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Research Article

Some Notes on the Poincaré-Bertrand Formula

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The aim of this present paper is to establish the Poincaré-Bertrand formula for the double-layer potential on piecewise Lyapunov curve of integration.

1. Introduction

For repeated singular integrals, the celebrated Poincaré-Bertrand formula of Hardy [1] and Poincaré [2] plays a fundamentally important role in the theory of one-dimensional singular integral equations:

$$\frac{1}{\pi i}\int_{\Gamma_\tau}\frac{d\tau}{\tau-t}\cdot\frac{1}{\pi i}\int_{\Gamma_{\tau_1}}\frac{f(\tau,\tau_1)}{\tau_1-\tau}d\tau_1 = f(t,t)+\frac{1}{\pi i}\int_{\Gamma_{\tau_1}}d\tau_1\cdot\frac{1}{\pi i}\int_{\Gamma_\tau}\frac{f(\tau,\tau_1)}{(\tau-t)(\tau_1-\tau)}d\tau, \tag{1.1}$$

where Γ is a smooth curve in $\mathbb{R}^2$, t is a fixed point on Γ, and f lies on some appropriate function space.

There have been different extensions of this Poincaré-Bertrand formula for problems with different backgrounds. For example, Mitelman and Shapiro [3] established a Poincaré-Bertrand formula for quaternion singular integrals of Cauchy type over a smooth Lyapunov surface, Kytmanov [4] has an extension for the Bochner-Martinelli integral over a smooth manifolds.

Generalizations of the Poincaré-Bertrand theorem has been the subject of research in a number of papers (see [5, 6]). Important applications of the Poincaré-Bertrand theorem to nuclear physics, transport theory, condensed matter physics have been established by Davies et al. [6, 7].

Another important extension has been achieved by Hang and Jiang [8] on a smooth hypersurfaces in higher dimensions and for more recent references under different contexts, see for instance [9–14].

The Poincaré-Bertrand formula concerning two repeated integral operators of the double-layer potential on a piecewise Lyapunov curve in the plane is not generally known. Indeed, the author has been unable to find any explicit reference to such a result at all.

In this paper, we obtain the Poincaré-Bertrand formula for the double layer potential on a piecewise Lyapunov curve in the plane.

2. Preliminary Material

In this section, we provide some well-known facts from classical complex analysis to be used in this paper. For more information, we refer the reader to [15, 16].

2.1. The Cauchy-Type Integrals

We will denote by γ a closed curve in the complex plane $\mathbb{C}$ which contains a finite number of conical points. If the complement (in γ) of the union of conical points is a Lyapunov curve, then we shall refer to γ as piecewise Lyapunov curve in $\mathbb{C}$. Suppose that a domain Ω with boundary γ is given in the complex plane $\mathbb{C}$. Let Ω^+ and Ω^- be, respectively, the interior and exterior domains bounded by γ. Suppose that f is a continuous complex-valued function on γ.

The Cauchy-type integral of f will be denoted by $K[f]$ and defined by

$$K[f](z) := \frac{1}{2\pi i}\int_{\gamma}\frac{f(\zeta)}{\zeta - z}\,d\zeta, \quad z \notin \gamma. \tag{2.1}$$

We now define the singular Cauchy-type integral of f as

$$S[f](t) := \frac{1}{\pi i}\int_{\gamma}\frac{f(\zeta)}{\zeta - t}\,d\zeta \equiv \frac{1}{\pi i}\lim_{\varepsilon\to 0}\int_{\{\zeta\in\gamma:|\zeta - t|\geq\varepsilon\}}\frac{f(\zeta)}{\zeta - t}\,d\zeta, \quad t \in \gamma. \tag{2.2}$$

We say that f is of Hölder class with exponent λ, denoted by $H_\lambda(\gamma)$, where $0 < \lambda \leq 1$, if there exist a constant $c > 0$ such that

$$\left|f(z_1) - f(z_2)\right| \leq c|z_1 - z_2|^{\lambda}, \quad z_1, z_2 \in \gamma. \tag{2.3}$$

The following theorem gives the classical Sokhotski-Plemelj formulae.

