Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 907951, 5 pages
doi:10.1155/2012/907951

Review Article

Quasi-Contractive Mappings in Modular Metric Spaces

Yeol J. E. Cho,[1] Reza Saadati,[2] and Ghadir Sadeghi[3]

[1] *Department of Mathematics Education and RINS, Gyeongsang National University, jinju 660-701, Republic of Korea*
[2] *Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran*
[3] *Department of Mathematics and Computer Sciences, Sabzevar Tarbiat Moallem University, Sabzevar, Iran*

Correspondence should be addressed to Reza Saadati, rsaadati@eml.cc

Received 6 October 2011; Accepted 30 November 2011

Academic Editor: Rudong Chen

We prove the existence of fixed point and uniqueness of quasi-contractive mappings in modular metric spaces which was introduced by Ćirić

1. Introduction and Preliminaries

In this paper, we prove the existence and uniqueness of fixed points of quasi-contractive mappings in modular metric spaces which develop the theory of metric spaces generated by modulars. Throughout the paper $\mathfrak{X}$ is a nonempty set and $\lambda > 0$. The notion of a metric modular was introduced by Chistyakov [1] as follows.

Definition 1.1. A function $\omega : (0,\infty) \times \mathfrak{X} \times \mathfrak{X} \to [0,\infty]$ is said to be a metric modular on $\mathfrak{X}$ (or, simply, a modular if no ambiguity arises) if it satisfies three axioms:

(i) for any $x, y \in \mathfrak{X}$, $\omega_\lambda(x, y) = 0$ for all $\lambda > 0$ if and only if $x = y$;

(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ for all $\lambda > 0$, and $x, y \in \mathfrak{X}$;

(iii) $\omega_{\lambda+\mu}(x, y) \le \omega_\lambda(x, z) + \omega_\mu(z, y)$ for all $\lambda, \mu > 0$ and $x, y \in \mathfrak{X}$.

Definition 1.2. Let $(\mathfrak{X}, \omega)$ be a metric modular space.

(1) A sequence $\{x_n\}$ in $\mathfrak{X}_\omega$ is said to be ω-convergent to a point $x \in \mathfrak{X}$ if, for all $\lambda > 0$,

$$\omega_\lambda(x_n, x) \longrightarrow 0 \tag{1.1}$$

as $n \to \infty$.

(2) A subset $\mathfrak{C}$ of $\mathfrak{X}_\omega$ is said to be ω-closed if the ω-limit of a ω-convergent sequence of $\mathfrak{C}$ always belongs to $\mathfrak{C}$.

(3) A subset $\mathfrak{C}$ of $\mathfrak{X}_\omega$ is said to be ω-complete if every ω-Cauchy sequence in $\mathfrak{C}$ is ω-convergent and its ω-limit is in $\mathfrak{C}$.

Definition 1.3. The metric modular ω is said to have the Fatou property if

$$\omega_\lambda(x, y) \leq \liminf_{n\to\infty} \omega(x_n, y) \tag{1.2}$$

for all $y \in \mathfrak{X}_\omega$ and $\lambda \in (0, \infty)$, where $\{x_n\}$ ω-converges to x.

2. Main Results

Definition 2.1. Let $(\mathfrak{X}, \omega)$ be a metric modular space, and let $\mathfrak{C}$ be a nonempty subset of $\mathfrak{X}_\omega$. The self-mapping $T : \mathfrak{C} \to \mathfrak{C}$ is said to be quasi-contraction if there exists $0 < k < 1$ such that

$$\omega_\lambda(T(x), T(y)) \leq k \max\{\omega_\lambda(x, y), \omega_\lambda(x, T(x)), \omega_\lambda(y, T(y)), \omega_\lambda(x, T(y)), \omega_\lambda(T(x), y)\} \tag{2.1}$$

for any $x, y \in \mathfrak{X}$ and $\lambda \in (0, \infty)$.

