RECENT DEVELOPMENTS IN VOLATILITY MODELING AND APPLICATIONS

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In financial modeling, it has been constantly pointed out that volatility clustering and conditional nonnormality induced leptokurtosis observed in high frequency data. Financial time series data are not adequately modeled by normal distribution, and empirical evidence on the non-normality assumption is well documented in the financial literature (details are illustrated by Engle (1982) and Bollerslev (1986)). An ARMA representation has been used byThavaneswaran et al., in 2005, to derive the kurtosis of the various class of GARCH models such as power GARCH, non-Gaussian GARCH, nonstationary and random coefficient GARCH. Several empirical studies have shown that mixture distributions are more likely to capture heteroskedasticity observed in high frequency data than normal distribution. In this paper, some results on moment properties are generalized to stationary ARMA process with GARCH errors. Application to volatility forecasts and option pricing are also discussed in some detail.

1. Introduction

Recently, there has been growing interest in using nonlinear time series models in finance and economics (see Granger [13], He and Teräsvirta [15] and Heston [16] including others). Inference for nonlinear time series had been studied by Thavaneswaran and Abraham [20] and by Thavaneswaran and Heyde [23] using estimating function theory. A nonlinear model had been proposed by Abraham and Thavaneswaran [1] and using nonlinear state space formulation, filtering, and smoothing had been studied (see Granger [13] for more details). Many financial series, such as returns on stocks and foreign exchange rates, exhibit leptokurtosis and time-varying volatility. These two features have been the subject of extensive studies ever since Nicholls and Quinn [19], Engle [7], and G.-Rivera [8] reported them. Random coefficient autoregressive (RCA) models (Nicholls and Quinn [19]), the autoregressive conditional heteroscedastic (ARCH)

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model (Engle [7, 8]) and its generalization, the GARCH model (Bollerslev [4]) provide a convenient framework to study time-varying volatility in financial markets.Financial time series models for intra-day trading are typical examples of random coefficient GARCH models.

In practice, a common assumption in applying GARCH models to financial data is that the return series is conditionally normally distributed. We will refer to this as the normal GARCH model. It is well known that the normal GARCH model is part of the volatility clustering patterns typically exhibited in financial and economic time series. However, the kurtosis implied by the normal GARCH model tends to be far less than the sample kurtosis observed for most financial return series. For example, Bollerslev [4] finds evidence of conditional leptokurtosis in monthly S&P 500 Composite Index returns and advocates the use of the t-distribution. Thus, the nonnormal GARCH model is more appropriate with the large vleptokurtosis typically observed in asset returns.

In this paper, kurtosis for various class of RCA models is given in Section 2. In Section 3, we give expressions for the kurtosis of GARCH(p,q) and for various class of GARCH models. Previously, He and Teräsvirta [15] and Heston [16] examined the forth moment structure of the GARCH$(1,1)$ model with conditionally nonnormal innovations and they extended their results to the GARCH(p,q) model. In both of these papers, the kurtosis is expressed as a function of the underlying model parameters. We take a somewhat different approach by working with the well-known ARMA representations of the powers of the error term and we are able to extend their results to a broader class of models. For any random variable X with finite fourth moments, the kurtosis is defined by $E(X-\mu)^4/[\mathrm{Var}(X)]^2$. Application of GARCH kurtosis in volatility forecasting and in analytical approximation of option pricing are discussed in Section 4.

2. Random coefficient autoregressive models

Random coefficient autoregressive time series were introduced by Nicholls and Quinn [19] and some of their properties have been studied recently by Appadoo et al. [2]. RCA models exhibiting long-memory properties have been considered in Leipus and Surgailis [18]. A sequence of random variables $\{y_t\}$ is called an RCA(1) time series if it satisfies the equations

$$y_t = (\phi + b_t)\, y_{t-1} + e_t, \quad t \in Z, \tag{2.1}$$

where Z denotes the set of integers and

(i) $\binom{b_t}{e_t} \sim \left(\binom{0}{0}, \begin{pmatrix} \sigma_b^2 & 0 \\ 0 & \sigma_e^2 \end{pmatrix}\right)$,

(ii) $\phi^2 + \sigma_b^2 < 1$.

The sequences $\{b_t\}$ and $\{e_t\}$, respectively, are the errors in the model. According to Nicholls and Quinn [19], (ii) is a necessary and sufficient condition for the second-order stationarity of $\{y_t\}$. Thus, together with (i), it also ensures strict stationarity. Moreover, Feigin and Tweedie [9] showed that $Ey_t^{2k} < \infty$ for some $k \geq 1$ if the moments of the noise sequences satisfy $Ee_t^{2k} < \infty$ and $E(\phi + b_t)^{2k} < 1$ for the same k.

Let $\{y_t\}$ be a stationary Gaussian linear process with mean zero and variance σ_y^2. Then it can easily be shown that the joint moment generating function of the pair (y_t, y_{t-k}) is given by $m(u,v) = E(e^{uy_t+vy_{t-k}}) = \exp[(1/2)\sigma_y^2(u^2+v^2+2\rho_k uv)]$, where $\sigma_y^2 = \mathrm{Var}(y_t)$ and the autocorrelation is $\rho_k = \rho_k(y_t)$. Since $E[y_t^2 y_{t-k}^2] = \sigma_y^4(1+2\rho_k^2)$ and $\mathrm{Var}(y_t^2) = 2\sigma_y^4$, we have $\rho_k(y_t^2) = (E[y_t^2 y_{t-k}^2] - \sigma_y^4)/2\sigma_y^4 = \rho_k^2(y_t)$. That is, for any stationary Gaussian process $\{y_t\}$, the autocorrelation of the squared process $\{y_t^2\}$ is the square of the autocorrelation of $\{y_t\}$ and hence the autocorrelation of any stationary Gaussian process $\{y_t\}$ is larger than the autocorrelation of $\{y_t^2\}$ (i.e., $|\rho_k^y| \geq \rho_k^{y^2}$). The following theorem and corollary on autocorrelation function and kurtosis are from Appadoo et al. [2].

Theorem 2.1. *Let $\{y_t\}$ be an RCA(1) time series satisfying conditions (i) and (ii), and let γ_y be its covariance function. Then*

(a) *$Ey_t = 0$, $Ey_t^2 = \sigma_e^2/(1-\phi^2-\sigma_b^2)$, the kth lag autocovariance for y_t is given by $\gamma_y(k) = \phi^k\sigma_e^2/(1-\phi^2-\sigma_b^2)$ and the autocorrelation for y_t is $\rho_k = \phi^k$ for all $k \in Z$; That is, the usual AR(1) process has same autocorrelation as the RCA(1);*

(b) *if $\{b_t\}$ and $\{e_t\}$ are normally distributed random variables, then the kurtosis $K^{(y)}$ of the AR(1) process $\{y_t\}$ is given by $K^{(y)} = 3[1-(\sigma_b^2+\phi^2)^2]/[1-(\phi^4+6\phi^2\sigma_b^2+3\sigma_b^4)]$ and for an AR(1) process $K^{(y)}$ reduces to 3;*

(c) *the autocorrelation of y_t^2 is given by $\rho_k^{y^2} = (\phi^2+\sigma_b^2)^k$ and for an AR(1) process it turns out to be $\rho_k^{y^2} = \phi^{2k}$.*

Note. When $\sigma_b = 0$, the kurtosis of y_t reduces to that of a standard AR(1) process, which is equal to 3.

