag{4.19}$$

It can be shown that $X_u = 0$ and $T_u = 0$ (i.e., $X = X(x,t)$ and $T = T(x,t)$). Then, the above expression for the Lagrangian reduces to

$$\begin{aligned} L &= \frac{1}{2}\left[\frac{\left(dU^2/dt\right)(X_x^2 - T_x^2) + \left(dU^2/dx\right)(X_t^2 - T_t^2) + 2(dU/dx)(dU/dt)(T_t T_x - X_t X_x)}{(X_x T_t - X_t T_x)^2}\right] \\ &\quad \times (X_x T_t - X_t T_x) \\ &= \frac{1}{2}\left[\frac{\left(dU^2/dt\right)(X_x^2 - T_x^2) + \left(dU^2/dx\right)(X_t^2 - T_t^2) + 2(dU/dx)(dU/dt)(T_t T_x - X_t X_x)}{X_x T_t - X_t T_x}\right]. \end{aligned} \tag{4.20}$$

With $dU/dt = U_t + u_t U_u$ and $dU/dx = U_x + u_x U_u$,

$$\begin{aligned} L = \frac{1}{2(X_x T_t - X_t T_x)} &\Big[(U_t + u_t U_u)^2\left(X_x^2 - T_x^2\right) + (U_x + u_x U_u)^2\left(X_t^2 - T_t^2\right) \\ &+ 2(U_x + u_x U_u)(U_t + u_t U_u)(T_t T_x - X_t X_x)\Big] \end{aligned}$$

$$
\begin{aligned}
= \left[\frac{1}{2(X_xT_t - X_tT_x)}\right]\Big\{ & u_xu_t\left[2U_u^2(T_tT_x - X_tX_x)\right] \\
& + u_x^2\left[U_u^2\left(X_t^2 - T_t^2\right)\right] + u_t^2\left[U_u^2\left(X_x^2 - T_x^2\right)\right] \\
& + u_x\left[2U_uU_x\left(X_t^2 - T_t^2\right) + 2U_tU_u(T_tT_x - X_tX_x)\right] \\
& + u_t\left[2U_tU_u\left(X_x^2 - T_x^2\right) + 2U_uU_x(T_tT_x - X_tX_x)\right] \\
& + \left[U_t^2\left(X_x^2 - T_x^2\right) + U_x^2\left(X_t^2 - T_t^2\right) + 2U_tU_x(T_tT_x - X_tX_x)\right]\Big\}.
\end{aligned}
\tag{4.21}
$$

This is the general form for a Lagrangian equivalent to the Lagrangian of the wave equation, where $U = U(x,t,u)$, $X = X(x,t)$, and $T = T(x,t)$. Once we have a Lagrangian which we know to be equivalent to the Lagrangian $\overline{L}$ given by (4.2), we can reverse the process of the examples above and use the form of the Lagrangian in the previous equation in order to find the transformations that map the solutions of the standard wave equation (4.1) to the solutions of the equivalent Euler differential equation.

4.3.1. Finding Transformations: Example 1

Consider (4.9) with its Lagrangian

$$
L = -u_xu_t, \tag{4.22}
$$

which we found to be equivalent to (4.2). We use the form of the equivalent Lagrangian given above in order to find the transformations that map this Lagrangian to the Lagrangian (4.2). Substituting (4.22) for L and separating by monomials, we arrive at the equations:

$$
\begin{aligned}
u_tu_x &: -1 = \frac{2U_u^2(T_tT_x - X_tX_x)}{2(X_xT_t - X_tT_x)}, \\
u_x^2 &: 0 = \frac{U_u^2(X_t^2 - T_t^2)}{2(X_xT_t - X_tT_x)}, \\
u_t^2 &: 0 = \frac{U_u^2(X_x^2 - T_x^2)}{2(X_xT_t - X_tT_x)}, \\
u_x &: 0 = \frac{2U_uU_x(X_t^2 - T_t^2) + 2U_tU_u(T_tT_x - X_tX_x)}{2(X_xT_t - X_tT_x)}, \\
u_t &: 0 = \frac{2U_tU_u(X_x^2 - T_x^2) + 2U_uU_x(T_tT_x - X_tX_x)}{2(X_xT_t - X_tT_x)}, \\
1 &: 0 = \frac{U_t^2(X_x^2 - T_x^2) + U_x^2(X_t^2 - T_t^2) + 2U_tU_x(T_tT_x - X_tX_x)}{2(X_xT_t - X_tT_x)}.
\end{aligned}
\tag{4.23}
$$

Using software to solve this overdetermined system of equations, we get the transformation

$$X = f(x) + g(t), \qquad T = f(x) - g(t), \qquad U = u, \tag{4.24}$$

for f and g arbitrary functions of x and t, respectively. The well-known transformation $X = t + x$, $T = t - x$, $U = u$, used in (4.1), is in fact a special case.

