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Research Article

On the Positive Definite Solutions of a Nonlinear Matrix Equation

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The positive definite solutions of the nonlinear matrix equation $X^s + A^* f(X)A = Q$ are discussed. A necessary and sufficient condition for the existence of positive definite solutions for this equation is derived. Then, the uniqueness of the Hermitian positive definite solution is studied based on an iterative method proposed in this paper. Lastly the perturbation analysis for this equation is discussed.

1. Introduction

Denote the set of all $n \times n$ positive definite matrices by $P(n)$. In this paper, we consider the matrix equation

$$X^s + A^* f(X) A = Q, \quad (1)$$

where A is nonsingular, Q is a Hermitian positive definite matrix, s is a positive real number, f is a continuous map from $P(n)$ into $P(n)$, and f is either monotone (meaning that $0 \le X \le Y$ implies that $f(X) \le f(Y)$) or antimonotone (meaning that $0 \le X \le Y$ implies that $f(X) \ge f(Y)$).

Nonlinear matrix equation of the form (1) often arises in dynamic programming, control theory, stochastic filtering, statistics, and so on. In recent years, many authors have been much interested in studying this class of matrix equations [1–10].

Equation (1) has been investigated in some special cases. For the case $s = 1$, Ran and Reurings [1] derived some sufficient conditions for the existence and uniqueness of a positive definite solution of (1). In addition, an iterative method for obtaining Hermitian positive definite solutions of (1) with $f(X) = X^{-t}$ is proposed by Yueting [10]. Liu and Gao [9] proved the existence of the symmetric positive definite solutions of (1) with $f(X) = \pm X^{-t}$ and $Q = I$. Many other authors investigated (1) for particular choices of the map f [2–4, 6–8].

In Section 2, we will derive a necessary and sufficient condition for the existence of positive definite solutions of (1). In Section 3, we will propose an iterative method and investigate the uniqueness of the Hermitian positive definite solution. Finally, in Section 4, we will discuss the perturbation analysis of (1).

The following notations are used throughout this paper. For a positive definite matrix A, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ stand for the maximal and minimal eigenvalues of matrix A, respectively. A^* is the conjugate transpose of the matrix A, and A^{-*} is the inversion of A^*. $\|A\|$ denotes the spectral norm of A. $A > 0$ ($A \ge 0$) denotes that A is a positive definite (semidefinite) matrix, and $A > B$ ($A \ge B$) means that $A - B > 0$ ($A - B \ge 0$). The notation $L_{A,B}$ denotes the line segment joining A and B; that is,

$$L_{A,B} = \{tA + (1-t)B \mid t \in [0,1]\}. \quad (2)$$

2. On the Positive Definite Solutions of (1)

In this section, we will derive a necessary and sufficient condition for the existence of positive definite solutions of (1).

Lemma 1 (see [8]). *Assume that* $A \ge B \ge 0$ ($A > B \ge 0$)*; if* $r \in (0,1]$, *then* $A^r \ge B^r$ ($A^r > B^r$), *and if* $r \in [-1, 0)$, *then* $A^r \le B^r$ ($A^r < B^r$).

If there is unique Hermitian positive definite matrix T, such that $f(X) = T^2, X \in P(n)$, then we denote that $T = f^{1/2}(X)$.

Theorem 2. *Assume that f is a continuous map from $P(n)$ into $HP(n)$, where $HP(n)$ denotes the set of all $n \times n$ Hermitian positive definite matrices. Then,* (1) *has a Hermitian positive definite solution if and only if there is a nonsingular matrix W, such that $W^*W = WW^*$, and $A = (f^{1/2}(W^*W))^{-1} Z Q^{1/2}$, where*

$$\left(Q^{-1/2}\right)^* (W^s)^* (W^s) Q^{-1/2} + Z^* Z = I. \quad (3)$$

In this case, (1) *has a Hermitian positive definite solution $X = W^*W$.*

Proof. If X is a Hermitian positive definite solution of (1), then there is unique Hermitian positive definite matrix W, such that $X = W^2$ (see [11]). So, $f(X) = f(W^*W) \in HP(n)$, and therefore there is unique Hermitian positive definite matrix T, such that $f(W^*W) = T^2$. Substituting $X = W^2 = W^*W$ into (1) gives

