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Research Article

H_∞-Based Pinning Synchronization of General Complex Dynamical Networks with Coupling Delays

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This paper investigates the synchronization of complex dynamical networks with coupling delays and external disturbances by applying local feedback injections to a small fraction of nodes in the whole network. Based on H_∞ control theory, some delay-independent and -dependent synchronization criteria with a prescribed H_∞ disturbances attenuation index are derived for such controlled networks in terms of linear matrix inequalities (LMIs), which guarantee that by placing a small number of feedback controllers on some nodes, the whole network can be pinned to reach network synchronization. A simulation example is included to validate the theoretical results.

1. Introduction

Large real communication networked systems have become a hot research topic for a rather long time [1, 2]. Typical examples include the Internet, which is an enormous network of many routers connected by physical or wireless links with information packets flowing on them, and traffic and transportation network [3–5]. Recently, dynamical processes of the complex dynamical networks such as synchronization have been extensively investigated [6–21]. The synchronization discussed here is a kind of typical collective behaviors and basic motions in nature [22]. However, in the case where the whole network cannot synchronize by itself, some controllers may be designed and applied to force the network to synchronize the configuration and states. What is more, the real-world traffic networks normally have a large number of nodes; therefore, it is usually difficult to control a network by adding the controllers to all nodes.

In manipulating various networks, pinning control is a simple and cost-effective technique for control, stabilization, and synchronization [7–12]. Wang and Chen [7] introduced a uniform model of complex dynamical networks by considering dynamical elements of a network as nodes and exploited a pinning control technique for scale-free chaotic dynamical networks, where local feedback injections were applied to a small portion of nodes so as to control the entire network. Reference [8] provided a clear explanation on why significantly less local controllers were needed by the specifically selective pinning scheme, which pinned the most highly connected nodes in a scale-free network, than that required by the randomly pinning scheme, and why there was no significant difference between the two schemes for controlling random-graph networks. In [9, 10], the idea of pinning control was again used to stabilize complex dynamical networks with nonlinear couplings onto some homogenous states. Several adaptive synchronization criteria were given in [11] by using Lyapunov stability theory and pinning control method. Reference [12] further solved some fundamental problems on how the local controllers on the pinned nodes affect the global network synchronization. The common feature of the work in [7–12] is that there are no coupling delays in the network. Networks with coupling delays have also received a great deal of attention. The time delays are usually caused by finite speed of information processing and communication, and their existence make dynamical behaviors of the networks much more complicated [13]. Pinning control of general complex dynamical networks with time-delay was given in [14, 15].

Moreover, in real physical systems, some noises or external disturbances always exist that may cause instability and

poor performance and thereby destroying the synchronization performance. The H_∞ control theory has been seen as an effective tool to reduce the effect of the noises or disturbances in chaos synchronization [23–25]. Reference [23] proposed the H_∞ control concept to reduce the effect of the disturbance on the available output to within a prescribed level. References [24, 25] proposed two dynamic feedback approaches for H_∞ synchronization of chaotic systems with and without time-delay. Therefore, how to reduce the effect of the noises or disturbances in synchronization of complex dynamical networks should also be paid attention to. New H_∞ synchronization and state estimation problems were proposed for an array of coupled discrete time-varying stochastic complex networks over a finite horizon [17]. However, to the best of our knowledge, the pinning synchronization of complex networks with coupling delays and external disturbances has not yet been established, which motivates the present study.

In this paper, we aim to deal with the pinning synchronization for complex dynamical networks with coupling delays and external disturbances by H_∞ control theory. By linearizing the controlled network on the synchronization state, the synchronization problem can be viewed as a normal H_∞ control problem. Then by some necessary model transforms, both delay-independent and -dependent synchronization conditions with a given H_∞ disturbances attenuation index γ are derived in terms of linear matrix inequalities (LMIs), which guarantee that by placing a small number of feedback controllers on some nodes, the whole network can be pinned to reach network synchronization. Finally, simulation results show that the network reaches the desired synchronization performance under the pinning control when there exist coupling delays and external disturbances.