4.3.2. Finding Transformations: Example 2

Utilising the Lagrangian

$$L = \frac{-1}{2x^3}\left[u^2\left(4u_x^2x^2 - 4uu_x x + u^2 - 4u_t^2\right)\right], \tag{4.25}$$

which was constructed to be equivalent to the Lagrangian $\overline{L}$; this was done in Subsection 4.2 above, the procedure yields $X = f(t + \ln x) - g(t - \ln x)$, $T = f(t + \ln x) + g(t - \ln x)$, and $U = \pm u^2/x$, where f and g are arbitrary functions and x and t as well. If we choose f such that $f(t + \ln x) = (1/2)(t + \ln x)$ and g such that $g(t - \ln x) = -(1/2)(t - \ln x)$, we have the transformation as the one used in Example 2 of the previous section that resulted in the Lagrangian (4.25).

4.4. The Equivalence of the Dissipative Wave and Klein-Gordon Equations

The equation

$$U_{TT} + U_T - U_{XX} = 0, \tag{4.26}$$

is the one-dimensional case of the linear wave equation with dissipation (see [8]). This equation has the well-known Lagrangian, after multiplication by a variational factor,

$$\overline{L} = \frac{1}{2}e^T\left(U_T^2 - U_X^2\right). \tag{4.27}$$

We map $\overline{L}$ to the Lagrangian

$$L = \frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 - \frac{1}{2}uu_t + \frac{1}{8}u^2, \tag{4.28}$$

giving rise to the Euler equation:

$$u_{tt} - u_{xx} - \frac{1}{4}u = 0, \tag{4.29}$$

which we note to be a Klein-Gordon equation. The Noether symmetries $X = \xi\partial_x + \tau\partial_t + \phi\partial_u$ with gauge (f, g) satisfy

$$\begin{aligned}
0 = {}& \frac{u\phi}{4} - \frac{\phi u_t}{2} + \frac{1}{2}uu_x\xi_t - u_t u_x \xi_t + \frac{1}{2}uu_t u_x \xi_u - u_t^2 u_x \xi_u + u_x^3 \xi_u \\
&+ u_x^2 \xi_x + \frac{1}{2}uu_t\tau_t - u_t^2\tau_t + \frac{1}{2}uu_t^2\tau_u - u_t^3\tau_u \\
&+ u_t u_x^2 \tau_u + u_t u_x \tau_x - \frac{u\phi_t}{2} + u_t\phi_t - \frac{1}{2}uu_t\phi_u + u_t^2\phi_u - u_x^2\phi_u - u_x\phi_x \\
&+ \left(\frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 - \frac{1}{2}uu_t + \frac{1}{8}u^2\right)(\tau_t + u_t\tau_u + \xi_x + u_x\xi_u) - \left(f_t + u_t f_u + g_x + u_x g_u\right),
\end{aligned} \tag{4.30}$$

which separates into an overdetermined system of partial differential equations whose solution is

$$\begin{aligned}
&X_1 = \partial_t, \qquad f = 0, \qquad g = 0, \\
&X_2 = \partial_x, \qquad f = 0, \qquad g = 0, \\
&X_3 = t\partial_x + x\partial_t, \qquad f = 0, \qquad g = \frac{1}{4}u^2, \\
&X_\infty = F(x,t)\partial_u, \qquad f = -\frac{1}{2}uF + uF_t, \qquad g = -uF_x,
\end{aligned} \tag{4.31}$$

where F satisfies $(1/4)F + F_{xx} - F_{tt} = 0$. The Lie algebra is isomorphic to the Noether algebra corresponding to the Lagrangian $\overline{L}$ (see [9]). Hence, L and $\overline{L}$ are equivalent Lagrangians.

We can therefore use (2.9) in order to find the transformations $X = X(x, t, u)$, $T = T(x, t, u)$ and $U = U(x, t, u)$ that map L to $\overline{L}$.