Let $T : \mathfrak{C} \to \mathfrak{C}$ be a mapping, and let $\mathfrak{C}$ be a nonempty subset of $\mathfrak{X}_\omega$. For any $x \in \mathfrak{C}$, define the orbit

$$\mathcal{O}(x) = \left\{x, T(x), T^2(x), \ldots\right\} \tag{2.2}$$

and its ω-diameter by

$$\delta_\omega(x) = \operatorname{diam}(\mathcal{O}(x)) = \sup\{\omega_\lambda(T^n(x), T^m(x)) : n, m \in \mathbb{N})\}. \tag{2.3}$$

Lemma 2.2. *Let $(\mathfrak{X}, \omega)$ be a metric modular space, and let $\mathfrak{C}$ be a nonempty subset of $\mathfrak{X}_\omega$. Let $T : \mathfrak{C} \to \mathfrak{C}$ be a quasi-contractive mapping, and let $x \in \mathfrak{C}$ be such that $\delta_\omega(x) < \infty$. Then, for any $n \geq 1$, one has*

$$\delta_\omega(T(x)) \leq k^n \delta_\omega(x), \tag{2.4}$$

where k is the constant associated with the mapping of T. Moreover, one has

$$\omega_\lambda(T^n(x), T^{n+m}(x)) \leq k^n \delta_\omega(x) \tag{2.5}$$

for any $n, m \geq 1$ *and* $\lambda \in (0, \infty)$.

Proof. For each $n, m \geq 1$, we have

$$\omega_\lambda(T^n(x), T^m(y)) \leq k \max\Big\{\omega_\lambda\Big(T^{n-1}(x), T^{m-1}(y)\Big), \omega_\lambda\Big(T^{n-1}(x), T^n(x)\Big), \omega_\lambda\Big(T^{m-1}(y), T^m(y)\Big), \omega_\lambda\Big(T^{n-1}(x), T^m(y)\Big), \omega_\lambda\Big(T^n(x), T^{m-1}(y)\Big)\Big\} \tag{2.6}$$

for any $x, y \in \mathfrak{C}$ and $\lambda \in (0, \infty)$. This obviously implies that

$$\delta_\omega(T^n(x)) \leq k\delta_\omega\Big(T^{n-1}(x)\Big) \tag{2.7}$$

for any $n \geq 1$. Hence, for any $n \geq 1$, we have

$$\delta_\omega(T^n(x)) \leq k^n \delta_\omega(x). \tag{2.8}$$

Moreover, for any $n, m \geq 1$, we have

$$\omega_\lambda(T^n(x), T^{n+m}(x) \leq \delta_\omega(T^n(x)) \leq k^n \delta_\omega(x). \tag{2.9}$$

This completes the proof. □

The next lemma is helpful to prove the main result in this paper.

Lemma 2.3. *Let* $(\mathfrak{X}, \omega)$ *be a modular metric space, and let* $\mathfrak{C}$ *be a* ω*-complete nonempty subset of* $\mathfrak{X}_\omega$*. Let* $T : \mathfrak{C} \to \mathfrak{C}$ *be quasi-contractive mapping, and let* $x \in \mathfrak{C}$ *be such that* $\delta_\omega(x) < \infty$*. Then* $\{T^n(x)\}$ ω*-converges to a point* $\nu \in \mathfrak{C}$*. Moreover, one has*

$$\omega_\lambda(T^n(\mathrm{X}) - \nu) \leq k^n \delta_\omega(x) \tag{2.10}$$

for all $n \geq 1$ *and* $\lambda \in (0, \infty)$.

Proof. From Lemma 2.2, we know that $\{T^n(x)\}$ is a ω-Cauchy sequence in $\mathfrak{C}$. Since $\mathfrak{C}$ is ω-complete, then there exists $\nu \in \mathfrak{C}$ such that $\{T^n(x)\}$ ω-converges to ν. Since

$$\omega_\lambda(T^n(x), T^{n+m}(x)) \leq k^n \delta_\omega(x) \tag{2.11}$$

for any $n, m \geq 1$ and ω satisfies the Fatou property, and letting $m \to \infty$, we have

$$\omega_\lambda(T^n(x), \nu) \leq \liminf_{m \to \infty} \omega_\lambda(T^n(x), T^{n+m}(x)) \leq k^n \delta_\omega(x). \tag{2.12}$$

This completes the proof. □

Next, we prove that ν is, in fact, a fixed point of T and it is unique provided some extra assumptions.

