Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 230392, 17 pages
http://dx.doi.org/10.1155/2013/230392

Research Article

A New Computational Technique for Common Solutions between Systems of Generalized Mixed Equilibrium and Fixed Point Problems

Pongsakorn Sunthrayuth and Poom Kumam

Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bang Mod, Bangkok 10140, Thailand

Correspondence should be addressed to Poom Kumam; poom.kum@kmutt.ac.th

Received 18 February 2013; Accepted 1 April 2013

Academic Editor: Wei-Shih Du

We introduce a new iterative algorithm for finding a common element of a fixed point problem of amenable semigroups of nonexpansive mappings, the set solutions of a system of a general system of generalized equilibria in a real Hilbert space. Then, we prove the strong convergence of the proposed iterative algorithm to a common element of the above three sets under some suitable conditions. As applications, at the end of the paper, we apply our results to find the minimum-norm solutions which solve some quadratic minimization problems. The results obtained in this paper extend and improve many recent ones announced by many others.

1. Introduction

Throughout this paper, we denoted by $\mathbb{R}$ the set of all real numbers. We always assume that H is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$ and C is a nonempty, closed, and convex subset of H. P_C denotes the metric projection of H onto C. A mapping $T : C \to C$ is said to be *L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in C. \tag{1}$$

If $0 < L < 1$, then T is a contraction, and if $L = 1$, then T is a nonexpansive mapping. We denote by $\mathrm{Fix}(T)$ the set of all fixed points set of the mapping T; that is, $\mathrm{Fix}(T) = \{x \in C : Tx = x\}$.

A mapping $F : C \to H$ is said to be *monotone* if

$$\langle Fx - Fy, x - y\rangle \geq 0, \quad \forall x, y \in C. \tag{2}$$

A mapping $F : C \to H$ is said to be *strongly monotone* if there exists $\eta > 0$ such that

$$\langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C. \tag{3}$$

Let $\varphi : C \to \mathbb{R}$ be a real-valued function, $\Theta : C \times C \to \mathbb{R}$ an equilibrium bifunction, and $\Psi : C \to H$ a nonlinear mapping. The *generalized mixed equilibrium problem* is to find $x^* \in C$ such that

$$\Theta(x^*, y) + \varphi(y) - \varphi(x^*) + \langle\Psi x^*, y - x^*\rangle \geq 0, \quad \forall y \in C, \tag{4}$$

which was introduced and studied by Peng and Yao [1]. The set of solutions of problem (4) is denoted by $\mathrm{GMEP}(\Theta, \varphi, \Psi)$. As special cases of problem (4), we have the following results.

(1) If $\Psi = 0$, then problem (4) reduces to *mixed equilibrium problem*. Find $x^* \in C$ such that

$$\Theta(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \quad \forall y \in C, \tag{5}$$

which was considered by Ceng and Yao [2]. The set of solutions of problem (5) is denoted by $\mathrm{MEP}(\Theta)$.

(2) If $\varphi = 0$, then problem (4) reduces to *generalized equilibrium problem*. Find $x^* \in C$ such that

$$\Theta(x^*, y) + \langle\Psi x^*, y - x^*\rangle \geq 0, \quad \forall y \in C, \tag{6}$$

which was considered by S. Takahashi and W. Takahashi [3]. The set of solutions of problem (6) is denoted by $\mathrm{GEP}(\Theta, \Psi)$.

(3) If $\Psi = \varphi = 0$, then problem (4) reduces to *equilibrium problem*. Find $x^* \in C$ such that

$$\Theta(x^*, y) \geq 0, \quad \forall y \in C. \tag{7}$$

The set of solutions of problem (7) is denoted by $\mathrm{EP}(\Theta)$.

(4) If $\Theta = \varphi = 0$, then problem (4) reduces to *classical variational inequality problem*. Find $x^* \in C$ such that

$$\langle \Psi x^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{8}$$

The set of solutions of problem (8) is denoted by $\mathrm{VI}(C, \Psi)$. It is known that $x^* \in C$ is a solution of the problem (8) if and only if x^* is a fixed point of the mapping $P_C(I - \lambda\Psi)$, where $\lambda > 0$ is a constant and I is the identity mapping.

The problem (4) is very general in the sense that it includes several problems, namely, fixed point problems, optimization problems, saddle point problems, complementarity problems, variational inequality problems, minimax problems, Nash equilibrium problems in noncooperative games, and others as special cases. Numerous problems in physics, optimization, and economics reduce to find a solution of problem (4) (see, e.g., [4–9]). Several iterative methods to solve the fixed point problems, variational inequality problems, and equilibrium problems are proposed in the literature (see, e.g., [1–3, 10–18]) and the references therein.

Let $A_1, A_2 : C \to H$ be two mappings. Ceng and Yao [12] considered the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{gathered} G_2(x^*, x) + \langle A_2 y^*, x - x^* \rangle + \frac{1}{\lambda_2} \langle x^* - y^*, x - x^* \rangle \geq 0, \\ \forall x \in C, \\ G_1(y^*, y) + \langle A_1 x^*, y - y^* \rangle + \frac{1}{\lambda_1} \langle y^* - x^*, y - y^* \rangle \geq 0, \\ \forall y \in C, \end{gathered} \tag{9}$$

which is called *a general system of generalized equilibria*, where $\lambda_1 > 0$ and $\lambda_2 > 0$ are two constants. In particular, if $G_1 = G_2 = G$ and $A_1 = A_2 = A$, then problem (9) reduces to the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{gathered} G(x^*, x) + \langle A y^*, x - x^* \rangle + \frac{1}{\lambda_2} \langle x^* - y^*, x - x^* \rangle \geq 0, \\ \forall x \in C, \\ G(y^*, y) + \langle A x^*, y - y^* \rangle + \frac{1}{\lambda_1} \langle y^* - x^*, y - y^* \rangle \geq 0, \\ \forall y \in C, \end{gathered} \tag{10}$$

which is called *a new system of generalized equilibria*, where $\lambda_1 > 0$ and $\lambda_2 > 0$ are two constants.

If $G_1 = G_2 = \Theta$, $A_1 = A_2 = A$, and $x^* = y^*$, then problem (9) reduces to problem (7).

If $G_1 = G_2 = 0$, then problem (9) reduces to *a general system of variational inequalities*. Find $(x^*, y^*) \in C \times C$ such that

$$\begin{gathered} \langle \lambda_2 A_2 y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ \langle \lambda_1 A_1 x^* + y^* - x^*, y - y^* \rangle \geq 0, \quad \forall y \in C, \end{gathered} \tag{11}$$

where $\lambda_1 > 0$ and $\lambda_2 > 0$ are two constants, which is introduced and considered by Ceng et al. [19].

In 2010, Ceng and Yao [12] proposed the following relaxed extragradient-like method for finding a common solution of generalized mixed equilibrium problems, a system of generalized equilibria (9), and a fixed point problem of a k-strictly pseudocontractive self-mapping S on C as follows:

$$\begin{gathered} z_n = S_{r_n}^{(\Theta,\varphi)}(x_n - r_n \Psi x_n), \\ y_n = S_{\lambda_1}^{G_1}\left[S_{\lambda_2}^{G_2}(z_n - \lambda_2 A_2 z_n) - \lambda_1 A_1 S_{\lambda_2}^{G_2}(z_n - \lambda_2 A_2 z_n)\right], \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n y_n + \delta_n S y_n, \quad \forall n \geq 0, \end{gathered} \tag{12}$$

where $\Psi, A_1, A_2 : C \to H$ are α-inverse strongly monotone, $\widetilde{\alpha}_1$-inverse strongly monotone, and $\widetilde{\alpha}_2$-inverse strongly monotone, respectively. They proved strong convergence of the related extragradient-like algorithm (12) under some appropriate conditions $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\} \subset [0,1]$ satisfying $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$, for all $n \geq 0$, to $\widehat{x} = P_\Omega \widehat{x}$, where $\Omega = \mathrm{Fix}(S) \cap \mathrm{GMEP}(\Theta, \varphi, \Psi) \cap \mathrm{Fix}(K)$, with the mapping $K : C \to C$ defined by

$$\begin{gathered} Kx = S_{\lambda_1}^{G_1}\left[S_{\lambda_2}^{G_2}(x - \lambda_2 A_2 x) - \lambda_1 A_1 S_{\lambda_2}^{G_2}(x - \lambda_2 A_2 x)\right], \\ \forall x \in C. \end{gathered} \tag{13}$$

Very recently, Ceng et al. [11] introduced an iterative method for finding fixed points of a nonexpansive mapping T on a nonempty, closed, and convex subset C in a real Hilbert space H as follows:

$$x_{n+1} = P_C\left[\alpha_n \gamma V x_n + (I - \alpha_n \mu F) T x_n\right], \quad \forall n \geq 0, \tag{14}$$

where P_C is a metric projection from H onto C, V is an L-Lipschitzian mapping with a constant $L \geq 0$, and F is a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa, \eta > 0$ and $0 < \mu < 2\eta/\kappa^2$. Then, they proved that the sequences generated by (14) converge strongly to a unique solution of variational inequality as follows:

$$\langle (\mu F - \gamma V) x^*, x^* - x \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T). \tag{15}$$

In this paper, motivated and inspired by the previous facts, we first introduce the following problem of finding $(x_1^*, x_2^*, \ldots, x_M^*) \in C \times C \times \cdots \times C$ such that

$$\begin{aligned}
&G_M(x_1^*, x_1) + \langle A_M x_M^*, x_1 - x_1^* \rangle \\
&\quad + \frac{1}{\lambda_M} \langle x_1^* - x_M^*, x_1 - x_1^* \rangle \geq 0, \quad \forall x_1 \in C, \\
&G_{M-1}(x_M^*, x_M) + \langle A_{M-1} x_{M-1}^*, x_M - x_M^* \rangle \\
&\quad + \frac{1}{\lambda_{M-1}} \langle x_M^* - x_{M-1}^*, x_M - x_M^* \rangle \geq 0, \quad \forall x_M \in C, \\
&\qquad \vdots \\
&G_2(x_3^*, x_3) + \langle A_2 x_2^*, x_3 - x_3^* \rangle + \frac{1}{\lambda_2} \langle x_3^* - x_2^*, x_3 - x_3^* \rangle \geq 0, \\
&\qquad \forall x_3 \in C, \\
&G_1(x_2^*, x_2) + \langle A_1 x_1^*, x_2 - x_2^* \rangle + \frac{1}{\lambda_1} \langle x_2^* - x_1^*, x_2 - x_2^* \rangle \geq 0, \\
&\qquad \forall x_2 \in C,
\end{aligned} \tag{16}$$

which is called a *more general system of generalized equilibria in Hilbert spaces*, where $\lambda_i > 0$ for all $i \in \{1, 2, \ldots, M\}$. In particular, if $M = 2$, $x_1^* = x^*$, $x_2^* = y^*$, $x_1 = x$, and $x_2 = y$, then problem (16) reduces to problem (9). Finally, by combining the relaxed extragradient-like algorithm (12) with the general iterative algorithm (14), we introduce a new iterative method for finding a common element of a fixed point problem of a nonexpansive semigroup, the set solutions of a general system of generalized equilibria in a real Hilbert space. We prove the strong convergence of the proposed iterative algorithm to a common element of the previous three sets under some suitable conditions. Furthermore, we apply our results to finding the minimum-norm solutions which solve some quadratic minimization problem. The main result extends various results existing in the current literature.

2. Preliminaries

Let S be a semigroup. We denote by ℓ^∞ the Banach space of all bounded real-valued functionals on S with supremum norm. For each $s \in S$, we define the left and right translation operators $l(s)$ and $r(s)$ on $\ell^\infty(S)$ by

$$(l(s) f)(t) = f(st), \qquad (r(s) f)(t) = f(ts), \tag{17}$$

for each $t \in S$ and $f \in \ell^\infty(S)$, respectively. Let X be a subspace of $\ell^\infty(S)$ containing 1. An element μ in the dual space X^* of X is said to be a *mean* on X if $\|\mu\| = \mu(1) = 1$. It is well known that μ is a mean on X if and only if

$$\inf_{s \in S} f(s) \leq \mu(f) \leq \sup_{s \in S} f(s), \tag{18}$$

for each $f \in X$. We often write $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$.

Let X be a translation invariant subspace of $\ell^\infty(S)$ (i.e., $l(s)X \subset X$ and $r(s)X \subset X$ for each $s \in S$) containing 1. Then, a mean μ on X is said to be *left invariant* (resp., right invariant) if $\mu(l(s)f) = \mu(f)$ (resp., $\mu(r(s)f) = \mu(f)$) for each $s \in S$ and $f \in X$. A mean μ on X is said to be *invariant* if μ is both left and right invariant [20–22]. S is said to be *left* (resp., *right*) *amenable* if X has a left (resp., right) invariant mean. S is a amenable if S is left and right amenable. In this case, $\ell^\infty(S)$ also has an invariant mean. As is well known, $\ell^\infty(S)$ is amenable when S is commutative semigroup; see [23]. A net $\{\mu_\alpha\}$ of means on X is said to be *left regular* if

$$\lim_\alpha \|l_s^* \mu_\alpha - \mu_\alpha\| = 0, \tag{19}$$

for each $s \in S$, where l_s^* is the adjoint operator of l_s.