Consider an RCA model of the form $y_t = \theta_t y_{t-1} + e_t$, then the following corollary is true.

Corollary 2.2. *Let $\{y_t\}$ be an RCA(1) time series satisfying the stationarity conditions, and let ρ_y denote its correlation function. Then*

(a) *when $\theta_t = \phi + b_t$, $Ey_t = 0$, $Ey_t^2 = \sigma_e^2/(1-\phi^2-\sigma_b^2)$, the kth lag autocorrelation for y_t is given by $\rho_k^y = \{E(\phi+b_t)\}^k = \phi^k$,*

(b) *when $\theta_t = \mathrm{sgn}(b_t)$, where $b_t \sim N(0,\sigma_b^2)$, then $\rho_k^y = [1-2F(0)]^k$, where F is the cumulative distribution function of b_t, that is, when the coefficient θ_t is driven by a binary random variable $\{b_t\}$ taking values -1 and $+1$,*

(c) *when $\theta_t = (\phi + |b_t|^\alpha)$, where $b_t \sim N(\mu,\sigma_b^2)$, then the autocorrelation*

$$\rho_k^y = \left[\phi + \frac{(2\sigma_b^2)^{\alpha/2}}{\sqrt{\pi}}\Gamma\left(\frac{\alpha+1}{2}\right)\right]^k, \tag{2.2}$$

where $\Gamma(\cdot)$ is the Gamma function.

The following theorem for RCA models with correlated errors follows from Appadoo et al. [3].

Theorem 2.3. *Let $\{y_t\}$ be a correlated RCA(1) time series satisfying conditions (i) and (ii), and let γ_y be its covariance function. If $\{b_t\}$ and $\{e_t\}$ are correlated normally distributed random variables with correlation coefficient ρ, then the kurtosis $K^{(y)}$ of the RCA process*

$\{y_t\}$ *is given by*

$$K^{(y)} = \frac{6(\sigma_b^2+\phi_1^2)[1-\phi_1^3-3\phi_1\sigma_b^2]+72\phi_1^3\rho^2\sigma_b^2+3[1-(\phi_1^2+\sigma_b^2)][1-\phi_1^3-3\phi_1\sigma_b^2]}{[1-\phi_1^3-3\phi_1\sigma_b^2][1-6\phi_1^2\sigma_b^2-\phi_1^4-3\sigma_b^4]} \times[1-(\phi_1^2+\sigma_b^2)] \tag{2.3}$$

and for an $AR(1)$ *process,* $K^{(y)}$ *reduces to* 3 *and when* $\rho = 0$, *the kurtosis turns out to be the kurtosis in Theorem 2.1.*

The kurtosis of the classical RCA model is a special case of the correlated RCA model of Theorem 2.1. The correlated RCA model has a higher kurtosis than its uncorrelated counterpart and easy computation leads to the following inequality for kurtosis of the different type of RCA models, $K_{\mathrm{AR}}^{(y)} \le K_{\mathrm{RCA}}^{(y)} \le K_{\mathrm{CRCA}}^{(y)}$.

A sequence of random variables $\{y_t\}$ is called an RCA-MA(1) time series if it satisfies the equations

$$y_t = (\phi+b_t)y_{t-1}+e_t+\theta e_{t-1}, \quad t\in Z, \tag{2.4}$$

where Z denotes the set of integers and

(i) $\binom{b_t}{e_t} \sim \left(\binom{0}{0}, \begin{pmatrix}\sigma_b^2 & 0\\ 0 & \sigma_e^2\end{pmatrix}\right)$,

(ii) $\phi^2+\sigma_b^2<1$.

Lemma 2.4. *Let* $\{y_t\}$ *be the RCA-*MA(1) *time series model described by (2.4) and let* $\gamma_y^{(K)}$ *denote its covariance function. Then*

(a) $E(y_t)=0$,

(b) $\mathrm{Var}(y_t) = \sigma_e^2(1+\theta^2)/(1-\sigma_b^2-\phi^2)$,

(c) $\gamma_y(k) = \phi^k(\sigma_e^2(1+\theta^2)/(1-\sigma_b^2-\phi^2))$,

(d) *the autocorrelation and kurtosis of the process are given by*

$$\rho_k = \begin{cases} 1, & k=0,\\ \phi^k, & k\ne 0,\end{cases} \tag{2.5}$$

and $K^{(y)} = [6(1+\theta^2)^2(\phi^2+\sigma_b^2)+(6\theta^2+3+3\theta^4)(1-\sigma_b^2-\phi_1^2)](1-\sigma_b^2-\phi^2)/[1-(\phi^4+3\sigma_b^4+6\phi^2\sigma_b^2)][(1+\theta^2)^2]$.

Proof. The proof is somewhat similar to the proof of Theorem 2.1 and is, therefore, omitted. □

Example 2.5. For a simple ARCH model of the form $y_t = \epsilon_{t-1}^2\epsilon_t$, where ϵ_t is a Gaussian white noise with variance σ_ϵ^2, the kurtosis is given by $K^{(y)} = 35$. This clearly shows that even a simple volatility model $y_t = \epsilon_{t-1}^2\epsilon_t$ with Gaussian error term ϵ_t can generate very high peakedness which is very common for financial time series. Moreover, for a y_t^2 process, $y_t^2 = \epsilon_{t-1}^4\epsilon_t^2$, the correlation is given by, $\rho_k^{y_t^2} = 0.114285$ for $k=1$ and $\rho_k^{y_t^2} = 0$ for $k>1$.

Example 2.6. For a simple time series model of the form $y_t = \phi y_{t-2}\epsilon_{t-1} + \epsilon_t$, where ϵ_t is a Gaussian white noise with variance σ_ϵ^2, the variance of the process is given by $\mathrm{Var}(y_t) = \sigma_\epsilon^2/(1-\phi\sigma_\epsilon^2)$ and the kurtosis is given by $K^{(y)} = 3[2\phi^2\sigma_\epsilon^4 + 1 - \phi^2\sigma_\epsilon^2](1-\phi^2\sigma_\epsilon^2)/(1-3\phi^4\sigma_\epsilon^2)(\sigma_\epsilon^2)$.