$$(W^*W)^s + A^* f(W^*W) A = Q. \quad (4)$$

Then, we have

$$(W^s)^* (W^s) + A^* T^* T A = Q. \quad (5)$$

Q is Hermitian positive definite, and so

$$\left(Q^{-1/2}\right)^* (W^s)^* (W^s) Q^{-1/2} + \left(Q^{-1/2}\right)^* A^* T^* T A Q^{-1/2} = I; \quad (6)$$

that is

$$\begin{bmatrix} W^s Q^{-1/2} \\ T A Q^{-1/2} \end{bmatrix}^* \begin{bmatrix} W^s Q^{-1/2} \\ T A Q^{-1/2} \end{bmatrix} = I. \quad (7)$$

Let $Z = TAQ^{-1/2}$. Then, $A = T^{-1} Z Q^{1/2} = (f^{1/2}(W^*W))^{-1} Z Q^{1/2}$, and by (7) we know that $(Q^{-1/2})^* (W^s)^* (W^s) Q^{-1/2} + Z^* Z = I$.

Conversely, if $A = (f^{1/2}(W^*W))^{-1} Z Q^{1/2}$, let $X = W^*W$. Then,

$$\begin{aligned} X^s + A^* f(X) A &= (W^*W)^s + \left(Q^{1/2}\right)^* Z^* \left(f^{1/2}(W^*W)\right)^{-*} \\ &\quad \times f(W^*W) \left(f^{1/2}(W^*W)\right)^{-1} Z Q^{1/2} \\ &= (W^s)^* W^s + \left(Q^{1/2}\right)^* Z^* Z Q^{1/2} \\ &= \left(Q^{1/2}\right)^* \\ &\quad \times \left(\left(W^s Q^{-1/2}\right)^* \left(W^s Q^{-1/2}\right) + Z^* Z\right) Q^{1/2} \\ &= Q. \end{aligned} \quad (8)$$

So, $X = W^*W$ is a Hermitian positive definite solution of (1). □

If $s = 1$, then the restriction $W^*W = WW^*$ in Theorem 2 can be omitted.

3. Iterative Method

In order to discuss an iterative method for solving (1), we assume that for a given matrix B, the equation $f(X) = B$ always has a positive definite solution and its solution is easy to obtain. We are interested in the inverse iteration, and consider the following iterative method:

$$X_{k+1} = f^{-1}\left(A^{-*}\left(Q - X_k^s\right) A^{-1}\right), \quad k = 0, 1, 2, \ldots \quad (9)$$

In this section, we assume that A, Q, f in (1) satisfy $f^{-1}(A^{-*} Q A^{-1}) \le Q^{1/s}$.

Theorem 3. *Suppose that f^{-1} exists and that f is antimonotone. Let $s = 1$. Equation* (1) *has a positive definite solution in the interval $(0, \lambda_{\max}(Q) I)$ if and only if there is a number $\chi \in (0, \lambda_{\max}(Q))$, such that $X_k \le \chi I$ for all k. Moreover, in this case, the iteration* (9) *with $X_0 = 0$ converges to the smallest positive definite solution of* (1).

Proof. Since f^{-1} exists and f is anti-monotone, then f^{-1} is also anti-monotone. Assume that there is a number χ as in the theorem. Since $X_0 = 0$, one has

$$X_1 = f^{-1}\left(A^{-*} Q A^{-1}\right) \ge X_0 = 0. \quad (10)$$

Furthermore, we get

$$\begin{aligned} X_2 &= f^{-1}\left(A^{-*}(Q - X_1) A^{-1}\right) \\ &\ge f^{-1}\left(A^{-*}(Q - X_0) A^{-1}\right) = X_1. \end{aligned} \quad (11)$$

Now, if $X_k \ge X_{k-1}$, we have

$$\begin{aligned} X_{k+1} &= f^{-1}\left(A^{-*}(Q - X_k) A^{-1}\right) \\ &\ge f^{-1}\left(A^{-*}(Q - X_{k-1}) A^{-1}\right) = X_k. \end{aligned} \quad (12)$$