Theorem 2.1 (see [16]). *Let Ω be a bounded domain in $\mathbb{C}$ with a piecewise Lyapunov boundary, and let $f \in H_\lambda(\gamma)$, $0 < \lambda < 1$. Then the following limits exist:*

$$\lim_{\Omega^{\pm}\ni z\to t\in\gamma} K[f](z) =: K^{\pm}[f](t), \tag{2.4}$$

and moreover, the following equalities hold:

$$K^{+}[f](t)=\frac{1}{2}\left(\widetilde{S}[f](t)+f(t)\right),\quad t\in\gamma, \tag{2.5}$$

$$K^{-}[f](t)=\frac{1}{2}\left(\widetilde{S}[f](t)-f(t)\right),\quad t\in\gamma, \tag{2.6}$$

here,

$$\widetilde{S}[f](t):=\frac{\pi-\alpha(t)}{\pi}f(t)+S[f](t) \tag{2.7}$$

is the modified Cauchy singular integral, and $\alpha(t)$ is the angle between the one-sided tangents at point t.

As was proved in Privalov's book, page 199 [17] and in the article by Alekseev [18] and also in [19], the limit values $K^{\pm}[f](t)$ of the Cauchy-type integral satisfy a Hölder condition. Thus, we have that the modified singular integral operator $\widetilde{S}$ in (2.7) acting invariantly on H_λ, $0<\lambda<1$ and we have

$$\widetilde{S}^2=I, \tag{2.8}$$

where I is the identity operator.

Adding equalities (2.5) and (2.6), and subtracting them from each other, we arrive at the formulas:

$$\begin{gathered}K^{+}[f](t)-K^{-}[f](t)=f(t),\qquad t\in\gamma,\\ K^{+}[f](t)+K^{-}[f](t)=\widetilde{S}[f](t),\qquad t\in\gamma.\end{gathered} \tag{2.9}$$

Theorem 2.2 (follows from Lemma 4.3 of [10]). *Let γ be a piecewise Lyapunov curve. If $t,\zeta_1\in\gamma$, $t\neq\zeta_1$, then*

$$\int_\gamma \frac{d\zeta}{(\zeta-t)(\zeta_1-\zeta)}=0. \tag{2.10}$$

In two repeated Cauchy's principal integrals over piecewise Lyapunov curves, the order of integration can be changed according to the following Poincaré-Bertrand formula (see, e.g., [12]) for all $t\in\gamma$:

$$\frac{1}{2\pi i}\int_{\gamma_\zeta}\frac{d\zeta}{\zeta-t}\cdot\frac{1}{2\pi i}\int_{\gamma_{\zeta_1}}\frac{f(\zeta,\zeta_1)}{\zeta_1-\zeta}d\zeta_1=\frac{1}{2\pi i}\int_{\gamma_{\zeta_1}}d\zeta_1\cdot\frac{1}{2\pi i}\int_{\gamma_\zeta}\frac{f(\zeta,\zeta_1)}{(\zeta-t)(\zeta_1-\zeta)}d\zeta+\left(\frac{\alpha(t)}{2\pi}\right)^2 f(t,t), \tag{2.11}$$

where the integrals being understood in the sense of the Cauchy principal value, $\alpha(t)$ is the angle between the one-sided tangents at the point t, and f lies on some appropriate function space. Noting that

$$\frac{1}{(\zeta - t)(\zeta_1 - \zeta)} = \frac{1}{\zeta_1 - t}\left[\frac{1}{\zeta - t} - \frac{1}{\zeta - \zeta_1}\right], \tag{2.12}$$

we find that the formula (2.11) can be presented in the form:

$$\begin{aligned} &\frac{1}{\pi i}\int_{\gamma_\zeta} \frac{d\zeta}{\zeta - t} \cdot \frac{1}{\pi i}\int_{\gamma_{\zeta_1}} \frac{f(\zeta,\zeta_1)}{\zeta_1 - \zeta}\, d\zeta_1 \\ &\quad = \eta^2(t) f(t,t) + \frac{1}{\pi i}\int_{\gamma_{\zeta_1}} \frac{d\zeta_1}{\zeta_1 - t}\frac{1}{\pi i}\left[\int_{\gamma_\zeta} \frac{f(\zeta,\zeta_1)}{\zeta - t}\, d\zeta - \int_{\gamma_\zeta} \frac{f(\zeta,\zeta_1)}{\zeta - \zeta_1}\, d\zeta\right], \end{aligned} \tag{2.13}$$

where $\eta(t) := \alpha(t)/\pi$.