Example 2.7. For $y_t = \theta_1\epsilon_{t-1}^2\epsilon_t + \theta_2\epsilon_{t-2}^2\epsilon_t$, where ϵ_t is a Gaussian white noise with variance σ_ϵ^2, we have $K^{(y)} = [[315(\theta_1^4 + \theta_2^4) + 180(\theta_1^3\theta_2 + \theta_1\theta_2^3) + 162\theta_1^2\theta_2^2]/(3\theta_1^2 + 2\theta_1\theta_2 + 3\theta_2^2)^2]$. The derivation of the autocorrelation of y_t is similar to that of y_t in Example 2.5 and hence is omitted. Moreover, for any conditionally Gaussian process of the form, $y_t = f(\epsilon_{t-1},\dots,\epsilon_{t-k})\epsilon_t$, where f is a measurable function of $\epsilon_{t-1},\dots,\epsilon_{t-k}$, where ϵ_t is a zero mean Gaussian process, then $E(y_t^4) = EE(y_t^4 \mid y_{t-1}) = E[3E(y_t^2 \mid y_{t-1})]^2 \geq 3(E[E(y_t^2 \mid y_{t-1})]^2 = 3(E(y_t^2))^2$, and

$$K^{(y)} = \frac{E(y_t^4)}{E(y_t^2)^2} \geq 3. \tag{2.6}$$

Theorem 2.8. *Let $\{y_t\}$ be an RCA-sign time series satisfying conditions (i) and (ii), and let γ_y be its covariance function. The sign RCA(1) model is given by $y_t = (\phi + b_t + \Phi s_t)y_{t-1} + e_t$ and*

$$s_t = \begin{cases} +1 & \text{if } y_t > 0, \\ 0 & \text{if } y_t = 0, \\ -1 & \text{if } y_t < 0. \end{cases} \tag{2.7}$$

Then

(a) $Ey_t = 0$, $Ey_t^2 = \sigma_e^2/(1-(\phi^2+\sigma_b^2+\Phi^2))$, *the kth lag autocovariance for y_t is given by $\gamma_y(k) = \phi^k\sigma_e^2/(1-(\phi^2+\sigma_b^2+\Phi^2))$ and the autocorrelation for y_t is $\rho_k = \phi^k$ for all $k \in Z$. That is, the usual AR(1) process has same autocorrelation as the RCA(1),*

(b) *if $\{b_t\}$ and $\{e_t\}$ are normally distributed random variables, then the kurtosis $K^{(y)}$ of the RCA process $\{y_t\}$ is given by*

$$K^{(y)} = \left[\frac{3\left[1-\left(\phi^2+\sigma_b^2+\Phi^2\right)^2\right]}{1-\left(\phi^4+\Phi^4+3\sigma_b^4+6\left(\phi^2\sigma_b^2+\Phi^2\left(\phi^2+\sigma_b^2\right)\right)\right)}\right] \tag{2.8}$$

and for an AR(1) process $K^{(y)}$ reduces to 3,

(c) *the autocorrelation of y_t^2 is given by $\rho_k^{y^2} = (\phi^2+\sigma_b^2+\Phi^2)^k$ and for an AR(1) process it turns out to be $\rho_k^{y^2} = \phi^{2k}$.*

Granger and Teräsvirta [14] had introduce the sign models. Here, the RCA analogue of Grangers's model is considered. Sign volatility models are important as they allow for an asymmetric behavior of the conditional volatility with respect to negative (positive) shocks observed in most financial time series models. Proof of the theorem is somewhat similar to Appadoo et al. [2].

A sequence of random variables $\{y_t\}$ is called a sign-RCA-MA time series if it satisfies the equations, $y_t = (\phi + b_t + \Phi s_t)y_{t-1} + e_t + \theta e_{t-1}$, $t \in Z$.

Lemma 2.9. *Let* $\{y_t\}$ *be the RCA-sign model time series with* MA(1) *errors described by (2.4) and let* $\gamma_y(k)$ *denote its covariance function. Then*

(a) $E(y_t) = 0$,

(b) $\mathrm{Var}(y_t) = [(\sigma_e^2 + \theta^2\sigma_e^2)/(1 - (\Phi^2 + \phi^2 + \sigma_b^2))]$,

(c) $\gamma_y(k) = \phi^k[(\sigma_e^2 + \theta^2\sigma_e^2)/(1 - (\Phi^2 + \phi^2 + \sigma_b^2))]$,

(d) *the autocorrelation and kurtosis of the process are given by*

$$\rho_k = \begin{cases} 1, & k = 0, \\ \phi^k, & k \neq 0, \end{cases} \tag{2.9}$$

and $K^{(y)} = [3(1 - (\Phi^2 + \phi^2 + \sigma_b^2)^2)/[1 - (\phi^4 + \Phi^4 + 3\sigma_b^4) + 6(\phi^2\sigma_b^2 + \Phi^2(\phi^2 + \sigma_b^2))]]$. *When* $\theta = 0$ *and* $\Phi = 0$, *the kurtosis of the process turns out to be* $K^{(y)} = 3[1 - (\phi^2 + \sigma_b^2)^2]/[1 - (\phi^4 + 6\phi^2\sigma_b^2 + 3\sigma_b^4)]$.

Proof.

$$\begin{aligned} y_t &= (\phi + b_t + \Phi s_t)y_{t-1} + e_t + \theta e_{t-1}, \\ E(y_t) &= E[(\phi + b_t + \Phi s_t)y_{t-1} + e_t + \theta e_{t-1}] = 0, \\ E(y_t^2) &= \Phi^2 E(y_{t-1}^2) + \phi^2 E(y_{t-1}^2) + \sigma_b^2 E(y_{t-1}^2) + \sigma_e^2 + \theta^2\sigma_e^2, \\ E(y_t^2) &= \left[\frac{\sigma_e^2 + \theta^2\sigma_e^2}{1 - (\Phi^2 + \phi^2 + \sigma_b^2)}\right], \end{aligned}$$

$$\begin{aligned} &E(y_t^4) \\ &= \frac{3(\sigma_e^4 + 2\theta^2\sigma_e^4 + \sigma_e^4\theta^4) + 6\sigma_e^2[\phi^2 + \sigma_b^2 + \Phi^2 + \phi^2\theta^2 + \sigma_b^2\theta^2 + \Phi^2\theta^2]E[y_{t-1}^2]}{[1 - \phi^4 - 3\sigma_b^4 - 6\phi^2\sigma_b^2 - \Phi^4 - 6\phi^2\Phi^2 - 6\sigma_b^2\Phi^2]} \\ &= \frac{3(\sigma_e^4 + 2\theta^2\sigma_e^4 + \sigma_e^4\theta^4)[1 - (\Phi^2 + \phi^2 + \sigma_b^2)] + 6\sigma_e^2[\phi^2 + \sigma_b^2 + \Phi^2 + \phi^2\theta^2 + \sigma_b^2\theta^2 + \Phi^2\theta^2][\sigma_e^2 + \theta^2\sigma_e^2]}{[1 - \phi^4 - 3\sigma_b^4 - 6\phi^2\sigma_b^2 - \Phi^4 - 6\phi^2\Phi^2 - 6\sigma_b^2\Phi^2][1 - (\Phi^2 + \phi^2 + \sigma_b^2)]}, \\ &K^{(y)} \\ &= \left[\frac{3(\sigma_e^4 + 2\theta^2\sigma_e^4 + \sigma_e^4\theta^4)[1 - (\Phi^2 + \phi^2 + \sigma_b^2)] + 6\sigma_e^2[\phi^2 + \sigma_b^2 + \Phi^2 + \phi^2\theta^2 + \sigma_b^2\theta^2 + \Phi^2\theta^2][\sigma_e^2 + \theta^2\sigma_e^2]}{[1 - \phi^4 - 3\sigma_b^4 - 6\phi^2\sigma_b^2 - \Phi^4 - 6\phi^2\Phi^2 - 6\sigma_b^2\Phi^2][1 - (\Phi^2 + \phi^2 + \sigma_b^2)]}\right] \\ &= \left[\frac{3(1 - (\Phi^2 + \phi^2 + \sigma_b^2)^2)}{[1 - (\phi^4 + \Phi^4 + 3\sigma_b^4) + 6(\phi^2\sigma_b^2 + \Phi^2(\phi^2 + \sigma_b^2))]}\right]. \end{aligned} \tag{2.10}$$

The following lemma and theorem for a stationary process with volatility errors are given in Ghahramani and Thavaneswaran [11].