Then, the sequence $\{X_k\}$ is a monotonically nondecreasing sequence and bounded above by some positive definite matrix χI. Consequently, the sequence $\{X_k\}$ converges to a positive definite matrix X, which is a solution of (1); that is,

$$X = f^{-1}\left(A^{-*}(Q - X) A^{-1}\right). \quad (13)$$

Conversely, let (1) have a positive definite solution $X \in (0, \lambda_{\max}(Q) I)$, and let $\chi \in (0, \lambda_{\max}(Q))$ be the largest eigenvalue of X. In order to prove that $X_k \le \chi I$ for every X_k generated from (9), we will prove that $X_k \le X$. Consider

$$\begin{aligned} X_0 &= 0 < X, \\ X_1 &= f^{-1}\left(A^{-*}(Q - X_0) A^{-1}\right) \\ &\le f^{-1}\left(A^{-*}(Q - X) A^{-1}\right) = X. \end{aligned} \quad (14)$$

Assume that $X_k \le X$, for some fixed k, and then we have

$$\begin{aligned} X_{k+1} &= f^{-1}\left(A^{-*}(Q - X_k) A^{-1}\right) \\ &\le f^{-1}\left(A^{-*}(Q - X) A^{-1}\right) = X. \end{aligned} \quad (15)$$

So, $X_{k+1} \in (0, X)$. Then, the theorem is proved. □

Lemma 4 (see [1]). *Let $f : U \to M(n)(U \subset M(n)$ open) be differentiable at any point of U. Then,*

$$\|f(X) - f(Y)\| \le \sup_{Z \in L_{X,Y}} \|\mathscr{D} f(Z)\| \|X - Y\|, \tag{16}$$

for all $X, Y \in U$.

Lemma 5. *Suppose that f^{-1} exists and that f is anti-monotone. If (1) with $s \ge 1$ has a Hermitian positive definite solution X, then $X \in [f^{-1}(A^{-*}QA^{-1}), Q^{1/s}]$.*

Proof. Since f^{-1} exists and f is anti-monotone, then f^{-1} is also anti-monotone. Assume that (1) has a Hermitian positive definite solution X. Since f maps into $P(n)$, we have $f(X) > 0$ and $A^* f(X) A \ge 0$. Therefore, $X^s = Q - A^* f(X) A \le Q$. By Lemma 1 we know that $X \le Q^{1/s}$. On the other hand, we have $A^* f(X) A \le Q$, $f(X) \le A^{-*}QA^{-1}$. Then, we obtain $X \ge f^{-1}(A^{-*}QA^{-1})$ because f^{-1} is anti-monotone. □

Theorem 6. *Suppose that f^{-1} exists and that f is anti-monotone, and suppose that f, f^{-1} are differentiable at any point of $\Omega_1 = [f^{-1}(A^{-*}QA^{-1}), Q^{1/s}]$ and $\Omega_2 = [f(Q^{1/s}), A^{-*}QA^{-1}]$, respectively. Let*

$$M_1 = \sup_{Z \in \Omega_1} \|\mathscr{D} f(Z)\|, \qquad M_2 = \sup_{X \in \Omega_2} \|\mathscr{D} f^{-1}(X)\|, \tag{17}$$

and $a = M_1 M_2 \|A^{-1}\|^2 \|A\|^2$.

(i) *If (1) with $s > 1$ has a Hermitian positive definite solution X and $a < 1$, then X is the unique solution of (1).*

(ii) *Assume that there is a closed set $\Omega \subseteq \Omega_1$ satisfying that $g : \Omega \to \Omega$ and $g(X) = f^{-1}(A^{-*}(Q - X^s)A^{-1})$; if $a < 1$, then (1) with $s > 1$ has a unique solution X in Ω. Furthermore, one considers the iterative method (9) with $X_0 \in \Omega$. The sequence $\{X_k\}$ in (9) converges to the unique solution X; moreover,*

$$\begin{aligned} \|X_n - X\| &\le \frac{a^n}{1-a} \|X_1 - X_0\|, \\ \|X_n - X\| &\le \frac{a}{1-a} \|X_n - X_{n-1}\|. \end{aligned} \tag{18}$$