Theorem 2.4. *Let $T, \mathfrak{C}$, and x be as in Lemma 2.3. Suppose that $\omega_\lambda(\nu, T(\nu)) < \infty$ and $\omega_\lambda(x, T(x)) < \infty$ for all $\lambda \in (0, \infty)$. Then the ω-limit of $\{T^n(x)\}$ is a fixed point of T, that is, $T(\nu) = \nu$. Moreover, if ν^* is any fixed point of T in $\mathfrak{C}$ such that $\omega_\lambda(\nu, \nu^*) < \infty$ for all $\lambda \in (0, \infty)$, then one has $\nu = \nu^*$.*

Proof. We have

$$\omega_\lambda(T(x), T(\nu)) \leq k \max\{\omega_\lambda(x, \nu), \omega_\lambda(x, T(x)), \omega_\lambda(\nu, T(\nu)), \omega_\lambda(x, T(\nu)), \omega_\lambda(T(x), \nu)\}. \tag{2.13}$$

From Lemma 2.3, it follows that

$$\omega_\lambda(T(x), T(\nu)) \leq k \max\{\delta_\omega(x), \omega_\lambda(\nu, T(\nu)), \omega_\lambda(x, T(\nu))\}. \tag{2.14}$$

Suppose that, for each $n \geq 1$,

$$\omega_\lambda(\mathrm{T}^n(x), T(\nu)) \leq \max\{k^n \delta_\omega(x), k\omega_\lambda(\nu, T(\nu)), k^n \omega_\lambda(x, T(\nu))\}. \tag{2.15}$$

Then we have

$$\begin{aligned}
&\omega_\lambda\left(T^{n+1}(x), T(\nu)\right) \\
&\quad \leq k \max\left\{\omega_\lambda(T^n(x), \nu), \omega_\lambda\left(T^n(x), T^{n+1}(x)\right), \omega_\lambda(\nu, T(\nu)), \omega_\lambda(T^n(x), T(\nu)), \omega_\lambda\left(T^{n+1}(x), \nu\right)\right\}.
\end{aligned} \tag{2.16}$$

Hence we have

$$\omega_\lambda\left(T^{n+1}(x), T(\nu)\right) \leq k \max\{k^n \delta_\omega(x), k\omega_\lambda(\nu, T(\nu)), \omega_\lambda(T^n(x), T(\nu))\}. \tag{2.17}$$

Using our previous assumption, we get

$$\omega_\lambda\left(T^{n+1}(x), T(\nu)\right) \leq \max\left\{k^{n+1} \delta_\omega(x), k\omega_\lambda(\nu, T(\nu)), k^{n+1} \omega_\lambda(x, T(\nu))\right\}. \tag{2.18}$$

Thus, by induction, we have

$$\omega_\lambda(T^n(x), T(\nu)) \leq \max\{k^n \delta_\omega(x), k\omega_\lambda(\nu, T(\nu)), k^n \omega_\lambda(x, T(\nu))\} \tag{2.19}$$

for any $n \geq 1$ and $\lambda \in (0, \infty)$. Therefore, we have

$$\limsup_{n \to \infty} \omega_\lambda(T^n(x), T(x)) \leq \omega(\nu, T(\nu)) \tag{2.20}$$

for all $\lambda \in (0, \infty)$. Using the Fatou property for the metric modular ω, we get

$$\omega_\lambda(\nu, T(\nu)) \liminf_{n \to \infty} \omega_\lambda(T^n(x), T(\nu)) \leq k\omega(\nu, T(\nu)) \tag{2.21}$$

for all $\lambda \in (0, \infty)$. Since $k < 1$, we get $\omega_\lambda(\nu, T(\nu)) = 0$ for all $\lambda \in (0, \infty)$, and so $T(\nu) = \nu$.

Let ν^* be another fixed point of T such that $\omega_\lambda(\nu, \nu^*) < \infty$ for all $\lambda \in (0, \infty)$. Then we have

$$\omega_\lambda(\nu, \nu^*) = \omega_\lambda(T(\nu), T(\nu^*)) \leq k\omega_\lambda(\nu, \nu^*), \tag{2.22}$$

which implies that

$$\omega_\lambda(\nu, \nu^*) = 0 \tag{2.23}$$

for all $\lambda \in (0, \infty)$. Hence $\nu = \nu^*$. This complete the proof. □

Acknowledgments

The authors are thankful to the anonymous referees and the area editor Professor Rudong Chen for their critical remarks which helped greatly to improve the presentation of this paper. The first author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (Grant no. 2011-0021821).

References

[1] V. V. Chistyakov, "Modular metric spaces I: basic concepts," *Nonlinear Analysis*, vol. 74, pp. 1–14, 2010.