Throughout this paper, I_n denotes the $n \times n$ identity matrix; $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the n-dimensional and the $n\times m$-dimensional Euclidean spaces, respectively; $X < 0$ represents that X is the symmetric negative matrix; given a matrix Y, $\overline{\sigma}(Y)$ represents its largest singular value; the superscripts T and -1 stand for matrix transposition and matrix inverse; in symmetric block matrices, $*$ is used as an ellipsis for terms induced by symmetry; the notation $\otimes$ denotes the Kronecker product; the space of square-integrable vector functions over $[0, \infty)$ is denoted by $L_2[0, \infty)$, and for $w(t) \in L_2[0, \infty)$, its normalized energy is defined by $\|w(t)\|_2 = (\int_0^\infty w^T(t)w(t)dt)^{1/2}$.

2. Model Description and Preliminaries

Consider the following complex dynamical network model with a coupling delay

$$\begin{aligned} \dot{x}_i(t) &= f(x_i(t)) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j(t-d) \\ &\quad + Bw_i(t) + u_i(t), \\ z_i(t) &= Cx_i(t), \quad i = 1, 2, \ldots, N, \end{aligned} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function, $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T \in \mathbb{R}^n$ are the state variables of node i, and $u_i(t) \in \mathbb{R}^n$ is the control input. $w_i(t) \in \mathbb{R}^p$ is the external disturbance input that belongs to $L_2[0, \infty)$, and $z_i(t) \in \mathbb{R}^q$ is the output vector. $d \geq 0$ is the time delay (we assume that all delays are the same in the network), the constant $c > 0$ is coupling strength, $\Gamma = (\gamma_{ij}) \in \mathbb{R}^{n\times n}$ is a inner-coupling matrix, if some pairs (i, j), $1 \leq i, j \leq n$ with $\gamma_{ij} \neq 0$, then it means that the two coupled nodes are linked through their ith and jth state variables, respectively. $A = (a_{ij}) \in \mathbb{R}^{N\times N}$ is the outer-coupling matrix of the network, in which a_{ij} is defined as follows: if there is a connection between node i and node $j (j \neq i)$, then $a_{ij} = a_{ji} = 1$; otherwise, $a_{ij} = a_{ji} = 0$ $(j \neq i)$, and the diagonal elements of matrix A are defined by

$$\begin{aligned} a_{ii} &= -\sum_{j=1, j\neq i}^{N} a_{ij} = -\sum_{j=1, j\neq i}^{N} a_{ji}, \\ &\qquad i = 1, 2, \ldots, N. \end{aligned} \tag{2}$$

Suppose that network (1) is connected in the sense that there are no isolated clusters, that is, A is an irreducible matrix. B and C are known real matrices with appropriate dimensions.

Before stating the main results of this paper, some preliminaries need to be given for convenient analysis.

Lemma 1. *Suppose A is a real symmetric and irreducible matrix, in which $a_{ij} \geq 0$ $(j \neq i)$ and $a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij}$, nonzero matrix $D = \mathrm{diag}(d_1, d_2, \ldots, d_N)$ satisfies $d_i \geq 0$ $(1 \leq i \leq N)$. Let $B_1 = A - D$, then*

(i) *all the eigenvalues of B_1 are less than 0;*

(ii) *there exists an orthogonal matrix $U \in \mathbb{R}^{N\times N}$ such that $U^T BU = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$, where $\lambda_1, \lambda_2, \ldots, \lambda_N$ are the eigenvalues of B_1.*

The proof of Lemma 1 is omitted here since it can be deduced using linear algebra theory such as in [26].

Lemma 2 (see [19]). *Assume that a and b are vectors, then for any positive-definite matrix X, the following inequality holds:*

$$-2a^T b \leq \inf_{X>0} \left\{a^T Xa + b^T X^{-1} b\right\}. \tag{3}$$

Lemma 3 (see [27]). *The LMI*

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} > 0, \tag{4}$$

where $Q(x) = Q^T(x)$, $R(x) = R^T(x)$, and $S(x)$ depend affinely on x, is equivalent to $R(x) > 0$, $Q(x) - S(x)R^{-1}(x)S^T(x) > 0$.