Proof. (i) Assume that X and $\overline{X}$ are two different Hermitian positive definite solutions of (1), and by Lemma 5, $X, \overline{X} \in [f^{-1}(A^{-*}QA^{-1}), Q^{1/s}] = \Omega_1$; then, $f(X), f(\overline{X}) \in [f(Q^{1/s}), A^{-*}QA^{-1}] = \Omega_2$. Let $T_1 = A^{-*}(Q - X^s)A^{-1}, T_2 = A^{-*}(Q - \overline{X}^s)A^{-1}$, and

$$M_2 = \sup_{X \in \Omega_2} \|\mathscr{D} f^{-1}(X)\|. \tag{19}$$

From Lemma 4,

$$\begin{aligned} \|X - \overline{X}\| &= \Big\| f^{-1}\left(A^{-*}(Q - X^s)A^{-1}\right) \\ &\quad - f^{-1}\left(A^{-*}\left(Q - \overline{X}^s\right)A^{-1}\right)\Big\| \\ &\le \sup_{Z \in L_{T_1,T_2}} \|\mathscr{D} f^{-1}(Z)\| \Big\| A^{-*}(Q - X^s)A^{-1} \\ &\qquad - A^{-*}\left(Q - \overline{X}^s\right)A^{-1}\Big\| \\ &= \sup_{Z \in L_{f(X),f(\overline{X})}} \|\mathscr{D} f^{-1}(Z)\| \left\| A^{-*}\left(\overline{X}^s - X^s\right)A^{-1}\right\| \\ &\le \sup_{Z \in \Omega_2} \|\mathscr{D} f^{-1}(Z)\| \left\| A^{-*}\left(\overline{X}^s - X^s\right)A^{-1}\right\| \\ &\le M_2 \|A^{-1}\|^2 \left\|\overline{X}^s - X^s\right\|. \end{aligned} \tag{20}$$

Let

$$M_1 = \sup_{Z \in \Omega_1} \|\mathscr{D} f(Z)\|. \tag{21}$$

Then,

$$\begin{aligned} \left\|\overline{X}^s - X^s\right\| &= \left\| Q - A^* f\left(\overline{X}\right) A - Q + A^* f(X) A \right\| \\ &\le \|A\|^2 \left\| f(X) - f\left(\overline{X}\right)\right\| \\ &\le \|A\|^2 \sup_{Z \in L_{X,\overline{X}}} \|\mathscr{D} f(Z)\| \left\|X - \overline{X}\right\| \\ &\le \|A\|^2 \sup_{Z \in \Omega_1} \|\mathscr{D} f(Z)\| \left\|X - \overline{X}\right\| \\ &\le \|A\|^2 M_1 \left\|X - \overline{X}\right\|. \end{aligned} \tag{22}$$

By $a < 1$ we have

$$\begin{aligned} \left\|X - \overline{X}\right\| &\le M_2 \|A^{-1}\|^2 \|A\|^2 M_1 \left\|X - \overline{X}\right\| \\ &= a \left\|X - \overline{X}\right\| < \left\|X - \overline{X}\right\|, \end{aligned} \tag{23}$$

which is a contradiction; so, $X = \overline{X}$. That is, X is the unique solution of (1).

(ii) Let $Y_1, Y_2 \in \Omega \subseteq \Omega_1$. Then, $f(g(Y_1)), f(g(Y_2)) \in \Omega_2$; from Lemma 4,

$$\begin{aligned}
&\|g(Y_1) - g(Y_2)\| \\
&= \left\| f^{-1}\left(A^{-*}\left(Q - Y_1^s\right)A^{-1}\right) - f^{-1}\left(A^{-*}\left(Q - Y_2^s\right)A^{-1}\right)\right\| \\
&\leq \sup_{Z \in L_{f(g(Y_1)),f(g(Y_2))}} \left\|\mathscr{D} f^{-1}(Z)\right\| \left\|A^{-*}\left(Y_2^s - Y_1^s\right)A^{-1}\right\| \\
&\leq \sup_{Z \in \Omega_2} \left\|\mathscr{D} f^{-1}(Z)\right\| \left\|A^{-1}\right\|^2 \left\|Y_2^s - Y_1^s\right\| \\
&\leq M_2 \left\|A^{-1}\right\|^2 \left\|Q - A^* f(Y_2) A - Q + A^* f(Y_1) A\right\| \\
&\leq M_2 \left\|A^{-1}\right\|^2 \|A\|^2 \left\|f(Y_1) - f(Y_2)\right\| \\
&\leq M_2 \left\|A^{-1}\right\|^2 \|A\|^2 \sup_{Z \in L_{Y_1,Y_2}} \|\mathscr{D} f(Z)\| \left\|Y_1 - Y_2\right\| \\
&\leq M_2 \left\|A^{-1}\right\|^2 \|A\|^2 \sup_{Z \in \Omega_1} \|\mathscr{D} f(Z)\| \left\|Y_1 - Y_2\right\| \\
&\leq M_2 \left\|A^{-1}\right\|^2 \|A\|^2 M_1 \left\|Y_1 - Y_2\right\| \\
&= a \left\|Y_1 - Y_2\right\|.
\end{aligned} \tag{24}$$