Lemma 2.10. *For a volatility process of the form*

$$y_t - \mu = \phi(y_{t-1} - \mu) + \varepsilon_{t-1}^2\varepsilon_t \tag{2.11}$$

under the stationarity assumptions that $|\phi| < 1$, ε_t *symmetric i.i.d. with mean* 0 *and variance*

σ_ε^2 and finite eighth moments, then

(a)

$$E(y_t-\mu)^2=\frac{E(\varepsilon_{t-1}^4)E(\varepsilon_t^2)}{(1-\phi^2)}, \tag{2.12}$$

(b)

$$K^{(y)}=\frac{E[(y_t-\mu)^4]}{\mathrm{Var}(y_t)^2}=\left[\frac{6\phi^2(E(\varepsilon_{t-1}^4)E(\varepsilon_t^2))^2+E(\varepsilon_{t-1}^8)E(\varepsilon_t^4)(1-\phi^2)}{(1+\phi^2)(E(\varepsilon_{t-1}^4)E(\varepsilon_t^2))^2}\right], \tag{2.13}$$

(c) *if ε_t are assumed to be i.i.d. $N(0,\sigma_\varepsilon^2)$, then $E[\varepsilon_t^{2n}]=((2n)!/2^n(n!))\sigma_\varepsilon^{2n}$ and hence*

$$K^{(y)}=\left[\frac{35-29\phi^2}{(1+\phi^2)}\right]. \tag{2.14}$$

Theorem 2.11. *Let y_t be a second-order linear stationary process having an* MA *(moving average) representation of the form $y_t-\mu=\sum_{j=0}^{\infty}\psi_j a_{t-j}$, where a_t is an uncorrelated noise process with mean zero, variance σ_a^2, and kurtosis $K^{(a)}$. Then the variance and the kurtosis of y_t are*

$$\begin{aligned}&\text{(i) } \mathrm{Var}(y_t)=\sigma_a^2\sum_{j=0}^{\infty}\psi_j^2,\\ &\text{(ii) } K^{(y)}=\frac{K^{(a)}[\sum_{j=0}^{\infty}\psi_j^4]+6\sum_{i<j}^{\infty}\psi_i^2\psi_j^2}{(\sum_{j=0}^{\infty}\psi_j^2)^2},\end{aligned} \tag{2.15}$$

provided $\sum_{j=0}^{\infty}\psi_j^4<\infty$.

Proof of Theorem 2.11 follows by the properties of stationary processes. The following lemma provides the kurtosis of an RCA model with GARCH errors. □

Lemma 2.12. *Let $\{y_t\}$ be an RCA(1) time series satisfying conditions (i) and (ii). The RCA(1) model is given by*

$$y_t=(\phi+b_t)y_{t-1}+\varepsilon_{t-1}^2\varepsilon_t, \tag{2.16}$$

$e_t\sim N(0,\sigma_e^2)$, $b_t\sim N(0,\sigma_b^2)$. Then the following holds:

$$\begin{aligned}&\text{(a) } E(y_t)=0,\qquad E(y_t^2)=\left[\frac{3\sigma_\varepsilon^6}{(1-(\phi^2+\sigma_b^2))}\right],\\ &\text{(b) } E(y_t^4)=\left(\frac{9\sigma_\varepsilon^{12}}{(1-6\phi^2\sigma_b^2-3\sigma_b^4-\phi^4)}\right)\left(\frac{35-29(\sigma_b^2+\phi^2)}{1-(\sigma_b^2+\phi^2)}\right),\\ &\text{(c) } K^{(y)}=\left(\frac{(35-29(\sigma_b^2+\phi^2))(1-(\phi^2+\sigma_b^2))}{(1-3\sigma_b^2(2\phi^2+\sigma_b^2)-\phi^4)}\right).\end{aligned} \tag{2.17}$$

Proof.

$$\begin{aligned} E(y_t^2) &= E(y_{t-1}^2\phi^2) + E(y_{t-1}^2 b_t^2) + E(\varepsilon_{t-1}^4)E(\varepsilon_t^2) \\ &= \phi^2 E(y_{t-1}^2) + \sigma_b^2 E(y_{t-1}^2) + 3\sigma_\varepsilon^6. \end{aligned} \tag{2.18}$$

Thus we have

$$\begin{aligned} E(y_t^2) &= \left[\frac{3\sigma_\varepsilon^6}{(1-(\phi^2+\sigma_b^2))}\right], \\ E(y_t^4) &= 6\phi^2\sigma_b^2 E(y_{t-1}^4) + 315\sigma_\varepsilon^{12} + 3\sigma_b^4 E(y_{t-1}^4) + \phi^4 E(y_{t-1}^4) \\ &\quad + 18\sigma_b^2\sigma_\varepsilon^6 E(y_{t-1}^2) + 18\sigma_\varepsilon^6\phi^2 E(y_{t-1}^2), \end{aligned} \tag{2.19}$$

and we have

$$\begin{aligned} E(y_t^4) &= \left(\frac{9\sigma_\varepsilon^{12}}{(1-6\phi^2\sigma_b^2-3\sigma_b^4-\phi^4)}\right)\left(\frac{35-29(\sigma_b^2+\phi^2)}{1-(\sigma_b^2+\phi^2)}\right), \\ K^{(y)} = \frac{E[y_t^4]}{E[y_t^2]^2} &= \left(\frac{9\sigma_\varepsilon^{12}}{(1-6\phi^2\sigma_b^2-3\sigma_b^4-\phi^4)}\right)\left(\frac{35-29(\sigma_b^2+\phi^2)}{1-(\sigma_b^2+\phi^2)}\right)\left(\frac{(1-(\phi^2+\sigma_b^2))^2}{9\sigma_\varepsilon^{12}}\right) \\ &= \left(\frac{(35-29(\sigma_b^2+\phi^2))(1-(\phi^2+\sigma_b^2))}{(1-3\sigma_b^2(2\phi^2+\sigma_b^2)-\phi^4)}\right). \end{aligned} \tag{2.20}$$

When $\sigma_b^2 = 0$, the kurtosis of the process y_t turns out to the one reported by Ghahramani and Thavaneswaran [11]. Moreover, when $\sigma_b^2 = 0$, and $\phi = 0$ the kurtosis of the process y_t turns out to be 35. □