3. H_∞-Based Pinning Synchronization of Complex Dynamical Networks with a Coupling Time Delay

In this section, we present the synchronization of a network (1) to an isolate node, which is assumed as

$$\begin{aligned} \dot{s}(t) &= f(s(t)), \\ \overline{z}(t) &= Cs(t), \end{aligned} \tag{5}$$

in which $s(t)$ can be an equilibrium point, a periodic orbit, and even a chaotic orbit in the phase space. To achieve the above goal, we apply the pinning control strategy on a small fraction δ $(0 < \delta \leq 1)$ of the nodes in network (1). Suppose that nodes $i_1, i_2, \ldots, i_k$ are selected to be under pinning control, and $i_{k+1}, i_{l+2}, \ldots, i_N$ are the unselected nodes, where $k = [\delta N]$ stands for the smaller but nearest integer to the real number δN. Certainly, k is not less than 0 for pinning control. Then, the controlled network can be described as

$$\begin{aligned} \dot{x}_{i_l}(t) &= f\left(x_{i_l}(t)\right) + c\sum_{j=1}^{N} a_{i_l j}\Gamma x_j(t-d) \\ &\quad + Bw_{i_l}(t) + u_{i_l}, \quad l = 1, 2, \ldots, k, \\ \dot{x}_{i_l}(t) &= f\left(x_{i_l}(t)\right) + c\sum_{j=1}^{N} a_{i_l j}\Gamma x_j(t-d) \\ &\quad + Bw_{i_l}(t), \quad l = k+1,\ k+2, \ldots, N. \end{aligned} \tag{6}$$

For simplicity, we use the local linear negative feedback control law as follows:

$$\begin{aligned} u_{i_l} &= -cd_{i_l}\Gamma\left(x_{i_l}(t-d) - s(t-d)\right), \\ &\qquad l = 1, 2, \ldots, k, \end{aligned} \tag{7}$$

where the feedback gain $d_{i_l} > 0$.

Combining (6)-(7) and letting $d_{i_l} = 0$, $l = k+1, k+2, \ldots, N$, we have

$$\begin{aligned} \dot{x}_i(t) &= f\left(x_i(t)\right) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j(t-d) \\ &\quad - cd_i\Gamma\left(x_i(t-d) - s(t-d)\right) + Bw_i(t), \\ z_i(t) &= Cx_i(t), \quad i = 1, 2, \ldots, N. \end{aligned} \tag{8}$$

Denote

$$e_i(t) = x_i(t) - s(t), \quad i = 1, 2, \ldots, N. \tag{9}$$

The error dynamical system is then described by

$$\begin{aligned} \dot{e}_i(t) &= f\left(e_i(t) + s(t)\right) - f(s(t)) \\ &\quad + c\sum_{j=1}^{N} a_{ij}\Gamma e_j(t-d) - cd_i\Gamma e_i(t-d) + Bw_i(t), \\ \overline{z}_i(t) &= Ce_i(t), \quad i = 1, 2, \ldots, N. \end{aligned} \tag{10}$$

Linearizing the system (10) on the state $s(t)$ yields the following error dynamical system:

$$\begin{aligned} \dot{e}_i(t) &= J(t)e_i(t) + c\sum_{j=1}^{N} a_{ij}\Gamma e_j(t-d) \\ &\quad - cd_i\Gamma e_i(t-d) + Bw_i(t), \\ \overline{z}_i(t) &= Ce_i(t), \quad i = 1, 2, \ldots, N, \end{aligned} \tag{11}$$

where $J(t) \in \mathbb{R}^{n\times n}$ is the Jacobian of f on $s(t)$. Obviously, if all the errors $e_i(t)$ $(i = 1, 2, \ldots, N)$ uniformly asymptotically tend to zero, then the network (1) realizes synchronization.

Define the following matrices:

$$\begin{aligned} D &= \operatorname{diag}(d_1, d_2, \ldots, d_N) \in \mathbb{R}^{N\times N}, \\ e(t) &= \left(e_1^T(t), \ldots, e_N^T(t)\right)^T \in \mathbb{R}^{nN}, \\ w(t) &= \left(w_1^T(t), \ldots, w_N^T(t)\right)^T \in \mathbb{R}^{pN}, \\ z(t) &= \left(\overline{z}_1^T(t), \ldots, \overline{z}_N^T(t)\right)^T \in \mathbb{R}^{qN}. \end{aligned} \tag{12}$$

By using the Kronecker product, the error dynamical system (13) can be rewritten in the following matrix form:

$$\begin{aligned} \dot{e}(t) &= \left(I_N \otimes J(t)\right)e(t) + \left(cB_1 \otimes \Gamma\right) \\ &\quad \times e(t-d) + \left(I_N \otimes B\right)w(t), \\ z(t) &= \left(I_N \otimes C\right)e(t), \end{aligned} \tag{13}$$

where $B_1 = A - D$. For the system (13), the attenuating ability of its stability performance against external disturbances can be quantitatively measured by the H_∞ norm of the transfer function matrix $T_{zw}(s)$ from the external disturbance $w(t)$ to the controlled output $z(t)$, which is defined by [28]:

$$\begin{aligned} \left\|T_{zw}(s)\right\|_\infty &= \sup_{\nu\in R}\overline{\sigma}\left(T_{zw}(j\nu)\right) \\ &= \sup_{0\neq w(t)\in L_2[0,\infty)} \frac{\|z(t)\|_2}{\|w(t)\|_2}. \end{aligned} \tag{14}$$