The interval Ω is a complete metric space because it is a closed subset of $P(n)$. And $a < 1$; so, g is a contraction on Ω. Then, it follows from contractive mapping principle that the map g has a unique fixed point X in Ω. Furthermore, the sequence $\{X_k\}$ in (9) converges to the unique solution of (1); moreover,

$$\begin{aligned}
\left\|X_n - X\right\| &\leq \frac{a^n}{1-a}\left\|X_1 - X_0\right\|, \\
\left\|X_n - X\right\| &\leq \frac{a}{1-a}\left\|X_n - X_{n-1}\right\|.
\end{aligned} \tag{25}$$

□

4. Perturbation Analysis

Let

$$\widetilde{X}^s + \widetilde{A}^* f\left(\widetilde{X}\right)\widetilde{A} = \widetilde{Q} \tag{26}$$

be the perturbation equation of (1). Let A, $\widetilde{A}$ be nonsingular matrices and Q, $\widetilde{Q}$ be positive definite, and $0 < s < 1$. Suppose that $f^{-1} : P(n) \to P(n)$ exists and f is anti-monotone.

Lemma 7 (see [12]). *If $0 < r < 1$, the operators X, Y satisfy $X \geq aI$ and $Y \geq aI$ for some positive number a; then,*

$$\left\|X^r - Y^r\right\| \leq r a^{r-1} \|X - Y\|. \tag{27}$$

Theorem 8. *Let X, $\widetilde{X}$ be the positive definite solutions of (1) and its perturbation equation (26), respectively. Map f^{-1} is differentiable at any point of Ω_4 with*

$$\begin{aligned}
\Omega_4 = \Big\{ &\alpha X + (1-\alpha) Y \mid \alpha \in [0,1], \\
&X \in f\left(\left[f^{-1}\left(A^{-*}QA^{-1}\right), Q^{1/s}\right]\right), \\
&Y \in f\left(\left[f^{-1}\left(\widetilde{A}^{-*}\widetilde{Q}\widetilde{A}^{-1}\right), \widetilde{Q}^{1/s}\right]\right)\Big\}.
\end{aligned} \tag{28}$$

Let

$$M_4 = \sup_{Z \in \Omega_4} \left\|\mathscr{D} f^{-1}(Z)\right\|. \tag{29}$$

If $M_4 \|A^{-1}\| \|\widetilde{A}^{-1}\| s \widetilde{\lambda}^{s-1} < 1$, one has

$$\frac{\left\|X - \widetilde{X}\right\|}{\|X\|} \leq \frac{T}{\lambda_{\min}\left(f^{-1}\left(A^{-*}QA^{-1}\right)\right)\left(M_4^{-1} - \left\|A^{-1}\right\|\left\|\widetilde{A}^{-1}\right\| s \widetilde{\lambda}^{s-1}\right)}, \tag{30}$$

where $T = \|A^{-1} - \widetilde{A}^{-1}\|(\|Q - \widetilde{\lambda}^s I\|\|A^{-1}\| + \|\widetilde{Q} - \widetilde{\lambda}^s I\|\|\widetilde{A}^{-1}\|) + \|A^{-1}\|\|\widetilde{A}^{-1}\|\|Q - \widetilde{Q}\|$, $\widetilde{\lambda} = \min\{\lambda_{\min}(f^{-1}(A^{-*}QA^{-1})), \lambda_{\min}(f^{-1}(\widetilde{A}^{-*}\widetilde{Q}\widetilde{A}^{-1}))\}$.