Remark 2.3. It is possible, and indeed desirable, to consider the analogous formulas in other spaces than the Hölder space, for example, the Banach space $L_p(\gamma), p > 1$. If $f \in L_p(\gamma)$, $p > 1$, then the Sokhotski-Plemelj formulas and the Poincaré-Bertrand formula are valid almost everywhere on γ.

Remark 2.4. Another class of interesting examples is rectifiable curves. The class of rectifiable curves includes as proper subclasses many other important classes of curves, in particular, smooth (Lyapunov) curves, piecewise Lyapunov curves, and Lipschitz curves. Various properties and applications of the Cauchy type integral for hyperholomorphic functions along rectifiable curves (and domains with rectifiable boundary) can be found, for instance, in [20].

Our purpose is to study the Poincaré-Bertrand formula associated with double-layer potential for piecewise Lyapunov curve. Before introducing the main results, we need a few standard facts from potential theory. For a detailed exposition, we refer the reader to for example, [15, 21].

2.2. Simple and Double Potentials

Suppose that $f : \gamma \to \mathbb{C}$ is a continuous function, and we refer to the functions $u[f](z)$ and $v[f](z)$, given by the formulae:

$$\begin{aligned} u[f](z) &:= \frac{1}{2\pi}\int_\gamma f(\zeta) \ln \frac{1}{|\zeta - z|} ds_\zeta, \quad z \notin \gamma, \\ v[f](z) &:= \frac{1}{2\pi}\int_\gamma f(\zeta) \frac{\partial}{\partial \vec{n}(\zeta)} \ln \frac{1}{|\zeta - z|} ds_\zeta, \quad z \notin \gamma, \end{aligned} \tag{2.14}$$

as the simple- and double-layer potentials, respectively. Here $\partial/\partial\vec{n}(\zeta)$ denotes partial differentiation with respect to the outward directed normal unit vector to the curve γ at a point

ζ, and ds denotes arc-length on γ. Clearly, the simple-layer potential $u[f]$ and double layer potential $v[f]$ are holomorphic in the interior of Ω for any integrable f.

Another option (see, e.g., [16]) is to use the simple layer potential of the form:

$$w[f](z) := \frac{1}{2\pi}\int_{\gamma} f(\zeta)\frac{\partial}{\partial\vec{\tau}(\zeta)}\ln\frac{1}{|\zeta - z|}ds_{\zeta}, \quad z \notin \gamma, \tag{2.15}$$

where $\vec{\tau}(\zeta)$ is a unit tangent vector.

For $z = t \in \gamma$ define

$$\begin{aligned}\mathcal{V}[f](t) &:= -\frac{1}{\pi}\int_{\gamma} f(\zeta)\frac{\partial \ln|\zeta - t|}{\partial\vec{n}(\zeta)}ds_{\zeta} \equiv -\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\{\zeta\in\gamma:|\zeta-t|\geq\varepsilon\}} f(\zeta)\frac{\partial \ln|\zeta - t|}{\partial\vec{n}(\zeta)}ds_{\zeta}, \\ \mathcal{W}[f](t) &:= -\frac{1}{\pi}\int_{\gamma} f(\zeta)\frac{\partial \ln|\zeta - t|}{\partial\vec{\tau}(\zeta)}ds_{\zeta} \equiv -\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\{\zeta\in\gamma:|\zeta-t|\geq\varepsilon\}} f(\zeta)\frac{\partial \ln|\zeta - t|}{\partial\vec{\tau}(\zeta)}ds_{\zeta}.\end{aligned} \tag{2.16}$$

