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Research Article

On Characterizations of Fourier Frames and Tilings

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We give some characterizations of Fourier frames and tilings and obtain a more general form of characterizations of spectra and tilings.

1. Introduction

A countable family of elements $\{f_n\}_{n\in\mathbb{N}}$ in a separable Hilbert space H is called a frame if there are positive constants A, B such that

$$A\|f\|^2 \le \sum_{n\in\mathbb{N}} \left|\langle f, f_n\rangle\right|^2 \le B\|f\|^2 \tag{1}$$

for all $f \in H$. A and B are called frame bounds. The sequence is called a tight frame if $A = B$. The sequence is called Bessel if the second inequality above holds. In this case, B is called the Bessel bound. Frames were first introduced by Duffin and Schaeffer [1] in the context of nonharmonic Fourier series, and today they have applications in a wide range of areas. A frame can be considered as a generalized basis in the sense that every element in H can be written as a linear combination of the frame elements.

In this paper, we consider Fourier frames for a special separable Hilbert space. Let $\Omega \subset \mathbb{R}^d$ have positive Lebesgue measure $m(\Omega) > 0$ and let Λ be a discrete subset of $\mathbb{R}^d$. The inner product and the norm on $L^2(\Omega)$ are

$$\begin{gathered}\langle f(x), g(x)\rangle_\Omega = \frac{1}{m(\Omega)}\int_\Omega f(x)\overline{g(x)}\,dx,\\ \|f\|_\Omega^2 = \frac{1}{m(\Omega)}\int_\Omega |f(x)|^2\,dx.\end{gathered} \tag{2}$$

We write

$$\begin{gathered}e_\lambda(x) := e^{2\pi i\langle\lambda, x\rangle} \quad \text{for } x \in \mathbb{R}^d,\\ \mathscr{E}(\Lambda) := \{e_\lambda(x) : \lambda \in \Lambda\}.\end{gathered} \tag{3}$$

If $\mathscr{E}(\Lambda)$ is a frame or an orthonormal basis for $L^2(\Omega)$, then $\mathscr{E}(\Lambda)$ and (Ω, Λ) are called a Fourier frame and a spectral pair, respectively. In the case of the spectral pair, the Λ is then called a spectrum for Ω and Ω is called a spectral set. We follow the terminology of [2] and consider the packing and tiling in $\mathbb{R}^d$ by compact set Ω of the following kind.

A compact set Ω in $\mathbb{R}^d$ is a regular region if it has positive Lebesgue measure, is the closure of its interior Ω°, and has a boundary $\partial\Omega = \Omega \setminus \Omega^\circ$ of measure zero. If Ω is a regular region, then a discrete set Λ is a packing set for Ω if the sets $\{\Omega + \lambda : \lambda \in \Lambda\}$ have disjoint interiors or the intersections $(\Omega + \lambda) \cap (\Omega + \mu)$ for $\lambda \neq \mu$ in Λ have measure zero. It is a tiling set if, further, the translates $\{\Omega + \lambda : \lambda \in \Lambda\}$ cover $\mathbb{R}^d$ up to measure zero. In these cases, we say that $\Omega + \Lambda$ is a packing or tiling of $\mathbb{R}^d$, respectively. Equivalently, we call (Ω, Λ) a packing pair or a tiling pair, respectively.

It is well known that spectral sets and tilings are connected by the following conjecture of Fuglede [3].

Spectral Set Conjecture. A set Ω in $\mathbb{R}^d$ is a spectral set if and only if it tiles $\mathbb{R}^d$ by translations.

Many people attempt to prove the spectral set conjecture for some special sets, although the conjecture is false in many cases (see [4–7]). For example, Jorgensen and Pedersen [8] conjectured that $([0,1]^n, \Lambda)$ is a spectral pair if and only if $([0,1]^n, \Lambda)$ is a tiling pair. They established the conjecture for dimension $n \le 3$ and for all n when Λ is a discrete periodic set. Iosevich and Pedersen [9] simultaneously and independently established the above-mentioned conjecture by a different approach based on a geometric argument. Kolountzakis [10] gave an alternative proof of this fact, which is based on a characterization of translational tiling by a Fourier analytic criterion. Lagarias et al. [2] related the spectra of sets Ω to tiling in the Fourier space and obtained the following characterization of spectra and tilings.