To the reach desired synchronization performance against external disturbances, we need to design the local linear negative feedback control law u_{i_l}, $l = 1, 2, \ldots, k$ in (7) such that the following are achieved:

(i) the error dynamical system (13) is asymptotically stable with $w(t) = 0$;

(ii) $\|T_{zw}(s)\|_\infty < \gamma$ holds for a prescribed H_∞ index $\gamma > 0$, or equivalently the system (13), satisfying the following dissipation inequality:

$$\begin{aligned} &\int_0^\infty \|z(t)\|^2 \mathrm{d}t \\ &\qquad < \gamma^2 \int_0^\infty \|w(t)\|^2 \mathrm{d}t, \quad \forall w \in L_2[0,\infty). \end{aligned} \tag{15}$$

In summary, the pinning synchronization control problem of complex dynamical networks with external disturbances is transformed into the above H_∞ control problem.

It is easy to see that the matrix B_1 satisfies Lemma 1; let $\lambda_N \le \cdots \le \lambda_2 \le \lambda_1 < 0$ be the eigenvalues of matrix B_1; thus there exists an orthogonal matrix $U \in \mathbb{R}^{N\times N}$ such that

$$U^T B_1 U = \Lambda = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}. \tag{16}$$

Perform the orthogonal transform

$$\begin{aligned} \widehat{e}(t) &= \left(U^T \otimes I_n\right) e(t), \\ \widehat{w}(t) &= \left(U^T \otimes I_p\right) w(t), \\ \widehat{z}(t) &= \left(U^T \otimes I_q\right) z(t). \end{aligned} \tag{17}$$

Then from (13), we have

$$\begin{aligned} \dot{\widehat{e}}(t) &= \left(I_N \otimes J(t)\right)\widehat{e}(t) + (c\Lambda \otimes \Gamma)\,\widehat{e}(t-d) \\ &\quad + \left(I_N \otimes B\right)\widehat{w}(t) \\ \widehat{z}(t) &= \left(I_N \otimes C\right)\widehat{e}(t). \end{aligned} \tag{18}$$

Further, it can be easily proved that $T_{\widehat{z}\widehat{w}}(s) = (U^T \otimes I_n) T_{zw}(s)(U \otimes I_n)$, which leads to $\|T_{zw}(s)\|_\infty = \|T_{\widehat{z}\widehat{w}}(s)\|_\infty$ according to the definition of H_∞ norm defined in (14).

We can rewrite (17) into the following equations:

$$\begin{aligned} \dot{\widehat{e}}_i(t) &= J(t)\,\widehat{e}_i(t) + c\lambda_i \Gamma \widehat{e}_i(t-d) + B\widehat{w}_i(t), \\ \widehat{z}_i(t) &= C\widehat{e}_i(t), \quad i = 1, 2, \ldots, N. \end{aligned} \tag{19}$$

By the definition of H_∞ norm given in (14), if $\| T_{\widehat{z}_i\widehat{w}_i}(s)\|_\infty < \gamma$ holds for all $i = 1, 2, \ldots, N$, then $\|T_{\widehat{z}\widehat{w}}(s)\|_\infty < \gamma$ follows. To summarize, the error dynamical system (13) is asymptotically stable and $\| T_{zw}(s)\|_\infty < \gamma$, if the N equations (18) are all asymptotically stable and satisfy the H_∞ index γ.