3. Elementary Observations

It is easy to verify that if f is real-valued then

$$S[f](t) = \mathcal{V}[f](t) + i\mathcal{W}[f](t), \tag{3.1}$$

where

$$\begin{aligned}\mathcal{V}[f](t) &= \operatorname{Re}\left[\frac{1}{\pi i}\int_{\gamma}\frac{f(\zeta)}{\zeta - t}\,d\zeta\right], \\ \mathcal{W}[f](t) &= \operatorname{Im}\left[\frac{1}{\pi i}\int_{\gamma}\frac{f(\zeta)}{\zeta - t}\,d\zeta\right].\end{aligned} \tag{3.2}$$

We have already noted that for real-valued function f we have

$$\mathcal{V}[f] = 2\operatorname{Re}\ K[f] = K[f] + ZK[f], \tag{3.3}$$

where the complex conjugation, denoted by "Z".

Thus, for f_1, f_2 real we have

$$\begin{aligned}\mathcal{V}[f_1 + if_2] &:= \mathcal{V}[f_1] + i\mathcal{V}[f_2] \\ &= K[f_1] + ZK[f_1] + i(K[f_2] + ZK[f_2]) \\ &= (K + ZKZ)[f_1 + if_2].\end{aligned} \tag{3.4}$$

Then, we may write

$$\mathcal{V} = K + ZKZ. \tag{3.5}$$

4. Main Results

This section is devoted to the main results of the paper.

Theorem 4.1 (Sokhotski-Plemelj Formulas). *Let γ be a piecewise Lyapunov curve and f a complex-valued function defined on γ which belongs to $H_\lambda(\gamma)$, $0 < \lambda < 1$. Then the following limits exist:*

$$\lim_{\Omega^\pm \ni z \to t \in \gamma} v[f](z) =: v^\pm[f](t), \tag{4.1}$$

and moreover, the Sokhotski-Plemelj formulas hold:

$$v^\pm[f](t) = \frac{1}{2}\left(\pm f(t) + \widetilde{V}[f](t)\right), \quad t \in \gamma, \tag{4.2}$$

where

$$\widetilde{V}[f](t) := \frac{\pi - \alpha(t)}{\pi} f(t) + V[f](t) \tag{4.3}$$

is the modified singular double layer potential, and $\alpha(t)$ is the angle between the one-sided tangents at point t, and the integral exists as an improper integral.

Proof. For a function $f = f_1 + i f_2 \in H_\lambda(\gamma, \mathbb{C})$ with f_1, f_2 real-valued, we can write:

$$K[f_1 + i f_2](z) = K[f_1](z) + iK[f_2](z). \tag{4.4}$$

Note that $v[f_1]$ is the real part of $K[f_1]$, and $v[f_2]$ is the real part of $K[f_2]$. Now the conclusion follows directly from Section 2 and the Sokhotski-Plemelj formulas (2.5) and (2.6). □

Suppose that density f in (2.2) is real-valued and belongs to $H_\mu(\gamma \times \gamma, \mathbb{R})$. By formula (2.13) we have for each $t \in \gamma$:

$$\begin{aligned}
&\frac{1}{\pi}\int_{\gamma_\zeta} \frac{\partial \ln|\zeta - t|}{\partial \vec{n}(\zeta)} ds_\zeta \frac{1}{\pi}\int_{\gamma_{\zeta_1}} f(\zeta,\zeta_1) \frac{\partial \ln|\zeta_1 - \zeta|}{\partial \vec{n}(\zeta_1)} ds_{\zeta_1} - \frac{1}{\pi}\int_{\gamma_\zeta} \frac{\partial \ln|\zeta - t|}{\partial \vec{\tau}(\zeta)} ds_\zeta \frac{1}{\pi}\int_{\gamma_{\zeta_1}} f(\zeta,\zeta_1) \frac{\partial \ln|\zeta_1 - \zeta|}{\partial \vec{\tau}(\zeta_1)} ds_{\zeta_1} \\
&\quad = \eta^2(t) f(t,t) + \frac{1}{\pi}\int_{\gamma_{\zeta_1}} \frac{\partial \ln|\zeta_1 - t|}{\partial \vec{n}(\zeta_1)} ds_{\zeta_1} \\
&\qquad \times \left(\frac{1}{\pi}\int_{\gamma_\zeta} f(\zeta,\zeta_1) \frac{\partial \ln|\zeta - t|}{\partial \vec{n}(\zeta)} ds_\zeta - \frac{1}{\pi}\int_{\gamma_\zeta} f(\zeta,\zeta_1) \frac{\partial \ln|\zeta - \zeta_1|}{\partial \vec{n}(\zeta)} ds_\zeta\right) \\
&\qquad - \frac{1}{\pi}\int_{\gamma_{\zeta_1}} \frac{\partial \ln|\zeta_1 - t|}{\partial \vec{\tau}(\zeta_1)} ds_{\zeta_1} \left(\frac{1}{\pi}\int_{\gamma_\zeta} f(\zeta,\zeta_1) \frac{\partial \ln|\zeta - t|}{\partial \vec{\tau}(\zeta)} ds_\zeta - \frac{1}{\pi}\int_{\gamma_\zeta} f(\zeta,\zeta_1) \frac{\partial \ln|\zeta - \zeta_1|}{\partial \vec{\tau}(\zeta)} ds_\zeta\right),
\end{aligned} \tag{4.5}$$