Theorem 1. *Let Ω be a regular region in $\mathbb{R}^d$ and let Λ be such that the set of exponentials $\mathscr{E}(\Lambda)$ is orthogonal for $L^2(\Omega)$. Suppose that D is a regular region with $m(\Omega)m(D) = 1$ such that $D + \Lambda$ is a packing of $\mathbb{R}^d$. Then, Λ is a spectrum for Ω if and only if $D + \Lambda$ is a tiling of $\mathbb{R}^d$.*

Li [11] presented an elementary approach to obtain a more general form of Theorem 1. Enlightened by the ideas from [11, 12], we give some characterizations of Fourier frames and tilings and extend several results in [2] and [11].

2. Main Results and Their Proofs

Throughout this section, let Ω and D be two regular regions in $\mathbb{R}^d$. By the definition of frames, we may get that the following lemma.

Lemma 2. *Let $\Delta \subset \mathbb{R}^d$ be a discrete set. If $\mathscr{E}(\Delta)$ is a frame for $L^2(\Omega)$ with frame bounds A, B, then*

$$A(m(\Omega))^2 \le \sum_{\delta\in\Delta} \left|\widehat{\chi_\Omega}(t-\delta)\right|^2 \le B(m(\Omega))^2, \quad \forall t \in \mathbb{R}^d, \tag{4}$$

where

$$\widehat{\chi_\Omega}(u) = \int_{\mathbb{R}^d} \chi_\Omega(x)\, e^{-2\pi i\langle u,x\rangle}\, dx, \quad u \in \mathbb{R}^d \tag{5}$$

is the Fourier transform of the characteristic function $\chi_\Omega(x)$.

Proof. By the frame inequality (1), for any $t \in \mathbb{R}^d$, we have

$$\begin{aligned} &\sum_{\delta\in\Delta} \left|\widehat{\chi_\Omega}(t-\delta)\right|^2 \\ &\quad= \sum_{\delta\in\Delta} \left|m(\Omega)\left\langle e_t(x), e_\delta(x)\right\rangle_\Omega\right|^2 \\ &\quad\le B(m(\Omega))^2 \left\|e_t(x)\right\|_\Omega^2 = B(m(\Omega))^2. \end{aligned} \tag{6}$$

Similarly, we get $A(m(\Omega))^2 \le \sum_{\delta\in\Delta} |\widehat{\chi_\Omega}(t-\delta)|^2$. □

Remark 3. In the case for $A = B$, if $\mathscr{E}(\Delta)$ is a tight frame for $L^2(\Omega)$ with the frame bound B, then

$$\sum_{\delta\in\Delta} \left|\widehat{\chi_\Omega}(t-\delta)\right|^2 = B(m(\Omega))^2, \quad \forall t \in \mathbb{R}^d. \tag{7}$$

If $\mathscr{E}(\Delta)$ is a Bessel sequence for $L^2(\Omega)$ with the Bessel bound B, then

$$\sum_{\delta\in\Delta} \left|\widehat{\chi_\Omega}(t-\delta)\right|^2 \le B(m(\Omega))^2, \quad \forall t \in \mathbb{R}^d. \tag{8}$$

Moreover, $\mathscr{E}(\Delta)$ is an orthonormal basis for $L^2(\Omega)$ if and only if

$$\sum_{\delta\in\Delta} \left|\widehat{\chi_\Omega}(t-\delta)\right|^2 = (m(\Omega))^2, \quad \forall t \in \mathbb{R}^d. \tag{9}$$

Since $|\widehat{\chi_\Omega}(u)| = |\widehat{\chi_\Omega}(-u)|$, for all $u \in \mathbb{R}^d$, if we substitute "−" for "+" in (4), (7), (8), and (9), all the above results also hold.