Assuming that $X_u = 0$ and $T_u = 0$ as before, we get

$$\begin{aligned}
L = {}& \left[\frac{e^T}{2(X_x T_t - X_t T_x)}\right]\Big\{u_x u_t\left[2U_u^2(T_t T_x - X_t X_x)\right] \\
&+ u_x^2\left[U_u^2\left(X_t^2 - T_t^2\right)\right] + u_t^2\left[U_u^2\left(X_x^2 - T_x^2\right)\right] \\
&+ u_x\left[2U_u U_x\left(X_t^2 - T_t^2\right) + 2U_t U_u(T_t T_x - X_t X_x)\right] \\
&+ u_t\left[2U_t U_u\left(X_x^2 - T_x^2\right) + 2U_u U_x(T_t T_x - X_t X_x)\right] \\
&+\left[U_t^2\left(X_x^2 - T_x^2\right) + U_x^2\left(X_t^2 - T_t^2\right) + 2U_t U_x(T_t T_x - X_t X_x)\right]\Big\}.
\end{aligned} \tag{4.32}$$

Substituting for L and then separating by derivative terms, we arrive at the system

$$
\begin{aligned}
u_t u_x : 0 &= \frac{2U_u^2 e^T (T_t T_x - X_t X_x)}{2(X_x T_t - X_t T_x)}, \\
u_x^2 : -\frac{1}{2} &= \frac{U_u^2 e^T (X_t^2 - T_t^2)}{2(X_x T_t - X_t T_x)}, \\
u_t^2 : \frac{1}{2} &= \frac{U_u^2 e^T (X_x^2 - T_x^2)}{2(X_x T_t - X_t T_x)}, \\
u_x : 0 &= \frac{e^T \left[2U_u U_x (X_t^2 - T_t^2) + 2U_t U_u (T_t T_x - X_t X_x)\right]}{2(X_x T_t - X_t T_x)}, \\
u_t : -\frac{1}{2}u &= \frac{e^T \left[2U_t U_u (X_x^2 - T_x^2) + 2U_u U_x (T_t T_x - X_t X_x)\right]}{2(X_x T_t - X_t T_x)}, \\
1 : \frac{1}{8}u^2 &= \frac{e^T \left[U_t^2 (X_x^2 - T_x^2) + U_x^2 (X_t^2 - T_t^2) + 2U_t U_x (T_t T_x - X_t X_x)\right]}{2(X_x T_t - X_t T_x)}.
\end{aligned}
\tag{4.33}
$$

For this overdetermined system of equations, the software yields the result

$$
\begin{aligned}
X &= f_1(t+x) - f_2(t-x), \\
T &= f_1(t+x) + f_2(t-x), \\
U &= -e^{-(1/2)t} u.
\end{aligned}
\tag{4.34}
$$

Notes

(1) the special case $f_1(t+x) = (1/2)(t+x)$, $f_2(t-x) = (1/2)(t-x)$, from which we get that the transformation

$$
X = x, \qquad T = t, \qquad U = -e^{-(1/2)t} u \tag{4.35}
$$

is a known transformation mapping (4.26) to (4.29), see Ibragimov [8]. The transformations also transform the Noether symmetries of $\overline{L}$ to those of L;

(2) this result is true for n-dimensional case.

4.5. Equivalent Lagrangians and Wave Equations on Spacetime Manifolds

The three-dimensional linear wave equation is the well-known second-order PDE:

$$
U_{TT} - U_{XX} - U_{YY} - U_{ZZ} = 0, \tag{4.36}
$$

see [8]. Here, T, X, Y, and Z are the independent variables, and $U = U(T, X, Y, Z)$ is the dependent variable. The usual Lagrangian for this equation is

$$\overline{L} = \frac{1}{2}U_T^2 - \frac{1}{2}U_X^2 - \frac{1}{2}U_Y^2 - \frac{1}{2}U_Z^2, \tag{4.37}$$

[8]. It is well known that the Lagrangian for the three-dimensional wave equation in spherical co-ordinates is given by

$$L = \frac{1}{2}x^2 \sin y\, u_t^2 - \frac{1}{2}\sin y\, x^2\; u_x^2 - \frac{1}{2}\sin y\, u_y^2 - \frac{1}{2\sin y}u_z^2. \tag{4.38}$$

L and $\overline{L}$ are naturally equivalent because they are Lagrangians of the same equation. However, it can also be verified that they generate isomorphic Noether algebras of point symmetries.

In this section, we apply the method of equivalent Lagrangians to L and $\overline{L}$, in an attempt to recover the transformation that maps $\overline{L}$ to L. We shall demonstrate that (2.9) of Definition 2.1 is satisfied by the transformation from Cartesian to polar coordinates, which is to be expected from our choice of Lagrangians.