3.1. Delay-Independent Condition

Theorem 4. *For a given index $\gamma > 0$, if there exist two symmetric positive-definite matrices $P, Q \in \mathbb{R}^{n\times n}$, such that*

$$\begin{bmatrix} J(t)^T P + PJ(t) & c\lambda_N P\Gamma & PB & C^T \\ * & -Q & 0 & 0 \\ * & * & -\gamma I & 0 \\ * & * & * & -\gamma I \end{bmatrix} > 0 \tag{20}$$

is satisfied. Then, the error dynamical system (13) *is asymptotically stable and $\| T_{zw}(s)\|_\infty < \gamma$ holds for all $d \in [0, \infty)$, which implies that network synchronization is reached asymptotically with H_∞ disturbance attenuation index γ.*

Proof. First, we study the stability of the system (18) without external disturbances, that is, $\widehat{w}_i(t) = 0$. Define a Lyapunov-Krasovskii function

$$V\left(\widehat{e}_i(t)\right) = \widehat{e}_i(t)^T P \widehat{e}_i(t) + \int_{t-d}^{t} \widehat{e}_i(\theta)^T Q \widehat{e}_i(\theta)\, d\theta \tag{21}$$

with positive definite matrices $P, Q \in \mathbb{R}^{n\times n}$. The time derivative of $V(\widehat{e}_i(t))$ is

$$\begin{aligned} \dot{V}\left(\widehat{e}_i(t)\right) &= \widehat{e}_i(t)^T \left[J^T(t)P + PJ(t) + Q\right]\widehat{e}_i(t) \\ &\quad + 2\widehat{e}_i(t-d)^T c\lambda_i \Gamma^T P \widehat{e}_i(t) \\ &\quad - \widehat{e}_i(t-d)^T Q \widehat{e}_i(t-d). \end{aligned} \tag{22}$$

From Lemma 2, we have

$$\begin{aligned} 2\widehat{e}_i(t-d)^T c\lambda_i \Gamma^T P \widehat{e}_i(t) &\le \widehat{e}_i(t-d)^T Q \widehat{e}_i(t-d) \\ &\quad + c^2\lambda_i^2 \widehat{e}_i(t)^T P\Gamma Q^{-1}\Gamma^T P \widehat{e}_i(t). \end{aligned} \tag{23}$$

So, we can obtain

$$\begin{aligned} \dot{V}\left(\widehat{e}_i(t)\right) &= \widehat{e}_i(t)^T \left[J^T(t)P + PJ(t) + Q \right. \\ &\quad \left. + c^2\lambda_i^2 P\Gamma Q^{-1}\Gamma^T P\right]\widehat{e}_i(t) \\ &\le \widehat{e}_i(t)^T \left[J^T(t)P + PJ(t) + Q \right. \\ &\quad \left. + c^2\lambda_N^2 P\Gamma Q^{-1}\Gamma^T P\right]\widehat{e}_i(t). \end{aligned} \tag{24}$$

By Lemma 3, inequality (19) implies

$$J^T(t)P + PJ(t) + Q + c^2\lambda_N^2 P\Gamma Q^{-1}\Gamma^T P < 0. \tag{25}$$

Thus, the system (18) is asymptotically stable when $\widehat{w}_i(t) = 0$.

Subsequently, we discuss the performance of system (18) with nonzero disturbance $\widehat{w}_i(t)$. Consider the cost function

$$J\left(\widehat{w}_i\right) = \int_0^\infty \left(\gamma^{-1}\widehat{z}_i(t)^T\widehat{z}_i(t) - \gamma\widehat{w}_i(t)^T\widehat{w}_i(t)\right)dt. \tag{26}$$

Under the assumption that the initial state is zero-valued, we have

$$\begin{aligned} J\left(\widehat{w}_i\right) &= \int_0^\infty \left(\gamma^{-1}\widehat{z}_i(t)^T\widehat{z}_i(t) - \gamma\widehat{w}_i(t)^T\widehat{w}_i(t)\right) \\ &\quad + \dot{V}\left(\widehat{e}_i(t)\right)dt - V\left(\widehat{e}_i(\infty)\right) \\ &\le \int_0^\infty \left(\gamma^{-1}\widehat{z}_i(t)^T\widehat{z}_i(t) - \gamma\widehat{w}_i(t)^T\widehat{w}_i(t)\right) \\ &\quad + \dot{V}\left(\widehat{e}_i(t)\right)dt \\ &\le \int_0^\infty \xi^T(t)\,\Theta\,\xi(t)\,dt, \end{aligned} \tag{27}$$

where

$$\begin{aligned} \Theta &= \begin{bmatrix} J^T(t)P + PJ(t) + Q + c^2\lambda_N^2 P\Gamma Q^{-1}\Gamma^T P + \gamma^{-1}C^T C & PB \\ * & -\gamma I \end{bmatrix}, \\ \xi(t) &= \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}. \end{aligned} \tag{28}$$