$$\frac{1}{\pi}\int_{\gamma_\zeta}\frac{\partial \ln|\zeta-t|}{\partial \vec{n}(\zeta)}ds_\zeta \frac{1}{\pi}\int_{\gamma_{\zeta_1}} f(\zeta,\zeta_1)\frac{\partial \ln|\zeta_1-\zeta|}{\partial \vec{\tau}(\zeta_1)}ds_{\zeta_1}+\frac{1}{\pi}\int_{\gamma_\zeta}\frac{\partial \ln|\zeta-t|}{\partial \vec{\tau}(\zeta)}ds_\zeta \frac{1}{\pi}\int_{\gamma_{\zeta_1}} f(\zeta,\zeta_1)\frac{\partial \ln|\zeta_1-\zeta|}{\partial \vec{n}(\zeta_1)}ds_{\zeta_1}$$
$$=\frac{1}{\pi}\int_{\gamma_{\zeta_1}}\frac{\partial \ln|\zeta_1-t|}{\partial \vec{n}(\zeta_1)}ds_{\zeta_1}\left(\frac{1}{\pi}\int_{\gamma_\zeta} f(\zeta,\zeta_1)\frac{\partial \ln|\zeta-t|}{\partial \vec{\tau}(\zeta)}ds_\zeta-\frac{1}{\pi}\int_{\gamma_\zeta} f(\zeta,\zeta_1)\frac{\partial \ln|\zeta-\zeta_1|}{\partial \vec{\tau}(\zeta)}ds_\zeta\right)$$
$$+\frac{1}{\pi}\int_{\gamma_{\zeta_1}}\frac{\partial \ln|\zeta_1-t|}{\partial \vec{\tau}(\zeta_1)}ds_{\zeta_1}\left(\frac{1}{\pi}\int_{\gamma_\zeta} f(\zeta,\zeta_1)\frac{\partial \ln|\zeta-\zeta_1|}{\partial \vec{n}(\zeta)}ds_\zeta-\frac{1}{\pi}\int_{\gamma_\zeta} f(\zeta,\zeta_1)\frac{\partial \ln|\zeta-\zeta_1|}{\partial \vec{n}(\zeta)}ds_\zeta\right). \tag{4.6}$$

Multiplying both sides of (2.10) by $f(\zeta_1)d\zeta_1$, integrating over γ with respect to ζ_1 and separate complex coordinates, the following equalities can be easily obtained:

$$\begin{aligned}
&\frac{1}{\pi}\int_{\gamma_{\zeta_1}} f(\zeta_1)\frac{\partial \ln|\zeta_1-t|}{\partial \vec{n}(\zeta_1)}ds_{\zeta_1}\left(\frac{1}{\pi}\int_{\gamma_\zeta}\frac{\partial \ln|\zeta-t|}{\partial \vec{n}(\zeta)}ds_\zeta-\frac{1}{\pi}\int_{\gamma_\zeta}\frac{\partial \ln|\zeta-\zeta_1|}{\partial \vec{n}(\zeta)}ds_\zeta\right)\\
&\quad-\frac{1}{\pi}\int_{\gamma_{\zeta_1}} f(\zeta_1)\frac{\partial \ln|\zeta_1-t|}{\partial \vec{\tau}(\zeta_1)}ds_{\zeta_1}\left(\frac{1}{\pi}\int_{\gamma_\zeta}\frac{\partial \ln|\zeta-t|}{\partial \vec{\tau}(\zeta)}ds_\zeta-\frac{1}{\pi}\int_{\gamma_\zeta}\frac{\partial \ln|\zeta-\zeta_1|}{\partial \vec{\tau}(\zeta)}ds_\zeta\right)=0,\\
&\frac{1}{\pi}\int_{\gamma_{\zeta_1}} f(\zeta_1)\frac{\partial \ln|\zeta_1-t|}{\partial \vec{n}(\zeta_1)}ds_{\zeta_1}\left(\frac{1}{\pi}\int_{\gamma_\zeta}\frac{\partial \ln|\zeta-t|}{\partial \vec{\tau}(\zeta)}ds_\zeta-\frac{1}{\pi}\int_{\gamma_\zeta}\frac{\partial \ln|\zeta-\zeta_1|}{\partial \vec{\tau}(\zeta)}ds_\zeta\right)\\
&\quad+\frac{1}{\pi}\int_{\gamma_{\zeta_1}} f(\zeta_1)\frac{\partial \ln|\zeta_1-t|}{\partial \vec{\tau}(\zeta_1)}ds_{\zeta_1}\left(\frac{1}{\pi}\int_{\gamma_\zeta}\frac{\partial \ln|\zeta-t|}{\partial \vec{n}(\zeta)}ds_\zeta-\frac{1}{\pi}\int_{\gamma_\zeta}\frac{\partial \ln|\zeta-\zeta_1|}{\partial \vec{n}(\zeta)}ds_\zeta\right)=0.
\end{aligned} \tag{4.7}$$

Assume that a function f depends on ζ_1 only, then using (4.5)–(4.9) we have

$$\begin{aligned}
&\frac{1}{\pi}\int_{\gamma_\zeta}\frac{\partial \ln|\zeta-t|}{\partial \vec{n}(\zeta\ )}ds_\zeta\ \frac{1}{\pi}\int_{\gamma_{\zeta_1}} f(\zeta_1)\frac{\partial \ln|\zeta_1-\zeta|}{\partial \vec{n}(\zeta_1)}ds_{\zeta_1}\\
&\quad-\frac{1}{\pi}\int_{\gamma_\zeta}\frac{\partial \ln|\zeta-t|}{\partial \vec{\tau}(\zeta)}ds_\zeta\ \frac{1}{\pi}\int_{\gamma_{\zeta_1}} f(\zeta_1)\frac{\partial \ln|\zeta_1-\zeta|}{\partial \vec{\tau}(\zeta_1)}ds_{\zeta_1}=\eta^2(t)f(t,t),\\
&\frac{1}{\pi}\int_{\gamma_\zeta}\frac{\partial \ln|\zeta-t|}{\partial \vec{n}(\zeta)}ds_\zeta\ \frac{1}{\pi}\int_{\gamma_{\zeta_1}} f(\zeta_1)\frac{\partial \ln|\zeta_1-\zeta|}{\partial \vec{\tau}(\zeta_1)}ds_{\zeta_1}\\
&\quad+\frac{1}{\pi}\int_{\gamma_\zeta}\frac{\partial \ln|\zeta-t|}{\partial \vec{\tau}(\zeta)}ds_\zeta\ \frac{1}{\pi}\int_{\gamma_{\zeta_1}} f(\zeta_1)\frac{\partial \ln|\zeta_1-\zeta|}{\partial \vec{n}(\zeta_1)}ds_{\zeta_1}=0.
\end{aligned} \tag{4.8}$$

Now, in terms of double layer potential $\mathcal{V}$ and simple-layer potential $\mathcal{W}$ we can easily represent the very important properties:

$$\begin{aligned}
&\mathcal{V}^2-\mathcal{W}^2=\eta^2 I,\\
&\mathcal{V}\mathcal{W}+\mathcal{W}\mathcal{V}=0.
\end{aligned} \tag{4.9}$$

Remark 4.2. When γ is a smooth Lyapunov curve, the properties (4.9) coincide with properties given in [22, Section 4.1.].