In the remainder of this paper, we assume that $\Theta \subset \mathbb{R}^d$ is a discrete subset and Λ and Γ are two finite subsets of $\mathbb{R}^d$ such that $\Theta + \Lambda$ and $\Theta + \Gamma$ are two direct sums.

Theorem 4. *If $\mathscr{E}(\Theta + \Lambda)$ is a frame for $L^2(\Omega)$ with frame bounds A, B, and $(D, \Theta + \Gamma)$ is a tiling pair, then*

$$\frac{\#\Lambda}{B\#\Gamma} \le m(\Omega)\, m(D) \le \frac{\#\Lambda}{A\#\Gamma}, \tag{10}$$

where # denotes the cardinality of some set.

Proof. Let $\Lambda := \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ and $\Gamma := \{\gamma_1, \gamma_2, \ldots, \gamma_q\}$ with $\#\Lambda = p$ and $\#\Gamma = q$. Since $\mathscr{E}(\Theta + \Lambda)$ is a frame for $L^2(\Omega)$ with frame bounds A, B, it follows from Lemma 2 that

$$A(m(\Omega))^2 \le \sum_{i=1}^{p} \sum_{\theta\in\Theta} \left|\widehat{\chi_\Omega}(t+\theta+\lambda_i)\right|^2 \le B(m(\Omega))^2, \quad \forall t \in \mathbb{R}^d. \tag{11}$$

Note that $(D, \Theta + \Gamma)$ is a tiling pair, from the Plancherel's formula on $L^2(\mathbb{R}^d)$, we have the following:

$$\begin{aligned} m(\Omega) &= \left\|\chi_\Omega\right\|_{\mathbb{R}^d}^2 = \left\|\widehat{\chi_\Omega}\right\|_{\mathbb{R}^d}^2 = \int_{\mathbb{R}^d} \left|\widehat{\chi_\Omega}(t)\right|^2 dt \\ &= \frac{1}{p} \int_{\mathbb{R}^d} \sum_{i=1}^{p} \left|\widehat{\chi_\Omega}(t+\lambda_i)\right|^2 dt \\ &= \frac{1}{p} \int_{\cup_{\theta\in\Theta, 1\le j\le q}(D+\theta+\gamma_j)} \sum_{i=1}^{p} \left|\widehat{\chi_\Omega}(t+\lambda_i)\right|^2 dt \\ &= \frac{1}{p} \sum_{j=1}^{q} \sum_{\theta\in\Theta} \int_D \sum_{i=1}^{p} \left|\widehat{\chi_\Omega}\left(t+\theta+\lambda_i+\gamma_j\right)\right|^2 dt \\ &= \frac{1}{p} \sum_{j=1}^{q} \int_D \sum_{\theta\in\Theta} \sum_{i=1}^{p} \left|\widehat{\chi_\Omega}\left(t+\theta+\lambda_i+\gamma_j\right)\right|^2 dt \\ &= \frac{1}{p} \sum_{j=1}^{q} \int_D \sum_{i=1}^{p} \sum_{\theta\in\Theta} \left|\widehat{\chi_\Omega}\left(t+\theta+\lambda_i+\gamma_j\right)\right|^2 dt \\ &\le \frac{q}{p} B(m(\Omega))^2 m(D). \end{aligned} \tag{12}$$

The bottom third equality holds by Lebesgue dominated convergence theorem and the last inequality follows from (11). Thus, $p/qB \leq m(\Omega)m(D)$. Similarly, we get $m(\Omega)m(D) \leq p/qA$. Hence, the proof is completed. □

Since an orthonormal basis is also a tight frame with frame bounds $A = B = 1$, we get the following corollary.