3. GARCH(p,q) processes

Consider the general class of GARCH(p,q) model for the time series y_t, where

$$\begin{aligned} y_t &= \sqrt{h_t}Z_t, \\ h_t &= \omega + \sum_{i=1}^{p}\alpha_i y_{t-i}^2 + \sum_{j=1}^{q}\beta_j h_{t-j}, \end{aligned} \tag{3.1}$$

where Z_t is a sequence of independent, identically distributed random variables with zero mean, unit variance. Let $u_t = y_t^2 - h_t$ be the martingale difference and let σ_u^2 be the variance of u_t, (3.1) could be written as

$$y_t^2 - u_t = \omega + \sum_{i=1}^{p} \alpha_i y_{t-i}^2 + \sum_{j=1}^{q} \beta_i h_{t-j}, \tag{3.2}$$

$$\left[1 - \sum_{i=1}^{p} \alpha_i B^i - \sum_{j=1}^{q} \beta_j B^j\right] y_t^2 = \omega - \sum_{j=1}^{q} \beta_j B^j u_i, \tag{3.3}$$

$$\phi(B) y_t^2 = \omega + \beta(B) u_t, \tag{3.4}$$

where, $\phi(B) = 1 - \sum_{i=1}^{r} \phi_i B^i$, $\phi_i = (\alpha_i + \beta_i)$, $\beta(B) = 1 - \sum_{j=1}^{q} \beta_j B^j$ and $r = \max(p,q)$.

We will make the following stationarity assumptions for y_t^2 which has an ARMA(r,q) representation.

(A.1) All the zeroes of the polynomial $\phi(B)$ lie outside the unit circle.

(A.2) $\sum_{i=0}^{\infty} \psi_i^2 < \infty$, where the $\psi_i' s$ are obtained from the relation $\psi(B)\ \phi(B) = \beta(B)$ with $\psi(B) = 1 + \sum_{i=1}^{\infty} \psi_i B^i$. The assumptions ensure that the $u_t' s$ are uncorrelated with zero mean and finite variance and that the y_t^2 process is weakly stationary. In this case, the autocorrelation function of y_t^2 will be exactly the same as that for a stationary ARMA(r,q) model. If the process $\{Z_t\}$ is normal, then the process $\{y_t\}$ defined by (3.1) is called a normal GARCH(p,q) process. The kurtosis of the GARCH process is denoted by $K^{(y)}$ when it exists. In order to calculate the GARCH kurtosis in terms of the ψ weights, we have the following theorem (Thavaneswaran et al. [22]).

Theorem 3.1. *For the* GARCH(p,q) *process specified by (3.4), under the stationarity assumptions and finite fourth moment, the kurtosis* $K^{(y)}$ *of the process is given by*

(a)

$$K^{(y)} = \frac{E(Z_t^4)}{E(Z_t^4) - [E(Z_t^4) - 1] \sum_{j=0}^{\infty} \psi_j^2}, \tag{3.5}$$

(b) (i) *the variance of the* y_t^2 *process is* $\gamma_0^{y^2} = \sum_{j=-\infty}^{\infty} \psi_j^2 \sigma_u^2$,

(ii) *the k-lag autocovariance of the* y_t^2 *process is*

$$\gamma_k^{y^2} = \sigma_u^2 \sum_{j=-\infty}^{\infty} \psi_{k+j} \psi_j \quad \textit{for } k \geq 1, \tag{3.6}$$

(c) $K^{(y)} = 3/(1 - 2\sum_{j=1}^{\infty} \psi_j^2)$ *for a normal* GARCH(p,q) *process.*

Theorem 3.1 has potential application in identifying a GARCH model and the marginal distribution of the error term in the model.

3.1. Power GARCH(p,q) **model.** Consider the power GARCH$(1,1)$ studied in [15]: $y_t = \sqrt{h_t} Z_t$, $h_t^\delta = \omega + \alpha_1 |y_{t-1}|^\delta + \beta_1 h_{t-1}^\delta$, $u_t = |y_t|^\delta - h_t^\delta$, $Z_t \sim (0,1)$, $E(|Z_t|^\delta) = 1$, $E(Z_t) = 0$, $E(|Z_t|^{2\delta}) = c$, $E(Z_t^2) = 1$. $E(y_t) = E(\sqrt{h_t} Z_t) = E(\sqrt{h_t}) E(Z_t) = 0$, $|y_t|^\delta - u_t = \omega + \alpha_1 |y_{t-1}|^\delta + \beta_1 h_{t-1}^\delta = \omega + \alpha_1 |y_{t-1}|^\delta + \beta_1 [|y_{t-1}|^\delta - u_{t-1}]$, $|y_t|^\delta - \alpha_1 |y_{t-1}|^\delta - |y_{t-1}|^\delta \beta_1 = \omega + u_t + \beta_1 y_{t-1}$, $[1 - (\alpha_1 + \beta_1) B] |y_t|^\delta = \omega + (1 - \beta_1 B) u_t$, $\phi(B) |y_t|^\delta = \omega + \theta(B) u_t$. This shows that the power

GARCH$(1,1)$ model could be represented as an ARMA$(1,1)$ for $|y_t|^\delta$. Under the stationarity conditions, it is easy to prove the following theorem for any power GARCH(p,q) process $|y_t|^\delta$.

Theorem 3.2. *The kurtosis of a power* GARCH(p,q) *process is given by*

(a)

$$K^{(y)} = \frac{E(Z_t^4)}{E(Z_t^4) - [E(Z_t^4) - 1]\sum_{j=o}^{\infty}\psi_j^2}, \tag{3.7}$$

(b) (i) *the variance of the y_t^δ process is* $\gamma_0^{y^\delta} = \sum_{j=-\infty}^{\infty}\psi_j^2\sigma_u^2$,

(ii) *the k-lag autocovariance of the y_t^δ process is*

$$\gamma_0^{y^\delta} = \sigma_u^2 \sum_{j=-\infty}^{\infty} \psi_{k+j}\psi_j \quad \textit{for } k \geq 1, \tag{3.8}$$

(c) $K^{(y)} = 3/(1 - 2\sum_{j=1}^{\infty}\psi_j^2)$ *for a normal* GARCH *process.*

3.2. Random coefficient ARCH(1) **model.** By analogy with the RCA models we introduce a class of RCA versions of the GARCH models. Consider the general class of GARCH(p,q) models for the time series y_t, where

$$y_t = \sqrt{h_t}Z_t, \qquad h_t = \omega + (\alpha_1 + b_{t-1})y_{t-1}^2, \tag{3.9}$$

and Z_t is a sequence of independent, identically distributed random variables with zero mean, unit variance. Let $u_t = y_t^2 - h_t$ be the martingale difference and let σ_u^2 be the variance of u_t. When we write the model as

$$y_t^2 = \omega + (\alpha_1 + b_{t-1})y_{t-1}^2 + u_t, \tag{3.10}$$

then the minimum mean square error forecast is not optimal for the random coefficient ARCH(1) model (3.9).