Let C be a nonempty, closed, and convex subset of H. A family $\mathcal{S} = \{T(s) : s \in S\}$ is called a *nonexpansive semigroup* on C if for each $s \in S$, the mapping $T(s) : C \to C$ is nonexpansive and $T(st) = T(ts)$ for each $s, t \in S$. We denote by $\mathrm{Fix}(\mathcal{S})$ the set of common fixed point of $\mathcal{S}$; that is,

$$\mathrm{Fix}(\mathcal{S}) = \bigcap_{s \in S} \mathrm{Fix}(T(s)) = \bigcap_{s \in S} \{x \in C : T(s)x = x\}. \tag{20}$$

Throughout this paper, the open ball of radius r centered at 0 is denoted by B_r, and for a subset D of H by $\overline{\mathrm{co}}\, D$, we denote the closed convex hull of D. For $\epsilon > 0$ and a mapping $T : D \to H$, the set of ϵ-approximate fixed point of T is denoted by $F_\epsilon(T, D)$; that is, $F_\epsilon(T, D) = \{x \in D : \|x - Tx\| \leq \epsilon\}$.

In order to prove our main results, we need the following lemmas.

Lemma 1 (see [23–25]). *Let f be a function of a semigroup S into a Banach space E such that the weak closure of $\{f(t) : t \in S\}$ is weakly compact, and let X be a subspace of $\ell^\infty(S)$ containing all the functions $t \mapsto \langle f(t), x^* \rangle$ with $x^* \in E^*$. Then, for any $\mu \in X^*$, there exists a unique element f_μ in E such that*

$$\langle f_\mu, x^* \rangle = \mu_t \langle f(t), x^* \rangle, \tag{21}$$

for all $x^* \in E^*$. Moreover, if μ is a mean on X, then

$$\int f(t)\, d\mu(t) \in \overline{\mathrm{co}} \{f(t) : t \in S\}. \tag{22}$$

One can write f_μ by $\int f(t) d\mu(t)$.

Lemma 2 (see [23–25]). *Let C be a closed and convex subset of a Hilbert space H, $\mathcal{S} = \{T(s) : s \in S\}$ a nonexpansive semigroup from C into C such that $\mathrm{Fix}(\mathcal{S}) \neq \emptyset$, and X a subspace of ℓ^∞ containing 1, the mapping $t \mapsto \langle T(t)x, y \rangle$ an element of X for each $x \in C$ and $y \in H$, and μ a mean on X.*

If one writes $T(\mu)x$ instead of $\int T_t x d\mu(t)$, then the following hold:

(i) *$T(\mu)$ is nonexpansive mapping from C into C;*

(ii) *$T(\mu)x = x$ for each $x \in \mathrm{Fix}(\mathcal{S})$;*

(iii) *$T(\mu)x \in \overline{\mathrm{co}} \{T_t x : t \in S\}$ for each $x \in C$;*

(iv) *if μ is left invariant, then $T(\mu)$ is a nonexpansive retraction from C onto* $\mathrm{Fix}(\mathcal{S})$.

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and let C be a nonempty, closed, and convex subset of H. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Also, a mapping $I : C \to C$ denotes the identity mapping. For every point $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C. \tag{23}$$

Such a projection P_C is called the *metric projection* of H onto C. We know that P_C is a firmly nonexpansive mapping of H onto C; that is,

$$\langle x - y, P_C x - P_C y\rangle \ge \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{24}$$

It is known that

$$z = P_C x \Longleftrightarrow \langle x - z, y - z\rangle \le 0, \quad \forall x \in H,\ y \in C. \tag{25}$$

In a real Hilbert space H, it is well known that

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle, \tag{26}$$

$$\|\lambda x + (1-\lambda) y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2, \tag{27}$$

for all $x, y \in H$ and $\lambda \in [0,1]$.

If $A : C \to H$ is α-inverse strongly monotone, then it is obvious that A is $1/\alpha$-Lipschitz continuous. We also have that, for all $x, y \in C$ and $\lambda > 0$,

$$\begin{aligned} &\|(I - \lambda A)x - (I - \lambda)y\|^2 \\ &= \|(x - y) - \lambda(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2\lambda\langle Ax - Ay, x - y\rangle + \lambda^2\|Ax - Ay\|^2 \\ &\le \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2. \end{aligned} \tag{28}$$

In particular, if $\lambda < 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping from C to H.

For solving the equilibrium problem, let us assume that the bifunction $\Theta : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $\Theta(x, x) = 0$ for all $x \in C$;

(A2) Θ is monotone, that is, $\Theta(x, y) + \Theta(y, x) \le 0$ for each $x, y \in C$;

(A3) Θ is upper semicontinuous, that is, for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} \Theta(tz + (1-t)x, y) \le \Theta(x, y); \tag{29}$$

(A4) $\Theta(x, \cdot)$ is convex and weakly lower semicontinuous for each $x \in C$;

(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for all $z \in C \setminus D_x$,

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0; \tag{30}$$

(B2) C is a bounded set.

Lemma 3 (see [1]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions $(A1)-(A4)$, and let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex function. For $r > 0$ and $x \in H$, define a mapping $S_r^{(\Theta,\varphi)} : H \to C$ as follows:*

$$\begin{aligned} S_r^{(\Theta,\varphi)}(x) &= \Big\{ y \in C : \Theta(y, z) + \varphi(z) - \varphi(y) \\ &\quad + \frac{1}{r}\langle y - z, z - x\rangle \ge 0,\ \forall z \in C \Big\}. \end{aligned} \tag{31}$$

Assume that either $(B1)$ or $(B2)$ holds. Then, the following hold:

(i) $S_r^{(\Theta,\varphi)} \ne \emptyset$ *for all $x \in H$ and $S_r^{(\Theta,\varphi)}$ is single valued;*

(ii) $S_r^{(\Theta,\varphi)}$ *is firmly nonexpansive, that is, for all $x, y \in H$,*

$$\left\|S_r^{(\Theta,\varphi)}x - S_r^{(\Theta,\varphi)}y\right\|^2 \le \left\langle S_r^{(\Theta,\varphi)}x - S_r^{(\Theta,\varphi)}y, x - y\right\rangle; \tag{32}$$

(iii) $\mathrm{Fix}(S_r^{(\Theta,\varphi)}) = \mathrm{MEP}(\Theta, \varphi)$;

(iv) $\mathrm{MEP}(\Theta, \varphi)$ *is closed and convex.*

Remark 4. If $\varphi = 0$, then $S_r^{(\Theta,\varphi)}$ is rewritten as S_r^{Θ} (see [12, Lemma 2.1] for more details).

Lemma 5 (see [26]). *Let $\{x_n\}$ and $\{l_n\}$ be bounded sequences in a Banach space X, and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n)l_n + \beta_n x_n$ for all integers $n \ge 0$ and $\limsup_{n\to\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \le 0$. Then, $\lim_{n\to\infty}\|l_n - x_n\| = 0$.*

Lemma 6 (Demiclosedness Principle [27]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \ne \emptyset$. If $\{x_n\}$ is a sequence in C that converges weakly to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$; in particular, if $y = 0$, then $x \in \mathrm{Fix}(T)$.*

Lemma 7 (see [28]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - \sigma_n)a_n + \delta_n, \tag{33}$$

where $\{\sigma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty}\sigma_n = \infty$;

(ii) $\limsup_{n\to\infty}(\delta_n/\sigma_n) \le 0$ *or* $\sum_{n=0}^{\infty}|\delta_n| < \infty$.

Then, $\lim_{n\to\infty}a_n = 0$.

The following lemma can be found in [29, 30]. For the sake of the completeness, we include its proof in a real Hilbert space version.

Lemma 8. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $F : C \to X$ be a κ-Lipschitzian and η-strongly monotone operator. Let $0 < \mu < 2\eta/\kappa^2$ and $\tau = \mu(\eta - \mu\kappa^2/2)$. Then, for each $t \in (0, \min\{1, 1/2\tau\})$, the mapping $S : C \to H$ defined by $S := I - t\mu F$ is a contraction with constant $1 - t\tau$.*

Proof. Since $0 < \mu < 2\eta/\kappa^2$ and $t \in (0, \min\{1, 1/2\tau\})$, this implies that $1 - t\tau \in (0, 1)$. For all $x, y \in C$, we have

$$\begin{aligned}
\|Sx - Sy\|^2 &= \|(I - t\mu F)x - (I - t\mu F)y\|^2 \\
&= \|(x - y)x - t\mu(Fx - Fy)\|^2 \\
&= \|x - y\|^2 - 2t\mu\langle Fx - Fy, x - y\rangle \\
&\quad + t^2\mu^2\|Fx - Fy\|^2 \\
&\le \|x - y\|^2 - 2t\mu\eta\|x - y\|^2 \\
&\quad + t^2\mu^2\kappa^2\|x - y\|^2 \\
&\le \left[1 - t\mu\left(2\eta - \mu\kappa^2\right)\right]\|x - y\|^2 \\
&= \left[1 - 2t\mu\left(\eta - \frac{\mu\kappa^2}{2}\right)\right]\|x - y\|^2 \\
&\le \left[1 - t\mu\left(\eta - \frac{\mu\kappa^2}{2}\right)\right]^2\|x - y\|^2 \\
&= (1 - t\tau)^2\|x - y\|^2.
\end{aligned} \tag{34}$$

It follows that

$$\|Sx - Sy\| \le (1 - t\tau)\|x - y\|. \tag{35}$$

Hence, we have that $S := I - t\mu F$ is a contraction with constant $1 - t\tau$. This completes the proof. □

Lemma 9. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $A_i : C \to H$ $(i = 1, 2, \ldots, M)$ be a finite family of α_i-inverse strongly monotone operator. Let $K : C \to C$ be a mapping defined by*

$$\begin{aligned}
Kx := S_{\lambda_M}^{G_M}(I - \lambda_M A_M) S_{\lambda_{M-1}}^{G_{M-1}} \\
\times (I - \lambda_{M-1}A_{M-1}) \cdots S_{\lambda_1}^{G_1}(I - \lambda_1 A_1)x, \quad \forall x \in C.
\end{aligned} \tag{36}$$

If $0 < \lambda_i < 2\alpha_i$ for all $i = 1, 2, \ldots, M$, then $K : C \to C$ is nonexpansive.

Proof. Put $Q^i = S_{\lambda_i}^{G_i}(I - \lambda_i A_i) S_{\lambda_{i-1}}^{G_{i-1}}(I - \lambda_{i-1}A_{i-1}) \cdots S_{\lambda_1}^{G_1}(I - \lambda_1 A_1)$ for $i = 1, 2, \ldots, M$ and $Q^0 = I$. Then, $K = Q^M$. For all $x, y \in C$, it follows from (28) that

$$\begin{aligned}
&\|Kx - Ky\| \\
&\quad = \|Q^M x - Q^M y\| \\
&\quad = \left\|S_{\lambda_M}^{G_M}(I - \lambda_M A_M)Q^{M-1}x - S_{\lambda_M}^{G_M}(I - \lambda_M A_M)Q^{M-1}y\right\| \\
&\quad \le \left\|(I - \lambda_M A_M)Q^{M-1}x - (I - \lambda_M A_M)Q^{M-1}y\right\| \\
&\quad \le \left\|Q^{M-1}x - Q^{M-1}y\right\| \\
&\quad \vdots \\
&\quad \le \|Q^0 x - Q^0 y\| \\
&\quad = \|x - y\|,
\end{aligned} \tag{37}$$

which implies that K is nonexpansive. This completes the proof. □

Lemma 10. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $A_i : C \to H$ $(i = 1, 2, \ldots, M)$ be a nonlinear mapping. For given $(x_1^*, x_2^*, \ldots, x_M^*) \in C \times C \times \cdots \times C$, where $x^* = x_1^*$, $x_i^* = S_{\lambda_{i-1}}^{G_{i-1}}(I - \lambda_{i-1}A_{i-1})x_{i-1}^*$, for $i = 2, 3, \ldots, M$, and $x_1^* = S_{\lambda_M}^{G_M}(I - \lambda_M A_M)x_M^*$. Then, $(x_1^*, x_2^*, \ldots, x_M^*)$ is a solution of the problem (16) if and only if x^* is a fixed point of the mapping K defined as in Lemma 9.*

Proof. Let $(x_1^*, x_2^*, \ldots, x_M^*) \in C \times C \times \cdots \times C$ be a solution of the problem (16). Then, we have

$$\begin{aligned}
&G_M(x_1^*, x_1) + \langle A_M x_M^*, x_1 - x_1^*\rangle \\
&\quad + \frac{1}{\lambda_M}\langle x_1^* - x_M^*, x_1 - x_1^*\rangle \ge 0, \quad \forall x_1 \in C, \\
&G_{M-1}(x_M^*, x_M) + \langle A_{M-1}x_{M-1}^*, x_M - x_M^*\rangle \\
&\quad + \frac{1}{\lambda_{M-1}}\langle x_M^* - x_{M-1}^*, x_M - x_M^*\rangle \ge 0, \quad \forall x_M \in C, \\
&\qquad \vdots \\
&G_2(x_3^*, x_3) + \langle A_2 x_2^*, x_3 - x_3^*\rangle \\
&\quad + \frac{1}{\lambda_2}\langle x_3^* - x_2^*, x_3 - x_3^*\rangle \ge 0, \quad \forall x_3 \in C, \\
&G_1(x_2^*, x_2) + \langle A_1 x_1^*, x_2 - x_2^*\rangle \\
&\quad + \frac{1}{\lambda_1}\langle x_2^* - x_1^*, x_2 - x_2^*\rangle \ge 0, \quad \forall x_2 \in C, \\
&\qquad \Updownarrow
\end{aligned}$$

$$x_1^* = S_{\lambda_M}^{G_M}\left(I-\lambda_M A_M\right)x_M^*$$
$$x_2^* = S_{\lambda_1}^{G_1}\left(I-\lambda_1 A_1\right)x_1^*,$$
$$\vdots$$
$$x_{M-1}^* = S_{\lambda_{M-2}}^{G_{M-2}}\left(I-\lambda_{M-2} A_{M-2}\right)x_{M-2}^*,$$
$$x_M^* = S_{\lambda_{M-1}}^{G_{M-1}}\left(I-\lambda_{M-1} A_{M-1}\right)x_{M-1}^*,$$
$$\Updownarrow$$
$$x^* = S_{\lambda_M}^{G_M}\left(I-\lambda_M A_M\right)S_{\lambda_{M-1}}^{G_{M-1}} \times\left(I-\lambda_{M-1}A_{M-1}\right)\cdots S_{\lambda_1}^{G_1}\left(I-\lambda_1 A_1\right)x^* = Kx^*. \tag{38}$$