As before, we make use of Definition 2.1, which relates two equivalent Lagrangians to each other by means of the transformation that maps one to the other. Substituting L and $\overline{L}$ into (2.9) gives us

$$\begin{aligned} &\frac{1}{2}x^2 \sin y\, u_t^2 - \frac{1}{2}\sin y\; x^2 u_x^2 - \frac{1}{2}\sin y\, u_y^2 - \frac{1}{2\sin y}\; u_z^2 \\ &\quad = \left(\frac{1}{2}U_T^2 - \frac{1}{2}U_X^2 - \frac{1}{2}U_Y^2 - \frac{1}{2}U_Z^2\right)J. \end{aligned} \tag{4.39}$$

We first calculate the Jacobian J. We assume that $T_x = T_y = T_z = X_t = Y_t = Z_t = 0$. In other words, $T = T(t)$, $X = X(x, y, z)$, $Y = Y(x, y, z)$, and $Z = Z(x, y, z)$. Furthermore, we assume that $T = t$. Hence $T_t = 1$. It follows that $J = X_x Y_y Z_z - X_x Y_z Z_y - X_y Y_x Z_z + X_y Y_z Z_x + X_z Y_x Z_y - X_z Y_y Z_x$. Then,

$$\begin{pmatrix} U_T \\ U_X \\ U_Y \\ U_Z \end{pmatrix} = \frac{1}{J}\begin{pmatrix} u_t\left(X_x Y_y Z_z - X_x Y_z Z_y - X_y Y_x Z_z + X_y Y_z Z_x + X_z Y_x Z_y - X_z Y_y Z_x\right) \\ u_z\left(Y_x Z_y - Y_y Z_x\right) - u_y\left(Y_x Z_z - Y_z Z_x\right) + u_x\left(Y_y Z_z - Y_z Z_y\right) \\ u_y\left(X_x Z_z - X_z Z_x\right) - u_z\left(X_x Z_y - X_y Z_x\right) - u_x\left(X_y Z_z - X_z Z_y\right) \\ u_z\left(X_x Y_y - X_y Y_x\right) - u_y\left(X_x Y_z - X_z Y_x\right) + u_x\left(X_y Y_z - X_z Y_y\right) \end{pmatrix}, \tag{4.40}$$

from which we can read the expressions for U_T, U_X, U_Y, and U_Z in terms of u_t, u_x, u_y, and u_z. Substituting these into (4.39), along with our expression for J, we have the equation

$$\begin{aligned}
&\frac{1}{2}x^2 \sin y\, u_t^2 - \frac{1}{2}\sin y\; x^2 u_x^2 - \frac{1}{2}\sin y\, u_y^2 - \frac{1}{2\sin y}\; u_z^2 \\
&\quad = \frac{1}{2}\left(-X_zY_yZ_x + X_yY_zZ_x + X_zY_xZ_y - X_xY_zZ_y - X_yY_xZ_z + X_xY_yZ_z\right) \\
&\qquad \times \left[u_t^2 - \frac{\left(u_zX_yY_x - u_yX_zY_x - u_zX_xY_y + u_xX_zY_y + u_yX_xY_z - u_xX_yY_z\right)^2}{\left(X_zY_yZ_x - X_yY_zZ_x - X_zY_xZ_y + X_xY_zZ_y + X_yY_xZ_z - X_xY_yZ_z\right)^2}\right. \\
&\qquad\quad - \frac{\left(u_zY_yZ_x - u_yY_zZ_x - u_zY_xZ_y + u_xY_zZ_y + u_yY_xZ_z - u_xY_yZ_z\right)^2}{\left(X_zY_yZ_x - X_yY_zZ_x - X_zY_xZ_y + X_xY_zZ_y + X_yY_xZ_z - X_xY_yZ_z\right)^2} \\
&\qquad\quad \left. - \frac{\left(u_zX_yZ_x - u_yX_zZ_x - u_zX_xZ_y + u_xX_zZ_y + u_yX_xZ_z - u_xX_yZ_z\right)^2}{\left(-X_zY_yZ_x + X_yY_zZ_x + X_zY_xZ_y - X_xY_zZ_y - X_yY_xZ_z + X_xY_yZ_z\right)^2}\right].
\end{aligned} \tag{4.41}$$

It can be shown, amongst others, a solution is given by $T = t$, $X = x\sin y\cos z$, $Y = x\sin y\sin z$, and $Z = x\cos y$.

This procedure is particularly useful in mapping variational equations, like the wave equation, between equivalent "curved manifolds."

5. Conclusion

In this paper, we have applied the notion of equivalent Lagrangians to determine transformations that map differential equations one to another in order to generate solutions, conservation laws, inter alia. An additional consequence of the procedure is recovering some well-known transformations like the mapping from the standard wave equation to the canonical form. The procedure, although cumbersome, holds for partial differential equations of any number of independent variables. Finally, transformation maps between variational equations (like the wave equation) on equivalent manifolds can be determined by the procedure.

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