By Lemma 3, inequality (19) is equivalent to $\Theta < 0$; thus, we have

$$\int_0^\infty \widehat{z}_i(t)^T\widehat{z}_i(t)\,dt \le \gamma^2 \int_0^\infty \widehat{w}_i(t)^T\widehat{w}_i(t)\,dt \tag{29}$$

for N equations in the system (18), which completes the proof. □

3.2. Delay-Dependent Condition

Lemma 5 (see [29]). *Consider the following time-delay system:*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-d) + B\omega(t), \\ \widehat{z}(t) &= Cx(t), \end{aligned} \tag{30}$$

where $d \in [0, d^*]$ is the constant time delay. Given a scalar $\gamma > 0$, the system is asymptotically stable and with H_∞ perform-ance index γ, if there exist symmetric positive-definite matrices $P_1, S, R = \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix}$ and matrices $P_2, P_3, W = \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix}$ such that

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & P_2^T B & d^*\left(W_1^T + P_1\right) & d^*\left(W_3^T + P_2^T\right) & -W_3^T A_d & C^T \\ * & \Xi_{22} & P_3^T B & d^* W_2^T & d^*\left(W_4^T + P_3^T\right) & -W_4^T A_d & 0 \\ * & * & -\gamma^2 I & 0 & 0 & 0 & 0 \\ * & * & * & -d^* R_1 & -d^* R_2 & 0 & 0 \\ * & * & * & * & -d^* R_3 & -S & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0 \tag{31}$$

holds, where

$$\begin{aligned} \Xi_{11} &= \left(A + A_d\right)^T P_2 + P_2^T\left(A + A_d\right) \\ &\quad + \left(W_3^T A_d + A_d^T W_3\right) + S, \\ \Xi_{12} &= P_1 - P_2^T + \left(A + A_d\right)^T P_2 + A_d^T W_4 \\ \Xi_{22} &= -P_3 - P_3^T + d^* A_d^T R_3 A_d. \end{aligned} \tag{32}$$

Theorem 6. *For a given index $\gamma > 0$, the error dynamical system* (13) *is asymptotically stable and $\|T_{zw}(s)\|_\infty < \gamma$ for all $d \in [0, d^*]$; that is, network synchronization is achieved asymptotically with H_∞ index γ under the pinning control* (7) *if there exist symmetric positive-definite matrices $P_1, S, R = \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix}$ and matrices $P_2, P_3, W = \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix}$ such that*

$$\Phi = \begin{bmatrix} \phi_{11} & \phi_{12} & P_2^T B & d^*\left(W_1^T + P_1\right) & d^*\left(W_3^T + P_2^T\right) & -c\lambda_i W_3^T \Gamma & C^T \\ * & \phi_{22} & P_3^T B & d^* W_2^T & d^*\left(W_4^T + P_3^T\right) & -c\lambda_i W_4^T \Gamma & 0 \\ * & * & -\gamma^2 I & 0 & 0 & 0 & 0 \\ * & * & * & -d^* R_1 & -d^* R_2 & 0 & 0 \\ * & * & * & * & -d^* R_3 & -S & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0 \tag{33}$$

is satisfied for $i = 1, 2, \ldots, N$, where

$$\begin{aligned} \phi_{11} &= \left(J(t) + c\lambda_i\Gamma\right)^T P_2 + P_2^T\left(J(t) + c\lambda_i\Gamma\right) \\ &\quad + \left(c\lambda_i W_3^T\Gamma + c\lambda_i\Gamma^T W_3\right) + S, \\ \phi_{12} &= P_1 - P_2^T + \left(J(t) + c\lambda_i\Gamma\right)^T P_2 + c\lambda_i\Gamma^T W_4 \\ \phi_{22} &= -P_3 - P_3^T + d^* c^2\lambda_i^2\Gamma^T R_3\Gamma. \end{aligned} \tag{34}$$

Proof. Due to Lemma 5, all N equations in the system (18) are asymptotically stable and satisfy the H_∞ performance index γ, which implies that the error dynamical system (13) is asymptotically stable and $\|T_{zw}(s)\|_\infty < \gamma$ for all $d \in [0, d^*]$. □

Remark 7. Due to the convex property of LMIs, actually only two LMIs associated with the largest and the smallest eigenvalues λ_1 and λ_N need to be verified. This helps to significantly reduce the computational burden caused by the number of nodes in network, especially when the number N is very large.