This completes the proof. □

3. Main Results

Theorem 11. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $\Theta_k : C\times C \to \mathbb{R}$ $(k = 1,2,\ldots,N)$ a finite family of bifunctions which satisfy (A1)–(A4), $\varphi_k : C \to \mathbb{R}$ $(k = 1,2,\ldots,N)$ a finite family of lower semicontinuous and convex functions, and $\Psi_k : C \to H$ $(k = 1,2,\ldots,N)$ a finite family of a μ_k-inverse strongly monotone mapping and $A_k : C \to H$ $(k = 1,2,\ldots,M)$ a finite family of an α_k-inverse strongly monotone mapping. Let S be a semigroup, and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $\operatorname{Fix}(\mathcal{S}) \neq \emptyset$. Let X be a left invariant subspace of $\ell^\infty(S)$ such that $1 \in X$ and the function $t \to \langle T(t)x, y\rangle$ is an element of X for $x \in C$ and $y \in H$. Let $\{\mu_n\}$ be a left regular sequence of means on X such that $\lim_{n\to\infty}\|\mu_{n+1}-\mu_n\| = 0$. Let $F : C \to H$ be a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa,\eta > 0$, and let $V : C \to H$ be an L-Lipschitzian mapping with a constant $L \geq 0$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \leq \gamma L < \tau$, where $\tau = \mu(\eta - \mu\kappa^2/2)$. Assume that $\mathcal{F} := \bigcap_{k=1}^N \operatorname{GMEP}(\Theta_k,\varphi_k,\Psi_k) \cap (K) \cap \operatorname{Fix}(\mathcal{S}) \neq \emptyset$, where K is defined as in Lemma 9. For given $x_1 \in C$, let $\{x_n\}$ be a sequence defined by*

$$\begin{aligned} u_n &= S_{r_{N,n}}^{(\Theta_N,\varphi_N)}\left(I-r_{N,n}\Psi_N\right)S_{r_{N-1,n}}^{(\Theta_{N-1},\varphi_{N-1})} \\ &\quad\times\left(I-r_{N-1,n}\Psi_{N-1}\right)\cdots S_{r_{1,n}}^{(\Theta_1,\varphi_1)}\left(I-r_{1,n}\Psi_1\right)x_n, \\ y_n &= S_{\lambda_M}^{G_M}\left(I-\lambda_M A_M\right)S_{\lambda_{M-1}}^{G_{M-1}} \\ &\quad\times\left(I-\lambda_{M-1}A_{M-1}\right)\cdots S_{\lambda_1}^{G_1}\left(I-\lambda_1 A_1\right)u_n, \\ x_{n+1} &= \beta_n x_n + \left(1-\beta_n\right)P_C\left[\alpha_n\gamma V x_n + \left(I-\alpha_n\mu F\right)T\left(\mu_n\right)y_n\right], \\ &\qquad \forall n \geq 1, \end{aligned} \tag{39}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0,1)$, and $\{r_{k,n}\}_{k=1}^N$ is a sequence such that $\{r_{k,n}\}_{k=1}^N \subset [a_k,b_k] \subset (0,2\gamma_k)$ satisfying the following conditions:

(C1) *$\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$;*

(C2) *$0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;*

(C3) *$\liminf_{n\to\infty} r_{k,n} > 0$ and $\lim_{n\to\infty}(r_{k,n}/r_{k,n+1}) = 1$ for all $k \in \{1,2,\ldots,N\}$.*

Then, the sequence $\{x_n\}$ defined by (39) converges strongly to $\hat{x} \in \mathcal{F}$ as $n \to \infty$, where $\hat{x}$ solves uniquely the variational inequality

$$\left\langle\left(\mu F-\gamma V\right)\hat{x}, \hat{x}-v\right\rangle \leq 0, \quad \forall v \in \mathcal{F}. \tag{40}$$

Equivalently, one has $\hat{x} = P_{\mathcal{F}}(I-\mu F+\gamma V)\hat{x}$.

Proof. Note that from condition (C1), we may assume, without loss of generality, that $\alpha_n \leq \min\{1, 1/2\tau\}$ for all $n \in \mathbb{N}$. First, we show that $\{x_n\}$ is bounded. Set

$$\begin{aligned} G_n^k &:= S_{r_{k,n}}^{(\Theta_k,\varphi_k)}\left(I-r_{k,n}\Psi_k\right)S_{r_{k-1,n}}^{(\Theta_{k-1},\varphi_{k-1})} \\ &\quad\times\left(I-r_{k-1,n}\Psi_{k-1}\right)\cdots S_{r_{1,n}}^{(\Theta_1,\varphi_1)}\left(I-r_{1,n}\Psi_1\right), \\ &\qquad \forall k \in \{1,2,\ldots,N\},\ n \in \mathbb{N}, \\ Q^i &:= S_{\lambda_i}^{G_i}\left(I-\lambda_i A_i\right)S_{\lambda_{i-1}}^{G_{i-1}}\left(I-\lambda_{i-1}A_{i-1}\right)\cdots S_{\lambda_1}^{G_1}\left(I-\lambda_1 A_1\right), \\ &\qquad \forall i \in \{1,2,\ldots,M\}, \end{aligned} \tag{41}$$

$G_n^0 = Q^0 = I$. Then, we have $u_n = G_n^N x_n$ and $y_n = Q^M u_n$. From Lemmas 3 and 9, we have that G_n^N and Q^M are nonexpansive. Take $x^* \in \mathcal{F}$; we have

$$\left\|u_n - x^*\right\| = \left\|G_n^N x_n - G_n^N x^*\right\| \leq \left\|x_n - x^*\right\|. \tag{42}$$

By Lemma 10, we have $x^* = Q^M x^*$. It follows from (42) that

$$\begin{aligned} \left\|y_n - x^*\right\| &= \left\|Q^M u_n - Q^M x^*\right\| \\ &\leq \left\|u_n - x^*\right\| \\ &\leq \left\|x_n - x^*\right\|. \end{aligned} \tag{43}$$

Set

$$z_n := P_C\left[\alpha_n\gamma V x_n + \left(I-\alpha_n\mu F\right)T\left(\mu_n\right)y_n\right], \quad \forall n \in \mathbb{N}. \tag{44}$$

Then, we can rewrite (39) as $x_{n+1} = \beta_n x_n + (1-\beta_n)z_n$. From Lemma 8 and (43), we have

$$\begin{aligned} &\left\|z_n - x^*\right\| \\ &\quad= \left\|P_C\left[\alpha_n\gamma V x_n + \left(I-\alpha\mu F\right)T\left(\mu_n\right)y_n\right] - P_C x^*\right\| \\ &\quad\leq \left\|\alpha_n\left(\gamma V x_n - \mu F x^*\right) + \left(I-\alpha_n\mu F\right)\left(T\left(\mu_n\right)y_n - x^*\right)\right\| \\ &\quad\leq \alpha_n\left\|\gamma V x_n - \mu F x^*\right\| + \left(1-\alpha_n\tau\right)\left\|T\left(\mu_n\right)y_n - x^*\right\| \\ &\quad\leq \alpha_n\gamma\left\|V x_n - V x^*\right\| + \alpha_n\left\|\gamma V x^* - \mu F x^*\right\| \\ &\qquad + \left(1-\alpha_n\tau\right)\left\|y_n - x^*\right\| \\ &\quad\leq \left(1-\alpha_n\left(\tau-\gamma L\right)\right)\left\|x_n - x^*\right\| + \alpha_n\left\|\gamma V x^* - \mu F x^*\right\|. \end{aligned} \tag{45}$$

It follows from (45) that

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|\beta_n(x_n-x^*)+(1-\beta_n)(z_n-x^*)\| \\ &\le \beta_n\|x_n-x^*\|+(1-\beta_n)\|z_n-x^*\| \\ &\le \beta_n\|x_n-x^*\|+(1-\beta_n) \\ &\quad\times[(1-\alpha_n(\tau-\gamma L))\|x_n-x^*\|+\alpha_n\|\gamma Vx^*-\mu Fx^*\|] \\ &= (1-\alpha_n(1-\beta_n)(\tau-\gamma L))\|x_n-x^*\| \\ &\quad+\alpha_n(1-\beta_n)(\tau-\gamma L)\frac{\|\gamma Vx^*-\mu Fx^*\|}{\tau-\gamma L}.\end{aligned} \tag{46}$$

By induction, we have

$$\|x_n-x^*\|\le\max\left\{\|x_1-x^*\|,\frac{\|\gamma Vx^*-\mu Fx^*\|}{\tau-\gamma L}\right\},\quad \forall n\ge 1. \tag{47}$$

Hence, $\{x_n\}$ is bounded, and so are $\{Vx_n\}$ and $\{FT(\mu_n)y_n\}$.

Next, we show that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{48}$$

Observe that

$$\lim_{n\to\infty}\|T(\mu_{n+1})y_n-T(\mu_n)y_n\|=0. \tag{49}$$

Indeed,

$$\begin{aligned}\|T(\mu_{n+1})y_n-T(\mu_n)y_n\| &= \sup_{\|z\|=1}|\langle T(\mu_{n+1})y_n-T(\mu_n)y_n,z\rangle| \\ &= \sup_{\|z\|=1}|(\mu_{n+1})_s\langle T(s)y_n,z\rangle-(\mu_n)_s\langle T(s)y_n,z\rangle| \\ &\le \|\mu_{n+1}-\mu_n\|\sup_{s\in S}\|T(s)y_n\|.\end{aligned} \tag{50}$$

Since $\{y_n\}$ is bounded and $\lim_{n\to\infty}\|\mu_{n+1}-\mu_n\|=0$, then (49) holds. We observe that

$$\begin{aligned}\|y_{n+1}-y_n\| &= \|Q^M u_{n+1}-Q^M u_n\| \\ &\le \|u_{n+1}-u_n\|.\end{aligned} \tag{51}$$

Let $\{\omega_n\}$ be a bounded sequence in C. Now, we show that

$$\lim_{n\to\infty}\|G_n^N\omega_n-G_{n+1}^N\omega_n\|=0. \tag{52}$$

For the previous purpose, put $D_n^k=S_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I-r_{k,n}\Psi_k)$, and we first show that

$$\lim_{n\to\infty}\|D_{n+1}^k\omega_n-D_n^k\omega_n\|=0,\quad \forall k\in\{1,2,\ldots,N\}. \tag{53}$$

In fact, since $D_n^k\omega_n\in\text{GMEP}(\Theta_k,\varphi_k,\Psi_k)$ and $D_{n+1}^k\omega_n\in\text{GMEP}(\Theta_k,\varphi_k,\Psi_k)$, we have

$$\begin{aligned}&\Theta_k(D_n^k\omega_n,y)+\varphi_k(y)-\varphi_k(D_n^k\omega_n) \\ &\quad+\langle\Psi_k\omega_n,y-D_n^k\omega_n\rangle \\ &\quad+\frac{1}{r_{k,n}}\langle y-D_n^k\omega_n,D_n^k\omega_n-\omega_n\rangle\ge 0, \\ &\qquad\forall y\in C,\end{aligned} \tag{54}$$

$$\begin{aligned}&\Theta_k(D_{n+1}^k\omega_n,y)+\varphi_k(y)-\varphi_k(D_{n+1}^k\omega_n) \\ &\quad+\langle\Psi_k\omega_n,y-D_{n+1}^k\omega_n\rangle \\ &\quad+\frac{1}{r_{k,n+1}}\langle y-D_{n+1}^k\omega_n,D_{n+1}^k\omega_n-\omega_n\rangle\ge 0, \\ &\qquad\forall y\in C.\end{aligned} \tag{55}$$

Substituting $y=D_{n+1}^k\omega_n$ in (54) and $y=D_n^k\omega_n$ in (55), then add these two inequalities, and using (A2), we obtain

$$\begin{aligned}&\left\langle D_{n+1}^k\omega_n-D_n^k\omega_n,\frac{1}{r_{k,n}}(D_n^k\omega_n-\omega_n)\right. \\ &\qquad\left.-\frac{1}{r_{k,n+1}}(D_{n+1}^k\omega_n-\omega_n)\right\rangle\ge 0.\end{aligned} \tag{56}$$

Hence,

$$\begin{aligned}&\left\langle D_{n+1}^k\omega_n-D_n^k\omega_n,D_n^k\omega_n-D_{n+1}^k\omega_n+D_{n+1}^k\omega_n\right. \\ &\qquad\left.-\omega_n-\frac{r_{k,n}}{r_{n,k+1}}(D_{n+1}^k\omega_n-\omega_n)\right\rangle\ge 0;\end{aligned} \tag{57}$$

we derive from (57) that

$$\begin{aligned}&\|D_{n+1}^k\omega_n-D_n^k\omega_n\|^2 \\ &\quad\le\left\langle D_{n+1}^k\omega_n-D_n^k\omega_n,\left(1-\frac{r_{k,n}}{r_{k,n+1}}\right)(D_{n+1}^k\omega_n-\omega_n)\right\rangle \\ &\quad\le\|D_{n+1}^k\omega_n-D_n^k\omega_n\|\left|1-\frac{r_{k,n}}{r_{k,n+1}}\right|\|D_{n+1}^k\omega_n-\omega_n\|,\end{aligned} \tag{58}$$

which implies that

$$\|D_{n+1}^k\omega_n-D_n^k\omega_n\|\le\left|1-\frac{r_{k,n}}{r_{k,n+1}}\right|\|D_{n+1}^k\omega_n-\omega_n\|. \tag{59}$$