Corollary 5. *If $\mathscr{E}(\Theta + \Lambda)$ is an orthonormal basis for $L^2(\Omega)$, and $(D, \Theta + \Gamma)$ is a tiling pair, then $m(\Omega)m(D) = \#\Lambda/\#\Gamma$.*

Lemma 6. *Let $\Theta+\Lambda$ be such that the set of exponentials $\mathscr{E}(\Theta+\Lambda)$ is a Bessel sequence for $L^2(\Omega)$ with the Bessel bound B. If $D + \Theta + \Gamma$ is a tiling of $\mathbb{R}^d$ with $m(\Omega)m(D) \leq \#\Lambda/B\#\Gamma$, then*

$$\sum_{\lambda\in\Lambda}\sum_{\theta\in\Theta}\left|\widehat{\chi_\Omega}(t+\theta+\lambda)\right|^2 = B(m(\Omega))^2, \quad \forall t \in \mathbb{R}^d. \tag{13}$$

Proof. Keep the assumptions on Λ and Γ in the above proof. Since $\mathscr{E}(\Theta+\Lambda)$ is a Bessel sequence for $L^2(\Omega)$ with the Bessel bound B, it follows from Remark 3 that

$$\sum_{i=1}^{p}\sum_{\theta\in\Theta}\left|\widehat{\chi_\Omega}(t+\theta+\lambda_i)\right|^2 \leq B(m(\Omega))^2, \quad \forall t \in \mathbb{R}^d. \tag{14}$$

Since $D + \Theta + \Gamma$ is a tiling of $\mathbb{R}^d$ and $m(\Omega)m(D) \leq \#\Lambda/B\#\Gamma = p/qB$, for any $y \in \mathbb{R}^d$, then we have

$$\begin{aligned} m(\Omega) &= \int_{\mathbb{R}^d}\left|\widehat{\chi_\Omega}(t)\right|^2 dt \\ &= \frac{1}{p}\int_{\mathbb{R}^d}\sum_{i=1}^{p}\left|\widehat{\chi_\Omega}(t+\lambda_i)\right|^2 dt \\ &= \frac{1}{p}\int_{\cup_{\theta\in\Theta,1\leq j\leq q}(D+y+\theta+\gamma_j)}\sum_{i=1}^{p}\left|\widehat{\chi_\Omega}(t+\lambda_i)\right|^2 dt \\ &= \frac{1}{p}\sum_{j=1}^{q}\sum_{\theta\in\Theta}\int_{D+y}\sum_{i=1}^{p}\left|\widehat{\chi_\Omega}\left(t+\theta+\lambda_i+\gamma_j\right)\right|^2 dt \\ &= \frac{1}{p}\sum_{j=1}^{q}\int_{D+y}\sum_{\theta\in\Theta}\sum_{i=1}^{p}\left|\widehat{\chi_\Omega}\left(t+\theta+\lambda_i+\gamma_j\right)\right|^2 dt \\ &= \frac{1}{p}\sum_{j=1}^{q}\int_{D+y}\sum_{i=1}^{p}\sum_{\theta\in\Theta}\left|\widehat{\chi_\Omega}\left(t+\theta+\lambda_i+\gamma_j\right)\right|^2 dt \\ &\leq \frac{q}{p}B(m(\Omega))^2 m(D) \leq m(\Omega), \end{aligned} \tag{15}$$

which yields

$$\sum_{i=1}^{p}\sum_{\theta\in\Theta}\left|\widehat{\chi_\Omega}(t+\theta+\lambda_i)\right|^2 = B(m(\Omega))^2 \tag{16}$$

for almost every t in $D+y$. Since y is arbitrary, (16) holds for almost every t in $\mathbb{R}^d$. By the continuity of the function on the left side of (16), we see that (16) holds for every t in $\mathbb{R}^d$. □

Theorem 7. *If $\mathscr{E}(\Theta+\Lambda)$ is orthogonal in $L^2(\Omega)$ and $(D, \Theta+\Gamma)$ is a tiling pair with $m(\Omega)m(D) = \#\Lambda/\#\Gamma$, then $\mathscr{E}(\Theta + \Lambda)$ is an orthonormal basis for $L^2(\Omega)$.*