Remark 8. According to Theorems 4 and 6, if the network is given, the synchronization conditions are determined by $J(t)$, Γ, B, C, the eigenvalues of matrix B_1, the coupling strength c, and the disturbance attenuation index γ. Hence, the stabilization of such networks using feedback control is determined by the dynamics of each uncoupled node, the inner-coupling matrix, the coupling matrix, the disturbance input matrix, the output matrix, and the feedback gain

matrix of the network. Generally, the number of controllers is preferred to be very small with the entire network size N. According to Lemma 1, B_1 is symmetric and negative even if D has only one nonzero element. So in such case, appropriate c, γ, and D may make Lemma 1 holds. It is concluded that such a complex dynamical network can be pinned to its equilibrium by using only one controller.

Remark 9. In fact, the multiagent systems are the special cases of complex networks, so if we choose all the agents to be pinned, the method developed in this paper can be extended to the multi-agent systems in [30] and the corresponding delay-independent and -dependent consensus criteria can be derived.

4. Simulation Results

In this section, we give an example to demonstrate the effectiveness of the proposed method.

We assume that the controlled network (1) consists of 10 identical Lorenz systems [31], which is described by

$$\begin{aligned} \dot{x}_i(t) &= f(x_i(t)) + c\sum_{j=1}^{10} a_{ij}\Gamma x_j(t-d) \\ &\quad + Bw_i(t) + u_i(t), \\ z_i(t) &= Cx_i(t), \quad i = 1, 2, \ldots, 10, \end{aligned} \tag{35}$$

where the inner-coupling matrix is $\Gamma = \mathrm{diag}(1, 1, 1)$, the coupling matrix is

$$A = \begin{bmatrix} -4 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -3 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & -2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -2 \end{bmatrix}, \tag{36}$$

$B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and $C = [1\ 0\ 0]$. In order to clearly reflect the effect of external disturbances to the synchronization performance, the disturbance is assumed to be $w_i(t) = 7\sin(0.2t)$. The node dynamics are then given by

$$\begin{aligned} \dot{x}_{i1} &= \sigma(x_{i2} - x_{i1}), \\ \dot{x}_{i2} &= rx_{i1} - x_{i2} - x_{i1}x_{i3}, \\ \dot{x}_{i3} &= x_{i1}x_{i2} - bx_{i3}, \end{aligned} \tag{37}$$

where σ is called Prandtl number and assumed as $\sigma > 1$, $r > 1$. There are three equilibriums which are the origin

$$\begin{aligned} p &= \left[\sqrt{b(r-1)} \quad \sqrt{b(r-1)} \quad r-1\right]^T, \\ q &= \left[-\sqrt{b(r-1)} \quad -\sqrt{b(r-1)} \quad r-1\right]^T. \end{aligned} \tag{38}$$

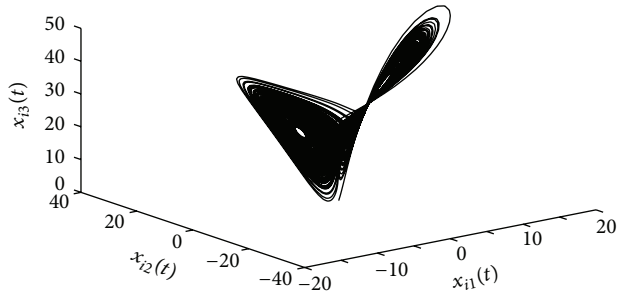

Figure 1: Lorenz chaos.

The Lorenz system is symmetrical with respect to the x_{i3} axis. Denote $r^* = \sigma(\sigma+b+3)/(\sigma-b-1)$. It is known that p and q are unstable and chaos occurs if $r > r^*$ Let $\sigma = 10$, $r = 28$, and $b = 8/3$, which means that p and q are unstable equilibria, and chaos occurs at the same time. A typical behavior of a Lorenz chaos system is the butterfly effect shown in Figure 1. Now the objective here is to stabilize this network (30) onto the unstable equilibrium p with H_∞ disturbance attenuation index γ by applying the local linear feedback pinning control (8).