Noticing that condition ($C3$) implies that (53) holds, from the definition of G_n^N and the nonexpansiveness of D_n^k, we have

$$\begin{aligned}
&\left\|G_n^N\omega_n - G_{n+1}^N\omega_n\right\| \\
&\quad= \left\|D_n^N G_n^{N-1}\omega_n - D_{n+1}^N G_{n+1}^{N-1}\omega_n\right\| \\
&\quad\le \left\|D_n^N G_n^{N-1}\omega_n - D_{n+1}^N G_n^{N-1}\omega_n\right\| \\
&\qquad+ \left\|D_{n+1}^N G_n^{N-1}\omega_n - D_{n+1}^N G_{n+1}^{N-1}\omega_n\right\| \\
&\quad\le \left\|D_n^N G_n^{N-1}\omega_n - D_{n+1}^N G_n^{N-1}\omega_n\right\| \\
&\qquad+ \left\|G_n^{N-1}\omega_n - G_{n+1}^{N-1}\omega_n\right\| \\
&\quad\le \left\|D_n^N G_n^{N-1}\omega_n - D_{n+1}^N G_n^{N-1}\omega_n\right\| \\
&\qquad+ \left\|D_n^{N-1} G_n^{N-2}\omega_n - D_{n+1}^{N-1} G_{n+1}^{N-2}\omega_n\right\| \\
&\qquad+ \left\|G_n^{N-2}\omega_n - G_{n+1}^{N-2}\omega_n\right\| \\
&\quad\le \left\|D_n^N G_n^{N-1}\omega_n - D_{n+1}^N G_n^{N-1}\omega_n\right\| \\
&\qquad+ \left\|D_n^{N-1} G_n^{N-2}\omega_n - D_{n+1}^{N-1} G_{n+1}^{N-2}\omega_n\right\| + \cdots \\
&\qquad+ \left\|D_n^2\omega_n - D_{n+1}^2\omega_n\right\| + \left\|D_n^1\omega_n - D_{n+1}^1\omega_n\right\|,
\end{aligned} \tag{60}$$

for which (52) follows by (53). Since $u_n = G_n^N x_n$ and $u_{n+1} = G_{n+1}^N x_{n+1}$, we have

$$\begin{aligned}
&\|u_n - u_{n+1}\| \\
&\quad= \left\|G_n^N x_n - G_{n+1}^N x_{n+1}\right\| \\
&\quad\le \left\|G_n^N x_n - G_{n+1}^N x_n\right\| + \left\|G_{n+1}^N x_n - G_{n+1}^N x_{n+1}\right\| \\
&\quad\le \left\|G_n^N x_n - G_{n+1}^N x_n\right\| + \|x_n - x_{n+1}\|.
\end{aligned} \tag{61}$$

Put a constant $M_1 > 0$ such that

$$\begin{aligned}
M_1 = \sup_{n\ge1} \{&\gamma\|Vx_{n+1}\| + \mu\|FT(t_{n+1})y_{n+1}\|, \\
&\gamma\|Vx_n\| + \mu\|FT(t_n)y_n\|\}.
\end{aligned} \tag{62}$$

From definition of $\{z_n\}$, we note that

$$\begin{aligned}
&\|z_{n+1} - z_n\| \\
&\quad= \|P_C[\alpha_{n+1}\gamma Vx_{n+1} + (I - \alpha_{n+1}\mu F)T(\mu_{n+1})y_{n+1}] \\
&\qquad -P_C[\alpha_n\gamma Vx_n + (I - \alpha_n\mu F)T(\mu_n)y_n]\| \\
&\quad\le \alpha_{n+1}\|\gamma Vx_{n+1} - \mu FT(\mu_{n+1})y_{n+1}\| \\
&\qquad+ \alpha_n\|\gamma Vx_{n+1} - \mu FT(\mu_n)y_n\| \\
&\qquad+ \|T(\mu_{n+1})y_{n+1} - T(\mu_n)y_n\| \\
&\quad\le \alpha_{n+1}\|\gamma Vx_{n+1} - \mu FT(\mu_{n+1})y_{n+1}\| \\
&\qquad+ \alpha_n\|\gamma Vx_{n+1} - \mu FT(\mu_n)y_n\| \\
&\qquad+ \|T(\mu_{n+1})y_{n+1} - T(\mu_{n+1})y_n\| \\
&\qquad+ \|T(\mu_{n+1})y_n - T(\mu_n)y_n\| \\
&\quad\le (\alpha_{n+1} + \alpha_n)M_1 + \|y_{n+1} - y_n\| \\
&\qquad+ \|T(\mu_{n+1})y_n - T(\mu_n)y_n\|.
\end{aligned} \tag{63}$$

It follows from (51), (61), and (63) that

$$\begin{aligned}
\|z_{n+1} - z_n\| \le\ &(\alpha_{n+1} + \alpha_n)M_1 + \left\|G_n^N x_n - G_{n+1}^N x_n\right\| \\
&+ \|x_{n+1} - x_n\| + \|T(\mu_{n+1})y_n - T(\mu_n)y_n\|.
\end{aligned} \tag{64}$$

From condition ($C1$), (49), and (52), we have

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0. \tag{65}$$

Hence, by Lemma 5, we obtain

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{66}$$

Consequently,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1 - \beta_n)\|z_n - x_n\| = 0. \tag{67}$$

From condition ($C1$), we have

$$\begin{aligned}
&\|z_n - T(\mu_n)y_n\| \\
&\quad= \|P_C[\alpha_n\gamma Vx_n + (I - \alpha_n\mu F)T(\mu_n)y_n] - P_C T(\mu_n)y_n\| \\
&\quad\le \alpha_n\|\gamma Vx_n - \mu FT(\mu_n)y_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{68}$$

From (66) and (68), we have

$$\begin{aligned}
\|x_n - T(\mu_n)y_n\| \le\ &\|x_n - z_n\| \\
&+ \|z_n - T(\mu_n)y_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{69}$$

Set $z_n = P_C v_n$, where $v_n = \alpha_n\gamma Vx_n + (I - \alpha_n\mu F)T(\mu_n)y_n$. From (25), we have

$$\begin{aligned}
\|z_n - x^*\| = \ &\langle v_n - x^*, z_n - x^*\rangle \\
&+ \langle P_C v_n - v_n, P_C v_n - x^*\rangle \\
\le\ &\langle v_n - x^*, z_n - x^*\rangle \\
=\ &\alpha_n\langle \gamma Vx_n - \mu Fx^*, z_n - x^*\rangle \\
&+ \langle (I - \alpha_n\mu F)(T(\mu_n)y_n - x^*), z_n - x^*\rangle
\end{aligned}$$

$$\begin{aligned}
&\leq (1-\alpha_n\tau)\|y_n-x^*\|\|z_n-x^*\| \\
&\quad + \alpha_n \langle \gamma V x_n - \mu F x^*, z_n - x^*\rangle \\
&\leq \frac{(1-\alpha_n\tau)}{2}\left(\|y_n-x^*\|^2+\|z_n-x^*\|^2\right) \\
&\quad + \alpha_n \langle \gamma V x_n - \mu F x^*, z_n - x^*\rangle \\
&\leq \frac{(1-\alpha_n\tau)}{2}\|y_n-x^*\|^2+\frac{1}{2}\|z_n-x^*\|^2 \\
&\quad + \alpha_n \langle \gamma V x_n - \mu F x^*, z_n - x^*\rangle .
\end{aligned} \tag{70}$$

It follows that

$$\begin{aligned}
\|z_n-x^*\|^2 &\leq \|y_n-x^*\|^2 + 2\alpha_n\langle \gamma V x_n - \mu F x^*, z_n - x^*\rangle \\
&\leq \|y_n-x^*\|^2 + 2\alpha_n\|\gamma V x_n - \mu F x^*\|\|z_n - x^*\| .
\end{aligned} \tag{71}$$

By the convexity of $\|\cdot\|^2$ and (71), we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&\quad = \|\beta_n(x_n-x^*)+(1-\beta_n)(z_n-x^*)\|^2 \\
&\quad \leq \beta_n\|x_n-x^*\|^2+(1-\beta_n)\|z_n-x^*\|^2 \\
&\quad \leq \beta_n\|x_n-x^*\|^2+(1-\beta_n) \\
&\qquad \times\left\{\|y_n-x^*\|^2+2\alpha_n\|\gamma V x_n-\mu F x^*\|\|z_n-x^*\|\right\} \\
&\quad \leq \beta_n\|x_n-x^*\|^2+(1-\beta_n)\|y_n-x^*\|^2+2\alpha_n(1-\beta_n)M_2,
\end{aligned} \tag{72}$$

where $M_2 > 0$ is an appropriate constant such that $M_2 = \sup_{n\geq 1}\{\|\gamma V x_n - \mu F x^*\|\|z_n - x^*\|\}$.

Next, we show that

$$\lim_{n\to\infty}\left\|G_n^{k-1}x_n - G_n^k x_n\right\| = 0, \quad \forall k\in\{1,2,\ldots,N\}. \tag{73}$$

From (28), we have

$$\begin{aligned}
&\left\|G_n^k x_n - x^*\right\|^2 \\
&\quad = \left\|S_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I-r_{k,n}\Psi_k)G_n^{k-1}x_n - S_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I-r_{k,n}\Psi_k)x^*\right\|^2 \\
&\quad \leq \left\|(I-r_{k,n}\Psi_k)G_n^{k-1}-(I-r_{k,n}\Psi_k)x^*\right\|^2 \\
&\quad \leq \left\|G_n^{k-1}x_n-x^*\right\|^2 + r_{k,n}(r_{k,n}-2\mu_k)\left\|\Psi_k G_n^{k-1}x_n-\Psi_k x^*\right\|^2 \\
&\quad \leq \|x_n-x^*\|^2 + r_{k,n}(r_{k,n}-2\mu_k)\left\|\Psi_k G_n^{k-1}x_n-\Psi_k x^*\right\|^2.
\end{aligned} \tag{74}$$

From (42), for all $k\in\{1,2,\ldots,N\}$, we note that

$$\begin{aligned}
\|y_n-x^*\|^2 &= \left\|Q^M u_n - x^*\right\| \leq \|u_n-x^*\|^2 \\
&= \left\|G_n^N x_n - x^*\right\|^2 \leq \left\|G_n^k x_n - x^*\right\|^2.
\end{aligned} \tag{75}$$

From (72) and (75), we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&\quad \leq \beta_n\|x_n-x^*\|^2+(1-\beta_n)\left\|G_n^N x_n-x^*\right\|^2 \\
&\qquad + 2\alpha_n(1-\beta_n)M_2 \\
&\quad \leq \beta_n\|x_n-x^*\|^2+(1-\beta_n)\left\|G_n^k x_n-x^*\right\|^2 \\
&\qquad + 2\alpha_n(1-\beta_n)M_2.
\end{aligned} \tag{76}$$

Substituting (74) into (72), we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&\quad \leq \beta_n\|x_n-x^*\|^2+(1-\beta_n) \\
&\qquad \times\left\{\|x_n-x^*\|^2+r_{k,n}(r_{k,n}-2\mu_k)\left\|\Psi_k G_n^{k-1}x_n-\Psi_k x^*\right\|^2\right\} \\
&\qquad + 2\alpha_n(1-\beta_n)M_2 \\
&\quad = \|x_n-x^*\|^2+(1-\beta_n)r_{k,n}(r_{k,n}-2\mu_k) \\
&\qquad \times\left\|\Psi_k G_n^{k-1}x_n-\Psi_k x^*\right\|^2 \\
&\qquad + 2\alpha_n(1-\beta_n)M_2,
\end{aligned} \tag{77}$$

which in turn implies that

$$\begin{aligned}
&(1-\beta_n)r_{k,n}(2\mu_k-r_{k,n})\left\|\Psi_k G_n^{k-1}x_n-\Psi_k x^*\right\|^2 \\
&\quad \leq \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+2\alpha_n(1-\beta_n)M_2 \\
&\quad \leq (\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\| \\
&\qquad + 2\alpha_n(1-\beta_n)M_2.
\end{aligned} \tag{78}$$

Since $\liminf_{n\to\infty}(1-\beta_n) > 0$, $0 < r_{k,n} < 2\mu_k$, for all $k \in \{1,2,\ldots,N\}$, from $(C1)$ and (67), we obtain that

$$\lim_{n\to\infty}\left\|\Psi_k G_n^{k-1}x_n-\Psi_k x^*\right\| = 0, \quad \forall k\in\{1,2,\ldots,N\}. \tag{79}$$