Proof. X, $\widetilde{X}$ are the positive definite solutions of (1) and (26), respectively. Let $T_1 = A^{-*}(Q - X^s)A^{-1}$, $T_2 = \widetilde{A}^{-*}(\widetilde{Q} - \widetilde{X}^s)\widetilde{A}^{-1}$. From Lemma 4,

$$\begin{aligned}
\left\|X - \widetilde{X}\right\| &= \left\| f^{-1}\left(A^{-*}\left(Q - X^s\right)A^{-1}\right) - f^{-1}\left(\widetilde{A}^{-*}\left(\widetilde{Q} - \widetilde{X}^s\right)\widetilde{A}^{-1}\right)\right\| \\
&\leq \sup_{Z \in L_{T_1,T_2}} \left\|\mathscr{D} f^{-1}(Z)\right\| \left\|A^{-*}\left(Q - X^s\right)A^{-1} - \widetilde{A}^{-*}\left(\widetilde{Q} - \widetilde{X}^s\right)\widetilde{A}^{-1}\right\|.
\end{aligned} \tag{31}$$

By Lemma 5,

$$\begin{aligned}
X &\in \left[f^{-1}\left(A^{-*}QA^{-1}\right), Q^{1/s}\right], \\
\widetilde{X} &\in \left[f^{-1}\left(\widetilde{A}^{-*}\widetilde{Q}\widetilde{A}^{-1}\right), \widetilde{Q}^{1/s}\right].
\end{aligned} \tag{32}$$

Let

$$\begin{aligned}
\Omega_4 = \Big\{ &\alpha X + (1-\alpha) Y \mid \alpha \in [0,1], \\
&X \in f\left(\left[f^{-1}\left(A^{-*}QA^{-1}\right), Q^{1/s}\right]\right), \\
&Y \in f\left(\left[f^{-1}\left(\widetilde{A}^{-*}\widetilde{Q}\widetilde{A}^{-1}\right), \widetilde{Q}^{1/s}\right]\right)\Big\}, \\
M_4 = &\sup_{Z \in \Omega_4} \left\|\mathscr{D} f^{-1}(Z)\right\|.
\end{aligned} \tag{33}$$

Then, we have

$$\sup_{Z\in L_{T_1,T_2}} \left\|\mathscr{D} f^{-1}(Z)\right\| = \sup_{Z\in L_{f(X),f(\widetilde{X})}} \left\|\mathscr{D} f^{-1}(Z)\right\| \leq \sup_{Z\in\Omega_4} \left\|\mathscr{D} f^{-1}(Z)\right\| \leq M_4. \tag{34}$$

Notice that

$$X \geq f^{-1}\left(A^{-*}QA^{-1}\right) \geq \lambda_{\min}\left(f^{-1}\left(A^{-*}QA^{-1}\right)\right) I,$$
$$\widetilde{X} \geq f^{-1}\left(\widetilde{A}^{-*}\widetilde{Q}\widetilde{A}^{-1}\right) \geq \lambda_{\min}\left(f^{-1}\left(\widetilde{A}^{-*}\widetilde{Q}\widetilde{A}^{-1}\right)\right) I. \tag{35}$$

Let $\widetilde{\lambda} = \min\{\lambda_{\min}(f^{-1}(A^{-*}QA^{-1})), \lambda_{\min}(f^{-1}(\widetilde{A}^{-*}\widetilde{Q}\widetilde{A}^{-1}))\}$. Then, $X, \widetilde{X} \geq \widetilde{\lambda} I$. By Lemma 1, $X^s, \widetilde{X}^s \geq \widetilde{\lambda}^s I$ since $0 < s < 1$. And from Lemma 7,