The proof of following lemma is straightforward.

Lemma 4.3. *Assume that γ is a piecewise Lyapunov curve. Then for $t \in \gamma$*

$$\frac{1}{\pi}\int_{\gamma_{\zeta_1}} \frac{\partial \ln|\zeta_1 - t|}{\partial \vec{n}(\zeta_1)} ds_{\zeta_1}\left(\frac{1}{\pi}\int_{\gamma_\zeta} \frac{\partial \ln|\zeta - t|}{\partial \vec{n}(\zeta)} ds_\zeta - \frac{1}{\pi}\int_{\gamma_\zeta} \frac{\partial \ln|\zeta - \zeta_1|}{\partial \vec{n}(\zeta)} ds_\zeta\right) = 0. \tag{4.10}$$

Thus, we remark also that (4.7) implies that

$$\frac{1}{\pi}\int_{\gamma_{\zeta_1}} \frac{\partial \ln|\zeta_1 - t|}{\partial \vec{\tau}(\zeta_1)} ds_{\zeta_1}\left(\frac{1}{\pi}\int_{\gamma_\zeta} \frac{\partial \ln|\zeta - t|}{\partial \vec{\tau}(\zeta)} ds_\zeta - \frac{1}{\pi}\int_{\gamma_\zeta} \frac{\partial \ln|\zeta - \zeta_1|}{\partial \vec{\tau}(\zeta)} ds_\zeta\right) = 0. \tag{4.11}$$

The following theorem is provided by using (4.11). Its proof is virtually identical to that of [10, Theorem 4.4], and is omitted.

Theorem 4.4. *Let γ be a piecewise Lyapunov curve. Then for $t \in \gamma$:*

$$\begin{aligned}
&\frac{1}{\pi}\int_{\gamma_\zeta} \frac{\partial \ln|\zeta - t|}{\partial \vec{\tau}(\zeta)} ds_\zeta \, \frac{1}{\pi}\int_{\gamma_{\zeta_1}} f(\zeta,\zeta_1)\frac{\partial \ln|\zeta_1 - \zeta|}{\partial \vec{\tau}(\zeta_1)} ds_{\zeta_1} \\
&= \frac{1}{\pi}\int_{\gamma_{\zeta_1}} \frac{\partial \ln|\zeta_1 - t|}{\partial \vec{\tau}(\zeta_1)} ds_{\zeta_1}\left(\frac{1}{\pi}\int_{\gamma_\zeta} f(\zeta,\zeta_1)\frac{\partial \ln|\zeta - t|}{\partial \vec{\tau}(\zeta)} ds_\zeta - \frac{1}{\pi}\int_{\gamma_\zeta} f(\zeta,\zeta_1)\frac{\partial \ln|\zeta - \zeta_1|}{\partial \vec{\tau}(\zeta)} ds_\zeta\right).
\end{aligned} \tag{4.12}$$

If we now take a $\mathbb{C}$-valued function f as $f_1 + if_2$ (see Section 3), then comparing Theorem 4.4 and formula (4.5) *we have one of our main results, analog of the Poincaré-Bertrand formula for double layer potential.*

Theorem 4.5 (Poincaré-Bertrand formula). *Let Ω be a bounded domain in $\mathbb{R}^2$ with piecewise Lyapunov curve. Assume that $f \in H_\lambda(\gamma \times \gamma, \mathbb{C})$, $0 < \lambda \leq 1$. Then for all $t \in \gamma$:*