On the other hand, from Lemma 3 and (26), we have

$$\begin{aligned}
&\left\|G_n^k x_n - x^*\right\|^2 \\
&\quad = \left\|S_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I-r_{k,n}\Psi_k)G_n^{k-1}x_n - S_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I-r_{k,n}\Psi_k)x^*\right\|^2 \\
&\quad \leq \left\langle (I-r_{k,n}\Psi_k)G_n^{k-1}x_n-(I-r_{k,n}\Psi_k)x^*, G_n^k x_n - x^*\right\rangle \\
&\quad = \frac{1}{2}\left(\left\|(I-r_{k,n}\Psi_k)G_n^{k-1}x_n-(I-r_{k,n}\Psi_k)x^*\right\|^2\right. \\
&\qquad + \left\|G_n^k x_n - x^*\right\|^2 \\
&\qquad - \left\|(I-r_{k,n}\Psi_k)G_n^{k-1}x_n-(I-r_{k,n}\Psi_k)x^*\right. \\
&\qquad \left.\left. -\left(G_n^k x_n-x^*\right)\right\|^2\right)
\end{aligned}$$

$$\leq \frac{1}{2}\left(\left\|G_n^{k-1}x_n - x^*\right\|^2 + \left\|G_n^k - x^*\right\|^2 - \left\|G_n^{k-1}x_n - G_n^k x_n - r_{k,n}\left(\Psi_k G_n^{k-1}x_n - \Psi_k x^*\right)\right\|^2\right), \tag{80}$$

which in turn implies that

$$\begin{aligned}
&\left\|G_n^k x_n - x^*\right\|^2 \\
&\quad \leq \left\|G_n^{k-1}x_n - x^*\right\|^2 \\
&\qquad - \left\|G_n^{k-1}x_n - G_n^k - r_{k,n}(\Psi_k G_n^{k-1} - \Psi_k x^*)\right\|^2 \\
&\quad = \left\|G_n^{k-1}x_n - x^*\right\|^2 - \left\|G_n^{k-1}x_n - G_n^k x_n\right\| \\
&\qquad - r_{k,n}^2\left\|\Psi_k G_n^{k-1}x_n - \Psi_k x^*\right\|^2 \\
&\qquad + 2r_{k,n}\left\langle G_n^{k-1}x_n - G_n^k x_n, \Psi_k G_n^{k-1}x_n - \Psi_k x^*\right\rangle \\
&\quad \leq \left\|G_n^{k-1}x_n - x^*\right\|^2 - \left\|G_n^{k-1}x_n - G_n^k x_n\right\|^2 \\
&\qquad + 2r_{k,n}\left\|G_n^{k-1}x_n - G_n^k x_n\right\|\left\|\Psi_k G_n^{k-1}x_n - \Psi_k x^*\right\| \\
&\quad \leq \left\|x_n - x^*\right\|^2 - \left\|G_n^{k-1}x_n - G_n^k x_n\right\|^2 \\
&\qquad + 2r_{k,n}\left\|G_n^{k-1}x_n - G_n^k x_n\right\|\left\|\Psi_k G_n^{k-1}x_n - \Psi_k x^*\right\|.
\end{aligned} \tag{81}$$

Substituting (81) into (76), we have

$$\begin{aligned}
&\left\|x_{n+1} - x^*\right\|^2 \\
&\quad \leq \beta_n\left\|x_n - x^*\right\|^2 + (1-\beta_n) \\
&\qquad \times \left\{\left\|x_n - x^*\right\|^2 - \left\|G_n^{k-1}x_n - G_n^k x_n\right\|^2 \right.\\
&\qquad\qquad \left. +2r_{k,n}\left\|G_n^{k-1}x_n - G_n^k x_n\right\|\left\|\Psi_k G_n^{k-1}x_n - \Psi_k x^*\right\|\right\} \\
&\qquad + 2\alpha_n(1-\beta_n)M_2 \\
&\quad = \left\|x_n - x^*\right\|^2 - (1-\beta_n)\left\|G_n^{k-1}x_n - G_n^k x_n\right\| \\
&\qquad + 2r_{k,n}(1-\beta_n)\left\|G_n^{k-1}x_n - G_n^k x_n\right\|\left\|\Psi_k G_n^{k-1}x_n - \Psi_k x^*\right\| \\
&\qquad + 2\alpha_n(1-\beta_n)M_2,
\end{aligned} \tag{82}$$

which in turn implies that

$$\begin{aligned}
&(1-\beta_n)\left\|G_n^{k-1}x_n - G_n^k x_n\right\| \\
&\quad \leq \left\|x_n - x^*\right\|^2 - \left\|x_{n+1} - x^*\right\|^2 \\
&\qquad + 2r_{k,n}(1-\beta_n)\left\|G_n^{k-1}x_n - G_n^k x_n\right\|\left\|\Psi_k G_n^{k-1}x_n - \Psi_k x^*\right\| \\
&\qquad + 2\alpha_n(1-\beta_n)M_2 \\
&\quad \leq (\left\|x_n - x^*\right\| + \left\|x_{n+1} - x^*\right\|)\left\|x_{n+1} - x_n\right\| \\
&\qquad + 2r_{k,n}(1-\beta_n)\left\|G_n^{k-1}x_n - G_n^k x_n\right\|\left\|\Psi_k G_n^{k-1}x_n - \Psi_k x^*\right\| \\
&\qquad + 2\alpha_n(1-\beta_n)M_2.
\end{aligned} \tag{83}$$

Since $\liminf_{n\to\infty}(1-\beta_n) > 0$, from $(C1)$, (67), and (79), we obtain that (73) holds. Consequently,

$$\begin{aligned}
\left\|x_n - u_n\right\| &= \left\|G_n^0 - G_n^N x_n\right\| \\
&\leq \left\|G_n^0 x_n - G_n^1 x_n\right\| + \left\|G_n^1 x_n - G_n^2 x_n\right\| + \cdots \\
&\quad + \left\|G_n^{N-1}x_n - G_n^N x_n\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{84}$$

Next, we show that

$$\lim_{n\to\infty}\left\|A_i Q^{i-1}u_n - A_i Q^{i-1}x^*\right\| = 0, \quad \forall i \in \{1, 2, \ldots, M\}. \tag{85}$$

From (28), we have

$$\begin{aligned}
&\left\|Q^M u_n - Q^M x^*\right\|^2 \\
&\quad = \left\|S_{\lambda_M}^{G_M}(I - \lambda_M A_M)Q^{M-1}u_n - S_{\lambda_M}^{G_M}(I - \lambda_M A_M)Q^{M-1}x^*\right\|^2 \\
&\quad \leq \left\|(I - \lambda_M A_M)Q^{M-1}u_n - (I - \lambda_M A_M)Q^{M-1}x^*\right\|^2 \\
&\quad \leq \left\|Q^{M-1}u_n - Q^{M-1}x^*\right\|^2 + \lambda_M(\lambda_M - 2\alpha_M) \\
&\qquad \times \left\|A_M Q^{M-1}u_n - A_M Q^{M-1}x^*\right\|^2.
\end{aligned} \tag{86}$$

By induction, we have

$$\begin{aligned}
&\left\|Q^M u_n - Q^M x^*\right\|^2 \\
&\quad \leq \left\|u_n - x^*\right\| + \sum_{i=1}^{M}\lambda_i(\lambda_i - 2\alpha_i)\left\|A_i Q^{i-1}u_n - A_i Q^{i-1}x^*\right\|^2 \\
&\quad \leq \left\|x_n - x^*\right\| + \sum_{i=1}^{M}\lambda_i(\lambda_i - 2\alpha_i)\left\|A_i Q^{i-1}u_n - A_i Q^{i-1}x^*\right\|^2.
\end{aligned} \tag{87}$$

From (72) and (75), we have

$$\begin{aligned}
\left\|x_{n+1} - x^*\right\|^2 &\leq \beta_n\left\|x_n - x^*\right\|^2 + (1-\beta_n)\left\|Q^M u_n - Q^M x^*\right\|^2 \\
&\quad + 2\alpha_n(1-\beta_n)M_2.
\end{aligned} \tag{88}$$

Substituting (87) into (88), we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
&\quad\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)\\
&\qquad\times\Big\{\|x_n-x^*\|^2\\
&\qquad\quad+\sum_{i=1}^{M}\lambda_i(\lambda_i-2\alpha_i)\left\|A_iQ^{i-1}u_n-A_iQ^{i-1}x^*\right\|^2\Big\}\\
&\qquad+2\alpha_n(1-\beta_n)M_2\\
&\quad=\|x_n-x^*\|^2\\
&\qquad+(1-\beta_n)\sum_{i=1}^{M}\lambda_i(\lambda_i-2\alpha_i)\left\|A_iQ^{i-1}u_n-A_iQ^{i-1}x^*\right\|^2\\
&\qquad+2\alpha_n(1-\beta_n)M_2,
\end{aligned}\tag{89}$$

which in turn implies that

$$\begin{aligned}
&(1-\beta_n)\sum_{i=1}^{M}\lambda_i(2\alpha_i-\lambda_i)\left\|A_iQ^{i-1}u_n-A_iQ^{i-1}x^*\right\|^2\\
&\quad\le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+2\alpha_n(1-\beta_n)M_2\\
&\quad\le(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|\\
&\qquad+2\alpha_n(1-\beta_n)M_2.
\end{aligned}\tag{90}$$

Since $\liminf_{n\to\infty}(1-\beta_n)>0$, from $(C1)$ and (67), we obtain that (85) holds.

On the other hand, from (24) and (26), we have

$$\begin{aligned}
&\left\|Q^Mu_n-Q^Mx^*\right\|^2\\
&\quad=\left\|S_{\lambda_M}^{G_M}(I-\lambda_MA_M)Q^{M-1}u_n-S_{\lambda_M}^{G_M}(I-\lambda_MA_M)Q^{M-1}x^*\right\|^2\\
&\quad\le\Big\langle(I-\lambda_MA_M)Q^{M-1}u_n-(I-\lambda_MA_M)Q^{M-1}x^*,\\
&\qquad Q^Mu_n-Q^Mx^*\Big\rangle\\
&\quad=\frac{1}{2}\Big(\left\|(I-\lambda_MA_M)Q^{M-1}u_n-(I-\lambda_MA_M)Q^{M-1}x^*\right\|^2\\
&\qquad+\left\|Q^Mu_n-Q^Mx^*\right\|^2\\
&\qquad-\big\|(I-\lambda_MA_M)Q^{M-1}u_n\\
&\qquad\quad-(I-\lambda_MA_M)x^*-(Q^Mu_n-Q^Mx^*)\big\|^2\Big)\\
&\quad\le\frac{1}{2}\Big(\left\|Q^{M-1}u_n-Q^{M-1}x^*\right\|^2+\left\|Q^Mu_n-Q^Mx^*\right\|^2\\
&\qquad-\big\|Q^{M-1}u_n-Q^Mu_n+Q^Mx^*-Q^{M-1}x^*\\
&\qquad\quad-\lambda_M\left(A_MQ^{M-1}u_n-A_MQ^{M-1}x^*\right)\big\|^2\Big),
\end{aligned}\tag{91}$$

which in turn implies that

$$\begin{aligned}
&\left\|Q^Mu_n-Q^Mx^*\right\|^2\\
&\quad\le\left\|Q^{M-1}u_n-Q^{M-1}x^*\right\|^2\\
&\qquad-\big\|Q^{M-1}u_n-Q^Mu_n+Q^Mx^*-Q^{M-1}x^*\\
&\qquad\quad-\lambda_M(A_MQ^{M-1}u_n-A_MQ^{M-1}x^*)\big\|^2\\
&\quad=\left\|Q^{M-1}u_n-Q^{M-1}x^*\right\|^2\\
&\qquad-\left\|Q^{M-1}u_n-Q^Mu_n-Q^Mx^*-Q^{M-1}x^*\right\|^2\\
&\qquad-\lambda_M^2\left\|A_MQ^{M-1}u_n-A_MQ^{M-1}x^*\right\|^2\\
&\qquad+2\lambda_M\Big\langle Q^{M-1}u_n-Q^Mu_n+Q^Mx^*-Q^{M-1}x^*,\\
&\qquad\quad A_MQ^{M-1}u_n-A_MQ^{M-1}x^*\Big\rangle\\
&\quad\le\left\|Q^{M-1}u_n-Q^{M-1}x^*\right\|^2\\
&\qquad-\left\|Q^{M-1}u_n-Q^Mu_n+Q^Mx^*-Q^{M-1}x^*\right\|^2\\
&\qquad+2\lambda_M\left\|Q^{M-1}u_n-Q^Mu_n+Q^Mx^*-Q^{M-1}x^*\right\|\\
&\qquad\times\left\|A_MQ^{M-1}u_n-A_MQ^{M-1}x^*\right\|.
\end{aligned}\tag{92}$$

By induction, we have

$$\begin{aligned}
&\left\|Q^Mu_n-Q^Mx^*\right\|^2\\
&\quad\le\|u_n-x^*\|\\
&\qquad-\sum_{i=1}^{M}\left\|Q^{i-1}u_n-Q^iu_n+Q^ix^*-Q^{i-1}x^*\right\|^2\\
&\qquad+\sum_{i=1}^{N}2\lambda_i\left\|Q^{i-1}u_n-Q^iu_n+Q^ix^*-Q^{i-1}x^*\right\|\\
&\qquad\quad\times\left\|A_iQ^{i-1}u_n-A_iQ^{i-1}x^*\right\|\\
&\quad\le\|x_n-x^*\|-\sum_{i=1}^{M}\left\|Q^{i-1}u_n-Q^iu_n+Q^ix^*-Q^{i-1}x^*\right\|^2\\
&\qquad+\sum_{i=1}^{N}2\lambda_i\left\|Q^{i-1}u_n-Q^iu_n+Q^ix^*-Q^{i-1}x^*\right\|\\
&\qquad\quad\times\left\|A_iQ^{i-1}u_n-A_iQ^{i-1}x^*\right\|.
\end{aligned}\tag{93}$$