$$\begin{aligned}
&\left\|A^{-*}(Q - X^s)A^{-1} - \widetilde{A}^{-*}\left(\widetilde{Q} - \widetilde{X}^s\right)\widetilde{A}^{-1}\right\| \\
&= \left\|A^{-*}(Q - X^s)\left(A^{-1} - \widetilde{A}^{-1}\right) + A^{-*}\left(Q - X^s - \widetilde{Q} + \widetilde{X}^s\right)\widetilde{A}^{-1} + \left(A^{-*} - \widetilde{A}^{-*}\right)\left(\widetilde{Q} - \widetilde{X}^s\right)\widetilde{A}^{-1}\right\| \\
&\leq \left\|A^{-1}\right\|\left\|Q - X^s\right\|\left\|A^{-1} - \widetilde{A}^{-1}\right\| + \left\|A^{-1}\right\|\left\|\widetilde{A}^{-1}\right\|\left(\left\|Q - \widetilde{Q}\right\| + \left\|\widetilde{X}^s - X^s\right\|\right) + \left\|A^{-1} - \widetilde{A}^{-1}\right\|\left\|\widetilde{A}^{-1}\right\|\left\|\widetilde{Q} - \widetilde{X}^s\right\| \\
&= \left\|A^{-1} - \widetilde{A}^{-1}\right\|\left(\left\|Q - X^s\right\|\left\|A^{-1}\right\| + \left\|\widetilde{Q} - \widetilde{X}^s\right\|\left\|\widetilde{A}^{-1}\right\|\right) + \left\|A^{-1}\right\|\left\|\widetilde{A}^{-1}\right\|\left\|Q - \widetilde{Q}\right\| + \left\|A^{-1}\right\|\left\|\widetilde{A}^{-1}\right\|\left\|\widetilde{X}^s - X^s\right\| \\
&\leq \left\|A^{-1} - \widetilde{A}^{-1}\right\|\left(\left\|Q - \widetilde{\lambda}^s I\right\|\left\|A^{-1}\right\| + \left\|\widetilde{Q} - \widetilde{\lambda}^s I\right\|\left\|\widetilde{A}^{-1}\right\|\right) + \left\|A^{-1}\right\|\left\|\widetilde{A}^{-1}\right\|\left\|Q - \widetilde{Q}\right\| + \left\|A^{-1}\right\|\left\|\widetilde{A}^{-1}\right\| s\widetilde{\lambda}^{s-1}\left\|\widetilde{X} - X\right\|.
\end{aligned} \tag{36}$$

Therefore,

$$\begin{aligned}
\left\|X - \widetilde{X}\right\| \leq M_4 \Big(&\left\|A^{-1} - \widetilde{A}^{-1}\right\| \times \left(\left\|Q - \widetilde{\lambda}^s I\right\|\left\|A^{-1}\right\| + \left\|\widetilde{Q} - \widetilde{\lambda}^s I\right\|\left\|\widetilde{A}^{-1}\right\|\right) \\
&+ \left\|A^{-1}\right\|\left\|\widetilde{A}^{-1}\right\|\left\|Q - \widetilde{Q}\right\| + \left\|A^{-1}\right\|\left\|\widetilde{A}^{-1}\right\| s\widetilde{\lambda}^{s-1}\left\|\widetilde{X} - X\right\|\Big).
\end{aligned} \tag{37}$$

That is,

$$\begin{aligned}
&\left(M_4^{-1} - \left\|A^{-1}\right\|\left\|\widetilde{A}^{-1}\right\| s\widetilde{\lambda}^{s-1}\right)\left\|\widetilde{X} - X\right\| \\
&\leq \left\|A^{-1} - \widetilde{A}^{-1}\right\|\left(\left\|Q - \widetilde{\lambda}^s I\right\|\left\|A^{-1}\right\| + \left\|\widetilde{Q} - \widetilde{\lambda}^s I\right\|\left\|\widetilde{A}^{-1}\right\|\right) + \left\|A^{-1}\right\|\left\|\widetilde{A}^{-1}\right\|\left\|Q - \widetilde{Q}\right\|.
\end{aligned} \tag{38}$$

If $M_4\|A^{-1}\|\|\widetilde{A}^{-1}\|s\widetilde{\lambda}^{s-1} < 1$, by $X \geq \lambda_{\min}(f^{-1}(A^{-*}QA^{-1}))I$ we have

$$\frac{\left\|X - \widetilde{X}\right\|}{\|X\|} \leq \frac{T}{\lambda_{\min}\left(f^{-1}\left(A^{-*}QA^{-1}\right)\right)\left(M_4^{-1} - \left\|A^{-1}\right\|\left\|\widetilde{A}^{-1}\right\| s\widetilde{\lambda}^{s-1}\right)}, \tag{39}$$

where

$$T = \left\|A^{-1} - \widetilde{A}^{-1}\right\|\left(\left\|Q - \widetilde{\lambda}^s I\right\|\left\|A^{-1}\right\| + \left\|\widetilde{Q} - \widetilde{\lambda}^s I\right\|\left\|\widetilde{A}^{-1}\right\|\right) + \left\|A^{-1}\right\|\left\|\widetilde{A}^{-1}\right\|\left\|Q - \widetilde{Q}\right\|. \tag{40}$$

□

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