$$\begin{aligned}
&\frac{1}{\pi}\int_{\gamma_\zeta} \frac{\partial \ln|\zeta - t|}{\partial \vec{n}(\zeta)} ds_\zeta \, \frac{1}{\pi}\int_{\gamma_{\zeta_1}} f(\zeta,\zeta_1)\frac{\partial \ln|\zeta_1 - \zeta|}{\partial \vec{n}(\zeta_1)} ds_{\zeta_1} \\
&= \eta^2(t) f(t,t) + \frac{1}{\pi}\int_{\gamma_{\zeta_1}} \frac{\partial \ln|\zeta_1 - t|}{\partial \vec{n}(\zeta_1)} ds_{\zeta_1}\left(\frac{1}{\pi}\int_{\gamma_\zeta} f(\zeta,\zeta_1)\frac{\partial \ln|\zeta - t|}{\partial \vec{n}(\zeta)} ds_\zeta - \frac{1}{\pi}\int_{\gamma_\zeta} f(\zeta,\zeta_1)\frac{\partial \ln|\zeta - \zeta_1|}{\partial \vec{n}(\zeta)} ds_\zeta\right),
\end{aligned} \tag{4.13}$$

where the integrals exist in the sense of the Cauchy principal value.

An easy consequence of Theorem 4.5 is the following corollary.

Corollary 4.6. *Let Ω be a bounded domain in $\mathbb{R}^2$ with piecewise Lyapunov curve. Suppose that $f(\zeta,\zeta_1) = f(\zeta) \in H_\lambda(\gamma\times\gamma,\mathbb{C})$, $0<\lambda\leq 1$. Then for all $t\in\gamma$:*

$$\frac{1}{\pi}\int_{\gamma_\zeta} f(\zeta)\frac{\partial \ln|\zeta - t|}{\partial \vec{n}(\zeta)}ds_\zeta\ \frac{1}{\pi}\int_{\gamma_{\zeta_1}} \frac{\partial \ln|\zeta_1 - \zeta|}{\partial \vec{n}(\zeta_1)}ds_{\zeta_1} = \eta^2(t)f(t). \tag{4.14}$$

Proof. Suppose that $f(\zeta,\zeta_1) = f(\zeta) \in H_\lambda(\gamma\times\gamma,\mathbb{C})$, $0<\lambda\leq 1$, then by using formula (4.10) we obtain (4.14). □

Note that in the previous theorems, we assumed that Ω was a bounded region in $\mathbb{R}^2$. Let now $\gamma = \mathbb{R}$ and we consider a function f on $\mathbb{R}$, of the class $L_p, p>1$. So, we have to understand $K[f]$ as the Lebesgue integral. In fact, the proof of the Poincaré-Bertrand formula is essentially local, and is valid almost everywhere. Thus, the following theorems hold.

Theorem 4.7. *If $p>1$, $f\in L_p(\mathbb{R}\times\mathbb{R},\mathbb{C})$, then, for almost all t:*

$$\begin{aligned}
&\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\partial \ln|\zeta - t|}{\partial \vec{n}(\zeta)}ds_\zeta\ \frac{1}{\pi}\int_{-\infty}^{+\infty} f(\zeta,\zeta_1)\frac{\partial \ln|\zeta_1 - \zeta|}{\partial \vec{n}(\zeta_1)}ds_{\zeta_1}\\
&= f(t,t)+\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\partial \ln|\zeta_1 - t|}{\partial \vec{n}(\zeta_1)}ds_{\zeta_1}\left(\frac{1}{\pi}\int_{-\infty}^{+\infty} f(\zeta,\zeta_1)\frac{\partial \ln|\zeta - t|}{\partial \vec{n}(\zeta)}ds_\zeta - \frac{1}{\pi}\int_{-\infty}^{+\infty} f(\zeta,\zeta_1)\frac{\partial \ln|\zeta - \zeta_1|}{\partial \vec{n}(\zeta)}ds_\zeta\right),
\end{aligned} \tag{4.15}$$

where the integrals exist in the sense of the Cauchy principal value.

Corollary 4.8. *Suppose that $f(\zeta,\zeta_1) = f(\zeta) \in L_p(\mathbb{R}\times\mathbb{R},\mathbb{C})$, $p>1$. Then for almost all t:*

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} f(\zeta)\frac{\partial \ln|\zeta - t|}{\partial \vec{n}(\zeta)}ds_\zeta\ \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\partial \ln|\zeta_1 - \zeta|}{\partial \vec{n}(\zeta_1)}ds_{\zeta_1} = f(t). \tag{4.16}$$

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