Substituting (93) into (88), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)\\
&\quad\times\Bigg\{\|x_n-x^*\|-\sum_{i=1}^{M}\left\|Q^{i-1}u_n-Q^iu_n+Q^ix^*-Q^{i-1}x^*\right\|^2\\
&\qquad+\sum_{i=1}^{M}2\lambda_i\left\|Q^{i-1}u_n-Q^iu_n+Q^ix^*-Q^{i-1}x^*\right\|\\
&\qquad\quad\times\left\|A_iQ^{i-1}u_n-A_iQ^{i-1}x^*\right\|\Bigg\}\\
&\quad+2\alpha_n(1-\beta_n)M_2\\
&\le\|x_n-x^*\|^2\\
&\quad-(1-\beta_n)\sum_{i=1}^{M}\left\|Q^{i-1}u_n-Q^iu_n+Q^ix^*-Q^{i-1}x^*\right\|^2\\
&\quad+(1-\beta_n)\\
&\quad\times\sum_{i=1}^{M}2\lambda_i\left\|Q^{i-1}u_n-Q^iu_n+Q^ix^*-Q^{i-1}x^*\right\|\\
&\qquad\times\left\|A_iQ^{i-1}u_n-A_iQ^{i-1}x^*\right\|\\
&\quad+2\alpha_n(1-\beta_n)M_2,
\end{aligned}\tag{94}$$

which in turn implies that

$$\begin{aligned}
(1-\beta_n)&\sum_{i=1}^{M}\left\|Q^{i-1}u_n-Q^iu_n+Q^ix^*-Q^{i-1}x^*\right\|^2\\
&\le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\
&\quad+(1-\beta_n)\sum_{i=1}^{M}2\lambda_i\left\|Q^{i-1}u_n-Q^iu_n+Q^ix^*-Q^{i-1}x^*\right\|\\
&\qquad\times\left\|A_iQ^{i-1}u_n-A_iQ^{i-1}x^*\right\|\\
&\quad+2\alpha_n(1-\beta_n)M_2\\
&\le(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|\\
&\quad+(1-\beta_n)\sum_{i=1}^{M}2\lambda_i\left\|Q^{i-1}u_n-Q^iu_n+Q^ix^*-Q^{i-1}x^*\right\|\\
&\qquad\times\left\|A_iQ^{i-1}u_n-A_iQ^{i-1}x^*\right\|\\
&\quad+2\alpha_n(1-\beta_n)M_2.
\end{aligned}\tag{95}$$

Since $\liminf_{n\to\infty}(1-\beta_n)>0$, from $(C1)$, (67), and (85), we obtain that

$$\lim_{n\to\infty}\left\|Q^{i-1}u_n-Q^iu_n+Q^ix^*-Q^{i-1}x^*\right\|=0,\quad \forall i\in\{1,2,\ldots,M\}.\tag{96}$$

Consequently,

$$\begin{aligned}
\|u_n-y_n\|&=\left\|Q^0u_n-Q^Mu_n\right\|\\
&\le\sum_{i=1}^{M}\left\|Q^{i-1}u_n-Q^iu_n+Q^ix^*-Q^{i-1}x^*\right\|\\
&\longrightarrow 0\quad\text{as } n\longrightarrow\infty.
\end{aligned}\tag{97}$$

It follows from (84) and (97) that

$$\begin{aligned}
\|x_n-y_n\|&\le\|x_n-u_n\|+\|u_n-y_n\|\\
&\longrightarrow 0\quad\text{as } n\longrightarrow\infty.
\end{aligned}\tag{98}$$

Next, we show that

$$\lim_{n\to\infty}\|x_n-T(t)x_n\|=0,\quad\forall t\in S.\tag{99}$$

Put

$$M^*=\max\left\{\|x_1-x^*\|,\frac{1}{\tau-\gamma L}\|\gamma Vx^*-\mu Fx^*\|\right\}.\tag{100}$$

Set $D=\{y\in C:\|y-x^*\|\le M^*\}$. We remark that D is nonempty, bounded, closed, and convex set, and $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are in D. We will show that

$$\limsup_{n\to\infty}\sup_{y\in D}\|T(\mu_n)y-T(t)T(\mu_n)y\|=0,\quad\forall t\in S.\tag{101}$$

To complete our proof, we follow the proof line as in [31] (see also [23, 32, 33]). Let $\epsilon>0$. By [34, Theorem 1.2], there exists $\delta>0$ such that

$$\overline{\text{co}}\,F_\delta(T(t);D)+B_\delta\subset F_\epsilon(T(t);D),\quad\forall t\in S.\tag{102}$$

Also by [34, Corollary 1.1], there exists a natural number N such that

$$\left\|\frac{1}{N+1}\sum_{i=0}^{N}T\left(t^is\right)y-T(t)\left(\frac{1}{N+1}\sum_{i=0}^{N}T\left(t^i\right)y\right)\right\|\le\delta,\tag{103}$$

for all $t, s \in S$ and $y \in D$. Let $t \in S$. Since $\{\mu_n\}$ is strongly left regular, there exists $n_0 \in \mathbb{N}$ such that $\|\mu_n - l_{t^i}^* \mu_n\| \leq \delta/(M^* + \|w\|)$ for all $n \geq n_0$ and $i = 1, 2, \ldots, N$. Then, we have

$$\begin{aligned} &\sup_{y\in D}\left\| T(\mu_n) y - \int \frac{1}{N+1}\sum_{i=0}^{N} T(t^i s) y d\mu_n(s)\right\| \\ &\quad = \sup_{y\in D}\sup_{\|z\|=1}\left| \langle T(\mu_n) y, z\rangle - \left\langle \int \frac{1}{N+1}\sum_{i=0}^{N} T(t^i s) y\, d\mu_n(s), z\right\rangle\right| \\ &\quad = \sup_{y\in D}\sup_{\|z\|=1}\left| \frac{1}{N+1}\sum_{i=0}^{N} (\mu_n)_s \langle T(s) y, z\rangle - \frac{1}{N+1}\sum_{i=0}^{N} (\mu_n)_s \left\langle T(t^i) y, z\right\rangle\right| \\ &\quad \leq \frac{1}{N+1} \times \sum_{i=0}^{N} \sup_{y\in D}\sup_{\|z\|=1} \left|(\mu_n)_s \langle T(s) y, z\rangle - (l_{t^i}^*\mu_n)_s \langle T(s) y, z\rangle\right| \\ &\quad \leq \max_{i=1,2,3,\ldots,N} \|\mu_n - l_{t^i}^*\mu_n\| (M^* + \|w\|) \leq \delta, \quad \forall n \geq n_0. \end{aligned} \tag{104}$$

On the other hand, by Lemma 2, we have

$$\begin{aligned} &\int \frac{1}{N+1}\sum_{i=0}^{N} T(t^i s) y\, d\mu_n(s) \\ &\quad \in \overline{\text{co}}\left\{ \frac{1}{N+1}\sum_{i=0}^{N} T(t^i) T(s) y : s \in S\right\}. \end{aligned} \tag{105}$$

Combining (103)–(105), we have

$$\begin{aligned} T(\mu_n) y &= \frac{1}{N+1}\sum_{i=1}^{N} T(t^i s) y\, d\mu_n(s) \\ &\quad + \left(T(\mu_n) y - \int \frac{1}{N+1}\sum_{i=1}^{N} T(t^i s) y\, d\mu_n(s)\right) \\ &\in \overline{\text{co}}\left\{ \frac{1}{N+1}\sum_{i=0}^{N} T(t^i)(T(s) y) : s \in S\right\} + B_\delta \\ &\subset \overline{\text{co}}\, F_\delta(T(t); D) + B_\delta, \end{aligned} \tag{106}$$

for all $y \in D$ and $n \geq n_0$. Therefore,

$$\limsup_{n\to\infty}\sup_{y\in D}\|T(\mu_n) y - T(t) T(\mu_n) y\| \leq \epsilon. \tag{107}$$

Since $\epsilon > 0$ is arbitrary, we obtain that (101) holds. Let $t \in S$ and $\epsilon > 0$. Then, there exists $\delta > 0$ satisfying (102). From (101) and condition $(C2)$, there exists $a, b \in (0, 1)$ such that $0 < a \leq \beta_n \leq b < 1$ and $T(\mu_n) y \in F_\delta(T(t); D)$ for all $y \in D$. From (69), there exists $k_0 \in \mathbb{N}$ such that $\|x_n - T(\mu_n) y_n\| < \delta/b$ for all $n > k_0$. Then, from (102) and (106), we have

$$\begin{aligned} x_{n+1} &= \beta_n x_n + (1-\beta_n) T(\mu_n) y_n \\ &= T(\mu_n) y_n + \beta_n (x_n - T(\mu_n) y_n) \\ &\in F_\delta(T(t); D) + B_\delta \subset F_\epsilon(T(t); D), \end{aligned} \tag{108}$$

for all $n > k_0$. Hence, $\limsup_{n\to\infty}\|x_n - T(t)x_n\| \leq \epsilon$. Since $\epsilon > 0$ is arbitrary, we obtain that (99) holds.

Next, we show that

$$\limsup_{n\to\infty}\langle \gamma V\hat{x} - \mu F\hat{x}, z_n - \hat{x}\rangle \leq 0, \tag{109}$$

where $\hat{x} = P_{\mathcal{F}}(I - \mu F + \gamma V)\hat{x}$. To show this, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma V\hat{x} - \mu F\hat{x}, x_n - \hat{x}\rangle = \lim_{i\to\infty}\left\langle \gamma V\hat{x} - \mu F\hat{x}, x_{n_i} - \hat{x}\right\rangle. \tag{110}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$. Now, we show that $v \in \mathcal{F}$.

(i) We first show that $v \in \text{Fix}(\mathcal{S})$. From (99), we have $\|x_n - T(t)x_n\| \to 0$ as $n \to \infty$ for all $t \in S$. Then, from Demiclosedness Principle 2.6, we get $v \in \text{Fix}(\mathcal{S})$.

(ii) We show that $v \in \text{Fix}(K)$, where K is defined as in Lemma 9. Then, from (97), we have

$$\begin{aligned} \|y_n - Ky_n\| &= \|Ku_n - Ky_n\| \leq \|u_n - y_n\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \end{aligned} \tag{111}$$

and from (98), we also have $\|x_n - Kx_n\| \to 0$. By Demiclosedness Principle 2.6, we get $v \in \text{Fix}(K)$.

(iii) We show that $v \in \bigcap_{k=1}^{N} \text{GMEP}(\Theta_k, \varphi_k, \Psi_k)$. Note that $G_n^k x_n = S_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I - r_{k,n}\Psi_k)G_n^{k-1}x_n$, for all $k \in \{1, 2, \ldots, N\}$. Then, we have

$$\begin{aligned} &\Theta_k\left(G_n^k x_n, y\right) + \varphi_k(y) - \varphi_k\left(G_n^k\right) \\ &\quad + \left\langle \Psi_k G_n^{k-1} x_n, y - G_n^k x_n\right\rangle \\ &\quad + \frac{1}{r_{k,n}}\left\langle y - G_n^k x_n, G_n^k x_n - G_n^{k-1} x_n\right\rangle \geq 0. \end{aligned} \tag{112}$$

Replacing n by n_i in the last inequality and using $(A2)$, we have

$$\begin{aligned} &\varphi_k(y) - \varphi_k\left(G_{n_i}^k x_{n_i}\right) + \left\langle \Psi_k G_{n_i}^{k-1} x_{n_i}, y - G_{n_i}^k x_{n_i}\right\rangle \\ &\quad + \frac{1}{r_{k,n}}\left\langle y - G_{n_i}^k x_{n_i}, G_{n_i}^k x_{n_i} - G_{n_i}^{k-1} x_{n_i}\right\rangle \geq \Theta_k\left(y, G_{n_i}^k x_{n_i}\right). \end{aligned} \tag{113}$$

Let $y_t = ty + (1-t)v$ for all $t \in (0,1]$ and $y \in C$. This implies that $y_t \in C$. Then, we have

$$\begin{aligned}
&\left\langle y_t - G_{n_i}^k x_{n_i}, \Psi_k y_t \right\rangle \\
&\quad \geq \varphi_k\left(G_{n_i}^k x_{n_i}\right) - \varphi_k\left(y_t\right) + \left\langle y_t - G_{n_i}^k x_{n_i}, \Psi_k y_t \right\rangle \\
&\quad\quad - \left\langle y_t - G_{n_i}^k x_{n_i}, \Psi_k G_n^{k-1} x_{n_i} \right\rangle \\
&\quad\quad - \left\langle y_t - G_{n_i}^k x_{n_i}, \frac{G_{n_i}^k x_{n_i} - G_{n_i}^{k-1} x_{n_i}}{r_{k,n_i}} \right\rangle + \Theta_k\left(y_t, G_{n_i}^k x_{n_i}\right) \\
&\quad = \varphi_k\left(G_{n_i}^k x_{n_i}\right) - \varphi_k\left(y_t\right) + \left\langle y_t - G_{n_i}^k x_{n_i}, \Psi_k y_t - \Psi_k G_{n_i}^k x_{n_i} \right\rangle \\
&\quad\quad + \left\langle y_t - G_{n_i}^k x_{n_i}, \Psi_k G_{n_i}^k x_{n_i} - \Psi_k G_{n_i}^{k-1} x_{n_i} \right\rangle \\
&\quad\quad - \left\langle y_t - G_{n_i}^k x_{n_i}, \frac{G_{n_i}^k x_{n_i} - G_{n_i}^{k-1} x_{n_i}}{r_{k,n_i}} \right\rangle + \Theta_k\left(y_t, G_{n_i}^k x_{n_i}\right).
\end{aligned} \tag{114}$$

From (73), we have $\|\Psi_k G_{n_i}^k x_{n_i} - \Psi_k G_{n_i}^{k-1} x_{n_i}\| \to 0$ as $i \to \infty$. Furthermore, by the monotonicity of Ψ_k, we obtain $\langle y_t - G_{n_i}^k x_{n_i}, \Psi_k y_t - \Psi_k G_{n_i}^k x_{n_i}\rangle \geq 0$. Then, from $(A4)$, we obtain

$$\left\langle y_t - v, \Psi_k y_t \right\rangle \geq \varphi_k(v) - \varphi_k\left(y_t\right) + \Theta_k\left(y_t, v\right). \tag{115}$$