Consider the network with a bounded coupling delay $0 \le d \le d^* = 0.02$, and the coupling strength is chosen as $c = 3$. Take the initial condition as $x_i(s) = 0$, $s \in [-0.02, 0]$. Three nodes are pinned and the feedback gains are designed as $d_1 = d_7 = d_9 = 2$; then a minimum value of $\gamma = 0.82$ is obtained by applying Theorem 6 with

$$P_1 = \begin{bmatrix} 0.7590 & -0.0697 & 0.0334 \\ -0.0697 & 0.5061 & -0.8253 \\ 0.0334 & -0.8253 & 4.0335 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0.7902 & 0.4187 & 0.9993 \\ -0.4681 & 1.0807 & -1.3012 \\ 0.0072 & -0.3312 & 1.3426 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 1.2139 \times 10^4 & -7.4842 \times 10^3 & 1.8941 \times 10^4 \\ 1.6926 \times 10^4 & 1.3199 \times 10^4 & -1.6672 \times 10^4 \\ -1.6925 \times 10^4 & -1.3198 \times 10^4 & 1.6672 \times 10^4 \end{bmatrix},$$

$$S = \begin{bmatrix} 2.8504 \times 10^5 & 2.6204 \times 10^5 & 2.6492 \times 10^5 \\ 2.6204 \times 10^5 & 1.3184 \times 10^6 & -2.0552 \times 10^6 \\ 2.6492 \times 10^5 & -2.0552 \times 10^6 & 7.4047 \times 10^6 \end{bmatrix},$$

$$R_3 = \begin{bmatrix} 3.9131 \times 10^8 & -3.9789 \times 10^7 & -4.0882 \times 10^7 \\ -3.9789 \times 10^7 & 3.8412 \times 10^8 & -2.5207 \times 10^7 \\ -4.0882 \times 10^7 & -2.5207 \times 10^7 & 4.1503 \times 10^8 \end{bmatrix}. \tag{39}$$

Figure 2 shows the control results when $d = 0.015$. Figure 3 shows the control results when we select 5 nodes as the controlled nodes. It can be seen that if we select more pinned nodes, it will cost less time to reach stable state.

5. Conclusions

This paper addressed the pinning synchronization problem for a group of complex dynamical networks with coupling

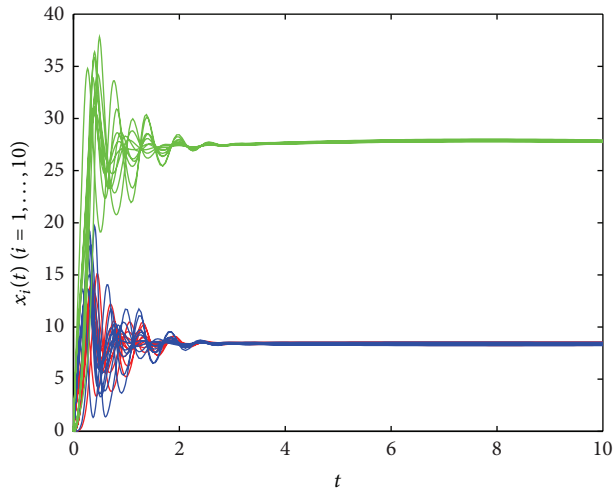

FIGURE 2: Control results of the network when $c = 3$, $d_1 = d_7 = d_9 = 3$, and $d = 0.015$.

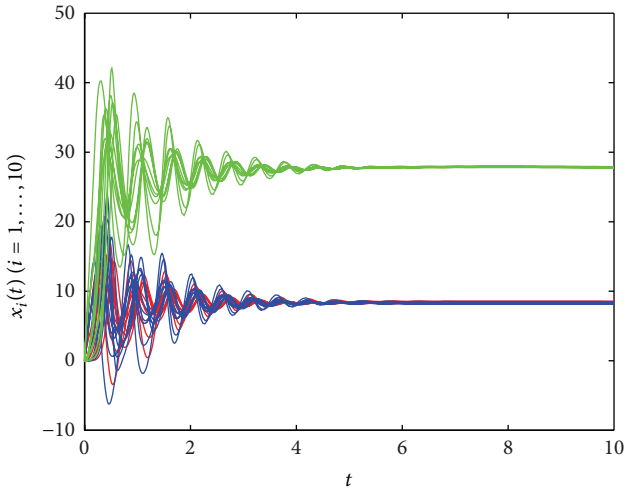

FIGURE 3: Control results of the network when $c = 3$, $d_1 = d_2 = d_4 = d_7 = d_9 = 3$, and $d = 0.015$.

delays and external disturbances, by transforming it into a normal H_∞ control problem. Specifically, delay-independent and delay-dependent conditions in terms of LMI were both derived to ensure the synchronization of networks with a prescribed H_∞ disturbance attenuation index. It deserves pointing out that the obtained results can be extended to complex dynamical network with coupling heterogeneous delays or time-varying delays by the same method introduced in this paper.

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