Proof. The proof is straightforward by the above lemma. □

Theorem 8. *Suppose that $D + \Theta + \Gamma$ is a packing of $\mathbb{R}^d$ with $\#\Lambda/A\#\Gamma \leq m(\Omega)m(D)$. If $\mathscr{E}(\Theta + \Lambda)$ is a frame for $L^2(\Omega)$ with the frame bounds A, B, then $D + \Theta + \Gamma$ is a tiling of $\mathbb{R}^d$.*

Proof. Since $\mathscr{E}(\Theta + \Lambda)$ is a frame for $L^2(\Omega)$ with the frame bounds A, B, then (11) holds. If $D + \Theta + \Gamma$ is a packing of $\mathbb{R}^d$, then it follows from (11) and $\#\Lambda/A\#\Gamma \leq m(\Omega)m(D)$ that

$$\begin{aligned} m(\Omega) &= \int_{\mathbb{R}^d}\left|\widehat{\chi_\Omega}(t)\right|^2 dt \\ &= \frac{1}{p}\int_{\mathbb{R}^d}\sum_{i=1}^{p}\left|\widehat{\chi_\Omega}(t+\lambda_i)\right|^2 dt \\ &\geq \frac{1}{p}\int_{\cup_{\theta\in\Theta,1\leq j\leq q}(D+\theta+\gamma_j)}\sum_{i=1}^{p}\left|\widehat{\chi_\Omega}(t+\lambda_i)\right|^2 dt \\ &= \frac{1}{p}\sum_{j=1}^{q}\sum_{\theta\in\Theta}\int_{D}\sum_{i=1}^{p}\left|\widehat{\chi_\Omega}\left(t+\theta+\lambda_i+\gamma_j\right)\right|^2 dt \\ &= \frac{1}{p}\sum_{j=1}^{q}\int_{D}\sum_{\theta\in\Theta}\sum_{i=1}^{p}\left|\widehat{\chi_\Omega}\left(t+\theta+\lambda_i+\gamma_j\right)\right|^2 dt \\ &= \frac{1}{p}\sum_{j=1}^{q}\int_{D}\sum_{i=1}^{p}\sum_{\theta\in\Theta}\left|\widehat{\chi_\Omega}\left(t+\theta+\lambda_i+\gamma_j\right)\right|^2 dt \\ &\geq \frac{q}{p}A(m(\Omega))^2 m(D) \geq m(\Omega). \end{aligned} \tag{17}$$

Thus, $D + \Theta + \Gamma$ is a tiling of $\mathbb{R}^d$. □

It is clear that the above theorem yields the following corollary.

Corollary 9. *Suppose that $D + \Theta + \Gamma$ is a packing of $\mathbb{R}^d$ with $\#\Lambda/\#\Gamma = m(\Omega)m(D)$. If $\mathscr{E}(\Theta + \Lambda)$ is an orthonormal basis for $L^2(\Omega)$, then $D + \Theta + \Gamma$ is a tiling of $\mathbb{R}^d$.*

Combining Theorem 7 with Corollary 9, we obtain a more general form of the theorem in [11] and Theorem 1.

Theorem 10. *Suppose that $\mathscr{E}(\Theta + \Lambda)$ is orthogonal in $L^2(\Omega)$, and $(D, \Theta + \Gamma)$ is a packing pair with $m(\Omega)m(D) = \#\Lambda/\#\Gamma$. Then, $(\Omega, \Theta + \Lambda)$ is a spectral pair if and only if $(D, \Theta + \Gamma)$ is a tiling pair.*

Example 11. Let p, q be two positive integers. Take the following:

$$\Omega = [0, 1], \qquad D = \left[0, \frac{p}{q}\right], \qquad \Theta = \{pk : k \in \mathbb{Z}\},$$

$$\Lambda = \{0, 1, \ldots, p-1\}, \qquad \Gamma = \left\{0, \frac{p}{q}, \frac{2p}{q}, \ldots, \frac{(q-1)p}{q}\right\}. \tag{18}$$

We see that $m(\Omega)m(D) = \#\Lambda/\#\Gamma$, $(\Omega, \Theta+\Lambda)$ is a spectral pair and $(D, \Theta+\Gamma)$ is a tiling pair.