Using $(A1)$, $(A4)$, and (115), we also obtain

$$\begin{aligned}
0 &= \Theta_k\left(y_t, y_t\right) + \varphi_k\left(y_t\right) - \varphi_k\left(y_t\right) \\
&\leq t\Theta_k\left(y_t, y\right) + (1-t)\Theta_k\left(y_t, v\right) + (1-t)\varphi_k(v) - \varphi_k\left(y_t\right) \\
&\leq t\left[\Theta_k\left(y_t, y\right) + \varphi_k(y) - \varphi_k\left(y_t\right)\right] + (1-t)\left\langle y_t - v, \Psi_k y_t\right\rangle \\
&= t\left[\Theta_k\left(y_t, y\right) + \varphi_k(y) - \varphi_k\left(y_t\right)\right] + (1-t)t\left\langle y - v, \Psi_k y_t\right\rangle,
\end{aligned} \tag{116}$$

and, hence,

$$0 \leq \Theta_k\left(y_t, y\right) + \varphi_k(y) - \varphi_k\left(y_t\right) + (1-t)\left\langle y - v, \Psi_k y_t\right\rangle. \tag{117}$$

Letting $t \to 0$ and using $(A3)$, we have, for each $y \in C$,

$$0 \leq \Theta_k(v, y) + \varphi_k(y) - \varphi_k(v) + \left\langle y - v, \Psi_k v\right\rangle. \tag{118}$$

This implies that $v \in \operatorname{GMEP}(\Theta_k, \varphi_k, \Psi_k)$. Hence, $v \in \bigcap_{k=1}^N \operatorname{GMEP}(\Theta_k, \varphi_k, \Psi_k)$. Therefore,

$$v \in \mathcal{F} := \bigcap_{k=1}^{N} \operatorname{GMEP}\left(\Theta_k, \varphi_k, \Psi_k\right) \cap \operatorname{Fix}(K) \cap \operatorname{Fix}(\mathcal{S}). \tag{119}$$

From (66) and (110), we obtain

$$\begin{aligned}
\limsup_{n\to\infty}\left\langle \gamma V\widehat{x} - \mu F\widehat{x}, z_n - \widehat{x}\right\rangle &= \limsup_{n\to\infty}\left\langle \gamma V\widehat{x} - \mu F\widehat{x}, x_n - \widehat{x}\right\rangle \\
&= \lim_{i\to\infty}\left\langle \gamma V\widehat{x} - \mu F\widehat{x}, x_{n_i} - \widehat{x}\right\rangle \\
&= \left\langle \gamma V\widehat{x} - \mu F\widehat{x}, v - \widehat{x}\right\rangle \leq 0.
\end{aligned} \tag{120}$$

Finally, we show that $x_n \to \widehat{x}$ as $n \to \infty$. Notice that $z_n = P_C v_n$, where $v_n = \alpha_n \gamma V x_n + (I - \alpha_n \mu F) T(\mu_n) y_n$. Then, from (25), we have

$$\begin{aligned}
\left\|z_n - \widehat{x}\right\|^2 &= \langle v_n - \widehat{x}, z_n - \widehat{x}\rangle + \langle P_C v_n - v_n, P_C v_n - \widehat{x}\rangle \\
&\leq \langle v_n - \widehat{x}, z_n - \widehat{x}\rangle \\
&= \alpha_n \gamma \langle V x_n - V\widehat{x}, z_n - \widehat{x}\rangle \\
&\quad + \alpha_n \langle \gamma V\widehat{x} - \mu F\widehat{x}, z_n - \widehat{x}\rangle \\
&\quad + \langle (I - \alpha_n \mu F)\left(T\left(\mu_n\right) y_n - \widehat{x}\right), z_n - \widehat{x}\rangle \\
&\leq \left(1 - \alpha_n\left(1 - \beta_n\right)(\tau - \gamma L)\right)\left\|x_n - \widehat{x}\right\|^2 \\
&\quad + \alpha_n \left\langle \gamma V\widehat{x} - \mu F\widehat{x}, z_n - \widehat{x}\right\rangle.
\end{aligned} \tag{121}$$

It follows from (121) that

$$\begin{aligned}
\left\|x_{n+1} - \widehat{x}\right\|^2 &\leq \beta_n\left\|x_n - \widehat{x}\right\|^2 + \left(1 - \beta_n\right)\left\|z_n - \widehat{x}\right\|^2 \\
&\leq \beta_n\left\|x_n - \widehat{x}\right\|^2 + \left(1 - \beta_n\right) \\
&\quad \times \left\{\left(1 - \alpha_n\left(1 - \beta_n\right)(\tau - \gamma L)\right)\left\|x_n - \widehat{x}\right\|^2 \right.\\
&\qquad \left. + \alpha_n \left\langle \gamma V\widehat{x} - \mu F\widehat{x}, z_n - \widehat{x}\right\rangle \right\} \\
&\leq \left(1 - \alpha_n\left(1 - \beta_n\right)(\tau - \gamma L)\right)\left\|x_n - \widehat{x}\right\|^2 \\
&\quad + \alpha_n\left(1 - \beta_n\right)\langle \gamma V\widehat{x} - \mu F\widehat{x}, z_n - \widehat{x}\rangle.
\end{aligned} \tag{122}$$

Put $\sigma_n := \alpha_n(1 - \beta_n)(\tau - \gamma L)$ and $\delta_n := \alpha_n(1 - \beta_n)\langle \gamma V\widehat{x} - \mu F\widehat{x}, z_n - \widehat{x}\rangle$. Then, (122) reduces to formula

$$\left\|x_{n+1} - \widehat{x}\right\|^2 \leq \left(1 - \sigma_n\right)\left\|x_n - \widehat{x}\right\|^2 + \delta_n. \tag{123}$$

It is easily seen that $\sum_{n=1}^{\infty} \sigma_n = \infty$, and (using (120))

$$\limsup_{n\to\infty} \frac{\delta_n}{\sigma_n} = \frac{1}{\tau - \gamma L} \limsup_{n\to\infty} \left\langle \gamma V\widehat{x} - \mu F\widehat{x}, z_n - \widehat{x}\right\rangle \leq 0. \tag{124}$$

Hence, by Lemma 7, we conclude that $x_n \to \widehat{x}$ as $n \to \infty$. This completes the proof. □

Using the results proved in [35] (see also [32]), we obtain the following results.

Corollary 12. *Let C, H, Θ_k, φ_k, Ψ_k, A_k, F, and V be the same as in Theorem 11. Let S and T be nonexpansive mappings on C with $ST = TS$. Assume that $\mathcal{F} := \operatorname{Fix}(S) \cap \operatorname{Fix}(T) \cap \bigcap_{n=1}^{\infty} \operatorname{GMEP}(\Theta_k, \varphi_k, \Psi_k) \cap \operatorname{Fix}(K) \neq \emptyset$, where K is defined as in*

Lemma 9. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_{k,n}\}_{k=1}^N$ be sequences satisfying (C1)–(C3). Then, the sequence $\{x_n\}$ defined by

$$\begin{aligned}
u_n &= S_{r_{N,n}}^{(\Theta_N,\varphi_N)}\left(I - r_{N,n}\Psi_N\right) S_{r_{N-1,n}}^{(\Theta_{N-1},\varphi_{N-1})} \\
&\quad \times \left(I - r_{N-1,n}\Psi_{N-1}\right)\cdots S_{r_{1,n}}^{(\Theta_1,\varphi_1)}\left(I - r_{1,n}\Psi_1\right)x_n, \\
y_n &= S_{\lambda_M}^{G_M}\left(I - \lambda_M A_M\right) S_{\lambda_{M-1}}^{G_{M-1}} \\
&\quad \times \left(I - \lambda_{M-1}A_{M-1}\right)\cdots S_{\lambda_1}^{G_1}\left(I - \lambda_1 A_1\right)u_n, \\
x_{n+1} &= \beta_n x_n + \left(1-\beta_n\right) \\
&\quad \times P_C\left[\alpha_n\gamma V x_n + \left(I - \alpha_n\mu F\right)\frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}S^iT^j y_n\right], \\
&\quad \forall n \geq 1,
\end{aligned} \tag{125}$$

converges strongly to $\hat{x} \in \mathcal{F}$, where $\hat{x}$ solves uniquely the variational inequality (40).

Corollary 13. *Let C, H, Θ_k, φ_k, Ψ_k, A_k, F, and V be the same as in Theorem 11. Let $\mathcal{S} = \{T(t) : t > 0\}$ be a strongly continuous nonexpansive semigroup on C. Assume that $\Omega := \mathrm{Fix}(\mathcal{S}) \cap \bigcap_{n=1}^{\infty}\mathrm{GMEP}(\Theta_k,\varphi_k,\Psi_k) \cap \mathrm{Fix}(K) \neq \emptyset$, where K is defined as in Lemma 9. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_{k,n}\}_{k=1}^N$ be sequences satisfying (C1)–(C3). Then, the sequence $\{x_n\}$ defined by*

$$\begin{aligned}
u_n &= S_{r_{N,n}}^{(\Theta_N,\varphi_N)}\left(I - r_{N,n}\Psi_N\right) S_{r_{N-1,n}}^{(\Theta_{N-1},\varphi_{N-1})} \\
&\quad \times \left(I - r_{N-1,n}\Psi_{N-1}\right)\cdots S_{r_{1,n}}^{(\Theta_1,\varphi_1)}\left(I - r_{1,n}\Psi_1\right)x_n, \\
y_n &= S_{\lambda_M}^{G_M}\left(I - \lambda_M A_M\right) S_{\lambda_{M-1}}^{G_{M-1}} \\
&\quad \times \left(I - \lambda_{M-1}A_{M-1}\right)\cdots S_{\lambda_1}^{G_1}\left(I - \lambda_1 A_1\right)u_n, \\
x_{n+1} &= \beta_n x_n + \left(1-\beta_n\right) \\
&\quad \times P_C\left[\alpha_n\gamma V x_n + \left(I - \alpha_n\mu F\right)\frac{1}{t_n}\int_0^{t_n} T(s)\, y_n ds\right], \\
&\quad \forall n \geq 1,
\end{aligned} \tag{126}$$

where $\{t_n\}$ is an increasing sequence in $(0,\infty)$ with $\lim_{n\to\infty}(t_n/t_{n+1}) = 1$, converges strongly to $\hat{x} \in \mathcal{F}$, where $\hat{x}$ solves uniquely the variational inequality (40).

Corollary 14. *Let C, H, Θ_k, φ_k, Ψ_k, A_k, F, and V be the same as in Theorem 11. Let $\mathcal{S} = \{T(t) : t > 0\}$ be a strongly continuous nonexpansive semigroup on C. Assume that $\Omega := \mathrm{Fix}(\mathcal{S}) \cap \bigcap_{n=1}^{\infty}\mathrm{GMEP}(\Theta_k,\varphi_k,\Psi_k) \cap \mathrm{Fix}(K) \neq \emptyset$, where K is defined as in Lemma 9. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_{k,n}\}_{k=1}^N$ be sequences satisfying (C1)–(C3). Then, the sequence $\{x_n\}$ defined by*

$$\begin{aligned}
u_n &= S_{r_{N,n}}^{(\Theta_N,\varphi_N)}\left(I - r_{N,n}\Psi_N\right) S_{r_{N-1,n}}^{(\Theta_{N-1},\varphi_{N-1})} \\
&\quad \times \left(I - r_{N-1,n}\Psi_{N-1}\right)\cdots S_{r_{1,n}}^{(\Theta_1,\varphi_1)}\left(I - r_{1,n}\Psi_1\right)x_n, \\
y_n &= S_{\lambda_M}^{G_M}\left(I - \lambda_M A_M\right) S_{\lambda_{M-1}}^{G_{M-1}} \\
&\quad \times \left(I - \lambda_{M-1}A_{M-1}\right)\cdots S_{\lambda_1}^{G_1}\left(I - \lambda_1 A_1\right)u_n, \\
x_{n+1} &= \beta_n x_n + \left(1-\beta_n\right) P_C \\
&\quad \times \left[\alpha_n\gamma V x_n + \left(I - \alpha_n\mu F\right)\right. \\
&\quad \left. \times \left(a_n\int_0^{\infty}\exp\left(-a_n s\right) T(s)\, y_n\right) ds\right], \quad \forall n \geq 1,
\end{aligned} \tag{127}$$

where $\{a_n\}$ is a decreasing sequence in $(0,\infty)$ with $\lim_{n\to\infty}a_n = 0$, converges strongly to $\hat{x} \in \Omega$, where $\hat{x}$ solves uniquely the variational inequality (40).

4. Some Applications

In this section, as applications, we will apply Theorem 11 to find minimum-norm solutions $\hat{x} = P_\Omega(0)$ of some variational inequalities. Namely, find a point $\hat{x}$ which solves uniquely the following quadratic minimization problem:

$$\|\hat{x}\|^2 = \min_{x\in\Omega}\|x\|^2. \tag{128}$$

Minimum-norm solutions have been applied widely in several branches of pure and applied sciences, for example, defining the pseudoinverse of a bounded linear operator, signal processing, and many other problems in a convex polyhedron and a hyperplane (see [36, 37]).

Recently, some iterative methods have been studied to find the minimum-norm fixed point of nonexpansive mappings and their generalizations (see, e.g. [38–49] and the references therein).

Using Theorem 11 and Corollaries 12, 13, and 14, we immediately have the following results, respectively.

Theorem 15. *Let C and H be the same as in Theorem 11. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $\mathcal{F} := \mathrm{Fix}(\mathcal{S}) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences satisfying (C1)–(C3). Then, the sequence $\{x_n\}$ defined by*

$$\begin{aligned}
x_{n+1} = \beta_n x_n + \left(1-\beta_n\right) P_C\left[\left(1-\alpha_n\right) T\left(\mu_n\right) x_n\right], \\
\forall n \geq 1,
\end{aligned} \tag{129}$$

converges strongly to $\hat{x} \in \mathcal{F}$, where $\hat{x} = P_{\mathcal{F}}(0)$ is the minimum-norm fixed point of $\mathcal{F}$, where $\hat{x}$ solves uniquely the quadratic minimization problem (128).