Lemma 3.3. *For the model $y_t = \sqrt{h_t}Z_t$, $h_t = \omega_0 + (\alpha_1 + b_{t-1})y_{t-1}^2$, where $Z_t \sim N(0,\sigma_Z^2)$ and $b_t \sim N(0,\sigma_a^2)$, the kurtosis is given by $K^{(y)} = 3[1 - \alpha_1^2\sigma_Z^4]/[1 - 3\sigma_Z^2(\alpha_1^2 + \sigma_b^2)]$. For an RCA-*GARCH *model in the form $y_t = \sqrt{h_t}Z_t$, $h_t = \omega_0 + (\alpha_1 + b_{t-1})y_{t-1}^2 + \beta_1 h_{t-1}$, $Z_t \sim N(0,\sigma_Z^2)$ and $b_t \sim N(0,\sigma_b^2)$. Under suitable stationary conditions, the kurtosis of y_t is given by*

$$K^{(y)} = \frac{3[1 - (\alpha_1^2\sigma_Z^2 + \beta_1)^2]}{[1 - 2\alpha_1\beta_1\sigma_Z^2 - 3\alpha_1^2\sigma_Z^2 - 3\sigma_a^2\sigma_Z^2 - \beta_1^2]}. \tag{3.11}$$

3.3. Volatility sign switching models. Consider the sign-switching GARCH(p,q) model by Fornari and Mele [10],

$$y_t = \sqrt{h_t} Z_t, \qquad y_t \mid I_{t-1} \sim N(0, h_t),$$
$$h_t = \omega + \sum_{i=1}^{p} \alpha_i y_{t-i}^2 + \sum_{i=1}^{q} \beta_j h_{t-j} + \sum_{x=1}^{m} \Phi_x s_{t-x},$$
$$s_t = \begin{cases} +1 & \text{if } y_t > 0, \\ 0 & \text{if } y_t = 0, \\ -1 & \text{if } y_t < 0, \end{cases} \tag{3.12}$$

where p, q, and $m > 0$, w, α_i, $(1 = 1,2,\dots,p)$, β_j, $(j = 1,2,\dots,q)$, and Φ_x, $(x = 1,2,\dots,m)$ are real parameters, satisfying the following conditions, $w > 0$, $\alpha_1 \geq 0$, $\beta_j \geq 0$. $|\sum_x \Phi_x \leq \omega|$ is important as it guarantees that the process $\{\sigma_t^2\}$ remains positive. Fornari and Mele [10] had derived the kurtosis for $p = 1$, $q = 1$, and $m = 1$. Here, we use an ARMA representation as in Thavaneswaran et al. [22] to derive the kurtosis of y_t. Consider the general class of GARCH(p,q) model for the time series y_t, where

$$y_t = \sqrt{h_t} Z_t,$$
$$h_t = \omega + \sum_{i=1}^{p} \alpha_i y_{t-i}^2 + \sum_{j=1}^{q} \beta_i h_{t-j} + \sum_{x=1}^{m} \Phi_x s_{t-x}, \tag{3.13}$$

where Z_t is a sequence of independent, identically distributed random variables with zero mean, unit variance. Let $u_t = y_t^2 - h_t$ be the martingale difference and let σ_u^2 be the variance of u_t, (3.13) could be written as

$$\phi(B) y_t^2 = \omega + \beta(B) u_t + \sum_{x=1}^{m} \Phi_x s_{t-x}, \tag{3.14}$$

where $\phi(B) = 1 - \sum_{i=1}^{r} \phi_i B^i$ and $\phi_i = (\alpha_i + \beta_i)$, $\beta(B) = 1 - \sum_{j=1}^{q} \beta_j B^j$ and $r = \max(p,q)$ $\operatorname{var}(y_t^2) = \sum_{i=0}^{\infty} \psi_i^2 \sigma_u^2$ and $\sum_{i=0}^{\infty} \psi_i^2 < \infty$, where the ψ_i's weights are obtained from the relation $\psi(B)\phi(B) = \beta(B)$ with $\psi(B) = 1 - \sum_{i=0}^{\infty} \psi_i B^i$.

Theorem 3.4. *For the* GARCH(p,q) *process specified by (3.13) and under the stationarity assumptions and finite fourth moment, the kurtosis* $K^{(y)}$ *of the process is given by*

(a)

$$K^{(y)} = \left[\frac{(E[h_t])^2 + \Phi^2}{(E[h_t])^2} \right] \left[\frac{E(Z_t^4)}{1 - [E(Z_t^4) - 1] \sum_{j=1}^{\infty} \psi_j^2} \right], \tag{3.15}$$

(b) (i) *the variance of the* y_t^2 *process is* $\gamma_0^{y^2} = \sum_{j=-\infty}^{\infty} \psi_j^2 \sigma_u^2$,
(ii) *the kth-lag autocovariance of the* y_t^2 *process is*

$$\gamma_k^{y^2} = \sigma_u^2 \sum_{j=-\infty}^{\infty} \psi_{k+j}\psi_j \quad \text{for } k \geq 1, \tag{3.16}$$

(iii) *the kth-lag autocorrelation is given by* $\rho_k^{y^2} = \gamma_k^{y^2}/\gamma_0^{y^2}$,
(c) $K^{(y)} = 3/(1 - 2\sum_{j=1}^{\infty} \psi_j^2)$ *for a normal* GARCH(p,q) *process.*

Example 3.5 (normal GARCH models). In this example, we show that the results for normal GARCH(1,1) model and ARCH(1) is a special case. (a) For the GARCH(1,1) model of the form: $y_t = \sqrt{h_t}Z_t$, $h_t = \omega + \alpha_1 y_{t-1}^2 + \beta_1 h_{t-1} + \Phi s_t$, $u_t = y_t^2 - h_t$, $y_t^2 - u_t = \omega + \alpha_1 y_{t-1}^2 + \beta_1(y_{t-1}^2 - u_{t-1}) + \Phi s_t$, $y_t^2 - \alpha_1 y_{t-1}^2 - \beta_1 y_{t-1}^2 = \omega + u_t - \beta_1 u_{t-1} + \Phi s_t$, $(1-\phi_1 B)y_t^2 = \omega + (1-\beta_1 B)u_t + \Phi s_t$, where $\psi_1 = \alpha_1$, $\psi_2 = \alpha_1(\alpha_1+\beta_1)$, $\psi_3 = \alpha_1(\alpha_1+\beta_1)^2, \ldots, \psi_j = \alpha_1(\alpha_1+\beta_1)^{(j-1)}$, $\sum_{j=1}^{\infty}\psi_j^2 = \alpha_1^2 + \alpha_1^2(\alpha_1+\beta_1)^2 + \cdots = \alpha_1^2/(1-(\alpha_1+\beta_1)^2)$. By part (c) of the theorem, $K^{(y)} = [(\omega^2 + \Phi^2(1-(\alpha+\beta))^2)/\omega^2][3[1-(\alpha+\beta)^2]/(1-(\alpha+\beta)^2 - 2\alpha^2)]$ and it turns out to be the same kurtosis formula reported by Fornari and Mele [10]. (b) For the ARCH(1) model of the form $y_t = \sqrt{h_t}Z_t$, $h_t = \omega + \alpha_1 y_{t-1}^2$, $u_t = y_t^2 - h_t$ if we set $\beta_1 = 0$ in (a), then $K^{(y)}$ turns out to be $K^{(y)} = [(\omega^2 + \Phi^2(1-\alpha)^2)/\omega^2][3(1-\alpha^2)/(1-3\alpha^2)]$.