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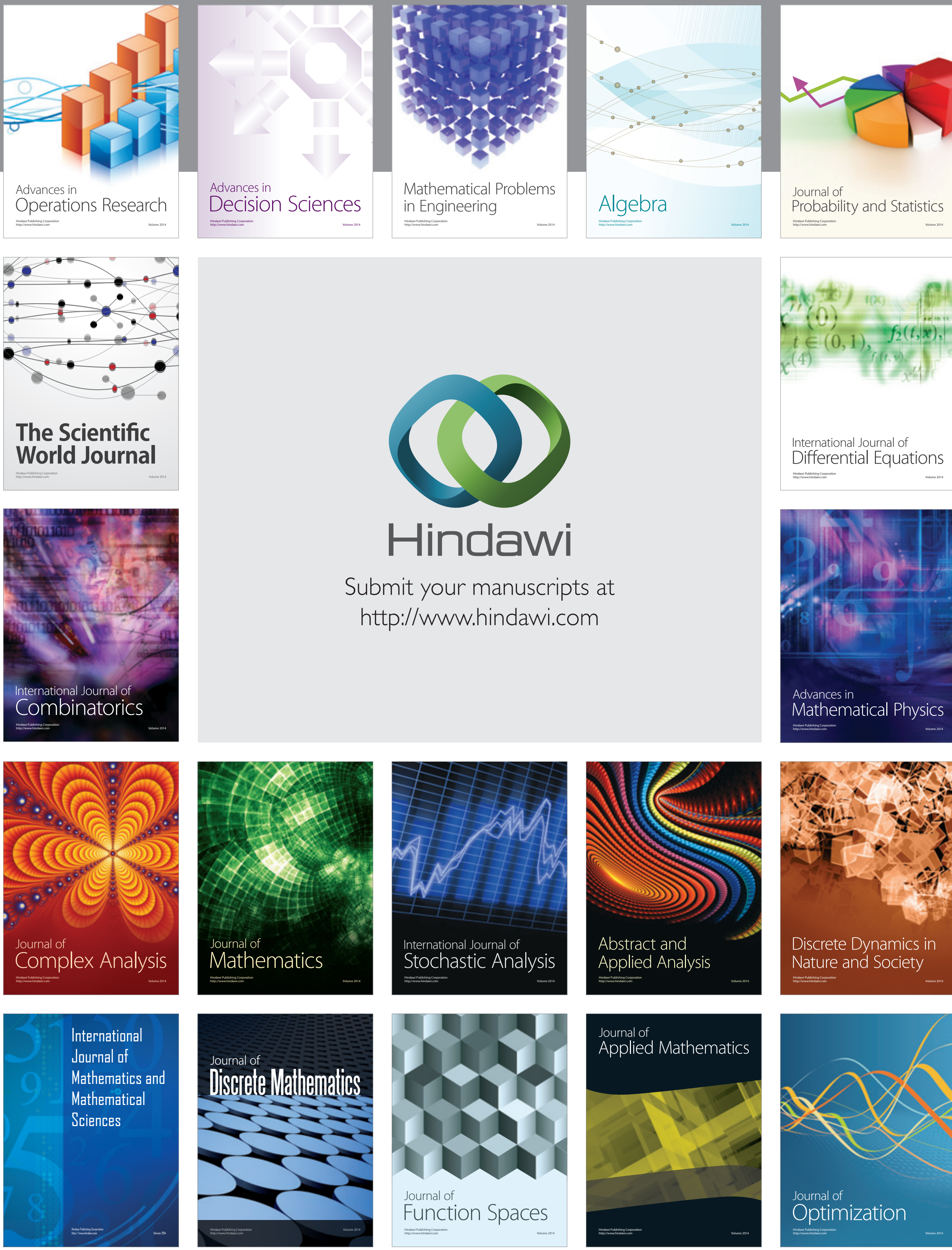
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