Theorem 16. *Let C and H be the same as in Corollary 12. Let S and T be nonexpansive mappings on C with $ST = TS$ such* *that* $\mathcal{F} := \mathrm{Fix}(S) \cap \mathrm{Fix}(T) \neq \emptyset$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences satisfying* $(C1)$–$(C3)$. *Then, the sequence* $\{x_n\}$ *defined by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) P_C \times \left[(1-\alpha_n) \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} S^i T^j x_n \right], \quad \forall n \geq 1, \tag{130}$$

converges strongly to $\hat{x} \in \mathcal{F}$, *where* $\hat{x} = P_{\mathcal{F}}(0)$ *is the minimum-norm fixed point of* $\mathcal{F}$, *where* $\hat{x}$ *solves uniquely the quadratic minimization problem* (128).

Theorem 17. *Let C and H be the same as in Corollary 13. Let* $\mathcal{S} = \{T(t) : t > 0\}$ *be a strongly continuous nonexpansive semigroup on C such that* $\mathcal{F} := \mathrm{Fix}(\mathcal{S}) \neq \emptyset$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences satisfying* $(C1)$–$(C3)$. *Then, the sequence* $\{x_n\}$ *defined by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) P_C \left[(1-\alpha_n) \frac{1}{t_n} \int_0^{t_n} T(s) x_n ds \right], \quad \forall n \geq 1, \tag{131}$$

where $\{t_n\}$ *is an increasing sequence in* $(0,\infty)$ *with* $\lim_{n\to\infty}(t_n/t_{n+1}) = 1$, *converges strongly to* $\hat{x} \in \mathcal{F}$, *where* $\hat{x} = P_{\mathcal{F}}(0)$ *is the minimum-norm fixed point of* $\mathcal{F}$, *where* $\hat{x}$ *solves uniquely the quadratic minimization problem* (128).

Theorem 18. *Let C and H be the same as in Corollary 14. Let* $\mathcal{S} = \{T(t) : t > 0\}$ *be a nonexpansive semigroup on C such that* $\mathcal{F} := \mathrm{Fix}(\mathcal{S}) \neq \emptyset$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences satisfying* $(C1)$–$(C3)$. *Then, the sequence* $\{x_n\}$ *defined by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) P_C \times \left[(1-\alpha_n) \left(a_n \int_0^{\infty} \exp(-a_n s) T(s) x_n ds \right) \right], \quad \forall n \geq 1, \tag{132}$$

where $\{a_n\}$ *is a decreasing sequence in* $(0,\infty)$ *with* $\lim_{n\to\infty} a_n = 0$, *converges strongly to* $\hat{x} \in \mathcal{F}$, *where* $\hat{x} = P_{\mathcal{F}}(0)$ *is the minimum-norm fixed point of* $\mathcal{F}$, *where* $\hat{x}$ *solves uniquely the quadratic minimization problem* (128).

Acknowledgment

The authors were supported by the Higher Education Research Promotion and National Research University Project of Thailand, Office of the Higher Education Commission (Grant no. NRU56000508).

References

[1] J.-W. Peng and J.-C. Yao, "A new hybrid-extragradient method for generalized mixed equilibrium problems, fixed point problems and variational inequality problems," *Taiwanese Journal of Mathematics*, vol. 12, no. 6, pp. 1401–1432, 2008.

[2] L.-C. Ceng and J.-C. Yao, "A hybrid iterative scheme for mixed equilibrium problems and fixed point problems," *Journal of Computational and Applied Mathematics*, vol. 214, no. 1, pp. 186–201, 2008.

[3] S. Takahashi and W. Takahashi, "Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space," *Nonlinear Analysis*, vol. 69, no. 3, pp. 1025–1033, 2008.

[4] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *The Mathematics Student*, vol. 63, no. 1–4, pp. 123–145, 1994.

[5] O. Chadli, N. C. Wong, and J. C. Yao, "Equilibrium problems with applications to eigenvalue problems," *Journal of Optimization Theory and Applications*, vol. 117, no. 2, pp. 245–266, 2003.

[6] O. Chadli, S. Schaible, and J. C. Yao, "Regularized equilibrium problems with application to noncoercive hemivariational inequalities," *Journal of Optimization Theory and Applications*, vol. 121, no. 3, pp. 571–596, 2004.

[7] I. V. Konnov, S. Schaible, and J. C. Yao, "Combined relaxation method for mixed equilibrium problems," *Journal of Optimization Theory and Applications*, vol. 126, no. 2, pp. 309–322, 2005.

[8] A. Moudafi and M. Théra, "Proximal and dynamical approaches to equilibrium problems," in *Ill-Posed Variational Problems and Regularization Techniques*, vol. 477 of *Lecture Notes in Economics and Mathematical Systems*, pp. 187–201, Springer, Berlin, Germany, 1999.

[9] L.-C. Zeng, S.-Y. Wu, and J.-C. Yao, "Generalized KKM theorem with applications to generalized minimax inequalities and generalized equilibrium problems," *Taiwanese Journal of Mathematics*, vol. 10, no. 6, pp. 1497–1514, 2006.

[10] P. Sunthrayuth and P. Kumam, "Viscosity approximation methods based on generalized contraction mappings for a countable family of strict pseudo-contractions, a general system of variational inequalities and a generalized mixed equilibrium problem in Banach spaces," *Mathematical and Computer Modelling*, 2013.

[11] L.-C. Ceng, Q. H. Ansari, and J.-C. Yao, "Some iterative methods for finding fixed points and for solving constrained convex minimization problems," *Nonlinear Analysis*, vol. 74, no. 16, pp. 5286–5302, 2011.

[12] L.-C. Ceng and J.-C. Yao, "A relaxed extragradient-like method for a generalized mixed equilibrium problem, a general system of generalized equilibria and a fixed point problem," *Nonlinear Analysis*, vol. 72, no. 3-4, pp. 1922–1937, 2010.

[13] L.-C. Ceng, Q. H. Ansari, and S. Schaible, "Hybrid extragradient-like methods for generalized mixed equilibrium problems, systems of generalized equilibrium problems and optimization problems," *Journal of Global Optimization*, vol. 53, no. 1, pp. 69–96, 2012.

[14] Y. Yao, Y.-C. Liou, and J.-C. Yao, "New relaxed hybrid-extragradient method for fixed point problems, a general system of variational inequality problems and generalized mixed equilibrium problems," *Optimization*, vol. 60, no. 3, pp. 395–412, 2011.

[15] V. Colao, G. Marino, and L. Muglia, "Viscosity methods for common solutions for equilibrium and hierarchical fixed point problems," *Optimization*, vol. 60, no. 5, pp. 553–573, 2011.

[16] K. R. Kazmi and S. H. Rizvi, "A hybrid extragradient method for approximating the common solutions of a variational inequality, a system of variational inequalities, a mixed equilibrium problem and a fixed point problem," *Applied Mathematics and Computation*, vol. 218, no. 9, pp. 5439–5452, 2012.

[17] P. Cholamjiak and S. Suantai, "Iterative methods for solving equilibrium problems, variational inequalities and fixed points of nonexpansive semigroups," *Journal of Global Optimization*, 2013.

[18] Y. Shehu, "Strong convergence theorem for nonexpansive semigroups and systems of equilibrium problems," *Journal of Global Optimization*, 2012.

[19] L.-C. Ceng, C.-Y. Wang, and J.-C. Yao, "Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities," *Mathematical Methods of Operations Research*, vol. 67, no. 3, pp. 375–390, 2008.

[20] A.T.-M. Lau, "Invariant means on almost periodic functions and fixed point properties," *The Rocky Mountain Journal of Mathematics*, vol. 3, pp. 69–76, 1973.

[21] A. T.-M. Lau, "Invariant means and fixed point properties of semigroup of nonexpansive mappings," *Taiwanese Journal of Mathematics*, vol. 12, no. 6, pp. 1525–1542, 2008.

[22] A. T.-M. Lau and W. Takahashi, "Invariant means and fixed point properties for non-expansive representations of topological semigroups," *Topological Methods in Nonlinear Analysis*, vol. 5, no. 1, pp. 39–57, 1995.

[23] A. T.-M. Lau, N. Shioji, and W. Takahashi, "Existence of nonexpansive retractions for amenable semigroups of nonexpansive mappings and nonlinear ergodic theorems in Banach spaces," *Journal of Functional Analysis*, vol. 161, no. 1, pp. 62–75, 1999.

[24] S. Saeidi, "Existence of ergodic retractions for semigroups in Banach spaces," *Nonlinear Analysis*, vol. 69, no. 10, pp. 3417–3422, 2008.

[25] W. Takahashi, "A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space," *Proceedings of the American Mathematical Society*, vol. 81, no. 2, pp. 253–256, 1981.

[26] T. Suzuki, "Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals," *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 227–239, 2005.

[27] K. Geobel and W. A. Kirk, *Topics on Metric Fixed-Point Theory*, vol. 28 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, 1990.

[28] H.-K. Xu, "Iterative algorithms for nonlinear operators," *Journal of the London Mathematical Society*, vol. 66, no. 1, pp. 240–256, 2002.

[29] P. Sunthrayuth and P. Kumam, "Iterative methods for variational inequality problems and fixed point problems of a countable family of strict pseudo-contractions in a q-uniformly smooth Banach space," *Fixed Point Theory and Applications*, vol. 2012, article 65, 2012.

[30] P. Sunthrayuth and P. Kumam, "Iterative algorithms approach to a general system of nonlinear variational inequalities with perturbed mappings and fixed point problems for nonexpansive semigroups," *Journal of Inequalities and Applications*, vol. 2012, article 133, 2012.

[31] S. Atsushiba and W. Takahashi, "Approximating common fixed points of nonexpansive semigroups by the Mann iteration process," *Annales Universitatis Mariae Curie-Skłodowska A*, vol. 51, pp. 1–16, 1997.

[32] A. T.-M. Lau, H. Miyake, and W. Takahashi, "Approximation of fixed points for amenable semigroups of nonexpansive mappings in Banach spaces," *Nonlinear Analysis*, vol. 67, no. 4, pp. 1211–1225, 2007.

[33] N. Shioji and W. Takahashi, "Strong convergence of averaged approximants for asymptotically nonexpansive mappings in Banach spaces," *Journal of Approximation Theory*, vol. 97, no. 1, pp. 53–64, 1999.

[34] R. E. Bruck, "On the convex approximation property and the asymptotic behavior of nonlinear contractions in Banach spaces," *Israel Journal of Mathematics*, vol. 38, no. 4, pp. 304–314, 1981.

[35] W. Takahashi, *Nonlinear Function Analysis*, Yokohama Publishers, Yokohama, 2000.

[36] P. J. S. G. Ferreira, "The existence and uniqueness of the minimum norm solution to certain linear and nonlinear problems," *Signal Processing*, vol. 55, no. 1, pp. 137–139, 1996.

[37] S. Fujishige, H. Sato, and P. Zhan, "An algorithm for finding the minimum-norm point in the intersection of a convex polyhedron and a hyperplane," *Japan Journal of Industrial and Applied Mathematics*, vol. 11, no. 2, pp. 245–264, 1994.

[38] S. Reich and H.-K. Xu, "An iterative approach to a constrained least squares problem," *Abstract and Applied Analysis*, vol. 8, pp. 503–512, 2003.

[39] X. Yang, Y.-C. Liou, and Y. Yao, "Finding minimum norm fixed point of nonexpansive mappings and applications," *Mathematical Problems in Engineering*, vol. 2011, Article ID 106450, 13 pages, 2011.

[40] Y. Cai, Y. Tang, and L. Liu, "Iterative algorithms for minimum-norm fixed point of non-expansive mapping in Hilbert space," *Fixed Point Theory and Applications*, vol. 2012, article 49, 2012.

[41] Y. Yao and H.-K. Xu, "Iterative methods for finding minimum-norm fixed points of nonexpansive mappings with applications," *Optimization*, vol. 60, no. 6, pp. 645–658, 2011.

[42] Y. Yao, R. Chen, and H.-K. Xu, "Schemes for finding minimum-norm solutions of variational inequalities," *Nonlinear Analysis*, vol. 72, no. 7-8, pp. 3447–3456, 2010.

[43] Y. Yao and Y.-C. Liou, "Some unified algorithms for finding minimum norm fixed point of nonexpansive semigroups in Hilbert spaces," *Analele Stiintifice ale Universitatii Ovidius Constanta*, vol. 19, no. 1, pp. 331–346, 2011.

[44] Y. Yao, M. A. Noor, and Y.-C. Liou, "Strong convergence of a modified extragradient method to the minimum-norm solution of variational inequalities," *Abstract and Applied Analysis*, vol. 2012, Article ID 817436, 9 pages, 2012.

[45] Y. Yao, Y.-C. Liou, and C.-P. Chen, "Algorithms construction for nonexpansive mappings and inverse-strongly monotone mappings," *Taiwanese Journal of Mathematics*, vol. 15, no. 5, pp. 1979–1998, 2011.

[46] Y. Yao and N. Shahzad, "New methods with perturbations for nonexpansive mappings in Hilbert spaces," *Fixed Point Theory and Applications*, vol. 2011, article 79, 2011.

[47] Y. Yao, V. Colao, G. Marino, and H.-K. Xu, "Implicit and explicit algorithms for minimum-norm fixed points of pseudocontractions in Hilbert spaces," *Taiwanese Journal of Mathematics*, vol. 16, no. 4, pp. 1489–1506, 2012.

[48] X. Yu, N. Shahzad, and Y. Yao, "Implicit and explicit algorithms for solving the split feasibility problem," *Optimization Letters*, vol. 6, no. 7, pp. 1447–1462, 2012.

[49] Y. Yao, Y.-C. Liou, and N. Shahzad, "Construction of iterative methods for variational inequality and fixed point problems," *Numerical Functional Analysis and Optimization*, vol. 33, no. 10, pp. 1250–1267, 2012.