Example 3.6. Consider, for example, the following model as elaborated by Fornari and Mele: [10] $y_t = \sqrt{h_t}Z_t$, $y_t \mid I_{t-1} \sim N(0,h_t)$, $h_t = \omega + \alpha y_{t-1}^2 + \beta h_{t-1} + s_{t-1}v_{t-1}$, $v_t = \delta_0 y_t^2 - \delta_1 h_t - \delta_2$, where v_t is a linear combination of the difference between the observed conditional volatility (y_t^2). The kurtosis of the process is given by

$$K^{(y)} = \left[\frac{(1-\alpha-\beta)^2(3\omega^2+3\delta_2^2) + [6\omega^2\beta + 6\omega^2\alpha - 6\omega\delta_0\delta_2 + 6\omega\delta_1\delta_2](1-\alpha-\beta)}{\omega^2[1-3\alpha^2-\beta^2-3\delta_0^2-\delta_1^2-2\alpha\beta+2\delta_0\delta_1]}\right]. \tag{3.17}$$

When $\delta_0 = \delta_1 = \delta_2 = 0$, the kurtosis of the process described above converge to the one reported by Thavaneswaran et al. [22]. $K^{(y)} = 3[1-(\alpha+\beta)^2]/(1-(\alpha+\beta)^2 - 2\alpha^2)$. Moreover, for an ARCH(1) model the kurtosis $K^{(y)}$ can be obtained by setting $\beta = 0$.

3.4. Sign RCA-GARCH volatility models. Let us state the following proposition, which will be needed in what follows. Consider the general class of GARCH(1,1) for the time series y_t, where $y_t = \sqrt{h_t}Z_t$, $h_t = \omega_0 + (\alpha_1 + b_{t-1} + \Phi s_{t-1})y_{t-1}^2 + \beta_1 h_{t-1}$, where $Z_t \sim N(0,\sigma_Z^2)$ and $a_t \sim N(0,\sigma_a^2)$. The kurtosis of the process is given by $K^{(y)} = 3[(1-(\alpha_1\sigma_Z^2+\beta_1)^2)/(1-3\sigma_Z^4(\Phi^2+\alpha_1^2+\sigma_b^2)+\sigma_Z^2(1+2\alpha_1\beta_1))]$ and when $\Phi = 0$, $K^{(y)}$ turns out to be the one in Lemma 3.3.

Theorem 3.7. *Suppose* y_t *is an RCA model with* GARCH(p,q) *innovations of the form*

$$\begin{gathered} y_t = (\phi + b_t)y_{t-1} + \varepsilon_t, \\ \varepsilon_t = \sqrt{h_t}Z_t, \\ h_t = \omega + \sum_{i=1}^{p}\alpha_i\varepsilon_{t-i}^2 + \sum_{j=1}^{q}\beta_j h_{t-j}, \end{gathered} \tag{3.18}$$

where b_t is an uncorrelated noise process with zero mean and with variance σ_b^2. Then, the following relationship holds:

$$\text{(i) } E(y_t^2) = \frac{E[h_t]}{(1-\phi^2-\sigma_b^2)},$$

$$\text{(ii) } E(y_t^4) = \left[\frac{6(\sigma_b^2+\phi^2)}{(1-\phi^2-\sigma_b^2)(1-6\phi^2\sigma_b^2-\phi^4-3\sigma_b^4)}\right](E[h_t])^2 + \left[\frac{3}{(1-6\phi^2\sigma_b^2-\phi^4-3\sigma_b^4)}\right]E(h_t^2),$$

$$\text{(iii) } K^{(y)} = \left[\frac{3(1-\phi^2-\sigma_b^2)[2(\sigma_b^2+\phi^2)(1-(\alpha+\beta)^2-2\alpha^2)+(1-\phi^2-\sigma_b^2)(1-(\alpha+\beta)^2)]}{(1-6\phi^2\sigma_b^2-\phi^4-3\sigma_b^4)(1-(\alpha+\beta)^2-2\alpha^2)}\right]. \tag{3.19}$$

Example 3.8. Let $\{y_t\}$ be a sign *RCA*-GARCH(1,1) time series satisfying conditions (i) and (ii) given by

$$y_t = (\phi + b_t + \Phi s_t)y_{t-1} + \varepsilon_t, \tag{3.20}$$

where

$$\begin{gathered} \varepsilon_t = \sqrt{h_t}Z_t, \\ h_t = \omega + \alpha_1\varepsilon_{t-1}^2 + \beta_1 h_{t-1}, \end{gathered} \tag{3.21}$$

where Z_t and b_t are sequences of independent, identically distributed random variables with zero mean, variance given by σ_Z^2 and σ_b^2, respectively, ω, α_1, β_1, and Φ are real parameters satisfying the following conditions, $\omega > 0$, $\alpha_1 \geq 0$, $\beta_1 \geq 0$. $|\Phi_x| \leq \omega$.

Note: $E(s_t^2) = 1$, and in order to calculate the kurtosis, we observe that $E(s_t^4) = 1$. Then, we have the following moment properties:

(a)

$$Ey_t = 0, \qquad E(y_t^2) = \frac{\omega\sigma_Z^2}{[1-(\phi^2+\sigma_b^2+\Phi^2)][1-(\alpha_1\sigma_Z^2+\beta_1)]}; \tag{3.22}$$

(b) if $\{b_t\}$ and $\{\varepsilon_t\}$ are normally distributed random variables then the kurtosis $K^{(y)}$ of the process $\{y_t\}$ is given by

$$K^{(y)} = \frac{3(1-(\alpha_1\sigma_Z^2+\beta_1)^2)[1-(\phi^2+\sigma_b^2+\Phi^2)]^2}{(1-3\alpha_1^2\sigma_Z^4-2\alpha_1\beta_1\sigma_Z^2-\beta_1^2)(1-6(\phi^2\Phi^2+\Phi^2\sigma_b^2+\phi^2\sigma_b^2)-\Phi^4-\phi^4-3\sigma_b^4)} + \frac{6(\Phi^2+\phi^2+\sigma_b^2)[1-(\phi^2+\sigma_b^2+\Phi^2)]}{(1-6(\phi^2\Phi^2+\Phi^2\sigma_b^2+\phi^2\sigma_b^2)-\Phi^4-\phi^4-3\sigma_b^4)}. \tag{3.23}$$

Example 3.9. Consider the RCA Sign-GARCH volatility models. Let us state the following proposition, which will be needed in what follows. Consider the general class of GARCH$(1,1)$ for the time series y_t, where

$$\begin{aligned} y_t &= (\phi + b_t) y_{t-1} + \varepsilon_t, \\ \varepsilon_t &= \sqrt{h_t} Z_t, \\ h_t &= \omega + (\alpha_1 + a_{t-1} + \Phi s_{t-1}) \varepsilon_{t-1}^2 + \beta_1 h_{t-1}, \end{aligned} \tag{3.24}$$

where $Z_t \sim N(0, \sigma_Z^2)$, $b_t \sim N(0, \sigma_b^2)$, and $a_t \sim N(0, \sigma_a^2)$,

$$s_t = \begin{cases} +1 & \text{if } y_t > 0, \\ 0 & \text{if } y_t = 0, \\ -1 & \text{if } y_t < 0, \end{cases} \tag{3.25}$$

$$E(h_t) = \frac{\omega}{[1 - (\alpha_1 \sigma_Z^2 + \beta_1)]} \text{ assuming } E(s_t) = 0,$$

$$E(h_t^2) = \frac{\omega^2 (\sigma_Z^2 \alpha_1 + \beta_1 + 1)}{[1 - \beta_1 (2\sigma_Z^2 \alpha_1 + \beta_1) - 3\sigma_Z^4 (\Phi^2 + \alpha_1^2 + \sigma_a^2)][1 - \sigma_Z^2 \alpha_1 - \beta_1]},$$

$$\operatorname{Var}(y_t) = \frac{\omega \sigma_Z^2}{[1 - (\phi^2 + \sigma_b^2)][1 - (\alpha_1 \sigma_Z^2 + \beta_1)]},$$

$$E[y_t^4] = \frac{3\sigma_Z^4}{(1 - 3\sigma_b^4 - \phi^4 - 6\phi^2 \sigma_b^2)} E[h_t^2] + \frac{6\sigma_Z^4 (\sigma_b^2 + \phi^2)}{(1 - 3\sigma_b^4 - \phi^4 - 6\phi^2 \sigma_b^2)[1 - (\phi^2 + \sigma_b^2)]} (E[h_t])^2. \tag{3.26}$$

The kurtosis of the process is given by

$$\begin{aligned} K^{(y)} = {} & \frac{3[1 - (\phi^2 + \sigma_b^2)]^2 [1 - (\alpha_1 \sigma_Z^2 + \beta_1)^2]}{[1 - \beta_1 (2\sigma_Z^2 \alpha_1 + \beta_1) - 3\sigma_Z^4 (\Phi^2 + \alpha_1^2 + \sigma_a^2)](1 - 3\sigma_b^4 - \phi^4 - 6\phi^2 \sigma_b^2)} \\ & + \frac{6(\sigma_b^2 + \phi^2)[1 - (\phi^2 + \sigma_b^2)]}{(1 - 3\sigma_b^4 - \phi^4 - 6\phi^2 \sigma_b^2)}. \end{aligned} \tag{3.27}$$

The proof of (3.27) parallels the proof of Theorem 2.3.

Note that when $\phi = 0$, $\sigma_b = 0$, $\sigma_a = 0$, and $\sigma_Z = 1$, the kurtosis converges to

$$K^{(y)} = \frac{3[1 - (\alpha_1 + \beta_1)^2]}{(1 - (\alpha_1 + \beta_1)^2) - 2\alpha_1^2} > 3. \tag{3.28}$$

3.5. Unconditional mixed distribution. The following lemma will be used in Theorem 3.11 to derive the kurtosis for the GARCH process.

Lemma 3.10. *For the mixture model given in Timmermann [24]. $y_t = \mu_{s_t} + \sigma_{s_t} Z_t$, where the Markov process S_t has the states $1,2,\ldots,k$ with steady state probabilities $\lambda_j = \pi_j$, $j = 1,2,\ldots,k$ and the unconditional mean and variance are given by μ_{s_t} and $\sigma^2_{s_t}$, respectively. The moments are given by*

(a) $\mu = E(y_t) = \sum_{j=1}^{k} \lambda_j \mu_j$,

(b) $m_2 = \mathrm{Var}(y_t) = \sigma^2 = \sum_{j=1}^{k} \lambda_j \sigma_j^2 + \sum_{j=1}^{k} \lambda_j (\mu_j - \mu)^2$,

(c) $m_3 = \sum_{j=1}^{k} \lambda_j (\mu_j - \mu)^3 + 3 \sum_{j=1}^{k} \sum_{i<j} \lambda_j \lambda_i (\sigma_j^2 - \sigma_i^2)(\mu_j - \mu)$,

(d) $m_n = E[(y_t - \mu)^n] = \sum_{i=1}^{k} \lambda_i \sum_{j=0}^{n} \binom{n}{j} E[Z_t^j] \sigma_i^j (\mu_i - \mu)^{n-j}$,

(e) *with regard to kurtosis, let Z be a k component mixed normal random variable but with $\mu_1 = \cdots = \mu_k = \mu$, so that $E(Z) = \sum_{j=1}^{k} \lambda_j \mu_j = \mu$. $\phi(u;\mu_j,\sigma_j^2) = (1/\sqrt{2\pi}\sigma_j)\exp\{-(1/2)((u-\mu_j)/\sigma_j)^2\}$, $j = 1,2,\ldots,k$. Then, from Jensen's inequality, $\sum_j \lambda_j \sigma_j^4 = \sum_j \lambda_j (\sigma_j^2)^2 > (\sum_j \lambda_j \sigma_j^2)^2$, so that the kurtosis of the mixture model* KM *is given by*

$$\mathrm{KM} = \frac{m_4}{m_2^2} = \frac{E[(Z-\mu)^4]}{[E[(Z-\mu)^2]]^2} = K^{(Z)} \frac{\sum_{j=1}^{k} \lambda_j \sigma_j^4}{\left(\sum_{j=1}^{k} \lambda_j \sigma_j^2\right)^2} \geq K^{(Z)}. \tag{3.29}$$

Theorem 3.11. *The kurtosis of* GARCH(p,q) *process having k component mixture distribution for Z_t is given by*

(a)

$$K^{(y)} = \frac{E(Z_t^4)}{E(Z_t^4) - [E(Z_t^4) - 1] \sum_{j=o}^{\infty} \psi_j^2}, \tag{3.30}$$

where $Z_t = (Z - \mu)/\sigma$ with μ is the mean of the mixture distribution, σ is the standard deviation of the mixture distribution as in Lemma 3.10 and $E(Z_t^4) = K_c \sum_j \lambda_j \sigma_j^4$, where K_c stands for the kurtosis of the component distribution,

(b) (i) *the variance of the y_t process is $\gamma_0^y = \sum_{j=-\infty}^{\infty} \psi_j^2 \sigma_u^2$, where*

$$\sigma_u^2 = \frac{\left(\omega/(1-\phi_1-\phi_2-\cdots)\right)^2}{1-(E(Z_t^2)-1)\sum_{j=1}^{\infty}\psi_j^2}[E(Z_t^4)-1], \tag{3.31}$$

(ii) *the lth-lag autocovariance of the y_t process is $\gamma_l^y = \sigma_u^2 \sum_{j=-\infty}^{\infty} \psi_{l+j}\psi_j$ for $l \geq 1$,*

(c) *for a k-component mixture* GARCH *process,*

$$\mathrm{KM}^{(y)} = \frac{K_c \sum_j \lambda_j \sigma_j^4}{K_c \sum_j \lambda_j \sigma_j^4 - [K_c \sum_j \lambda_j \sigma_j^4 - 1] \sum_{j=0}^{\infty} \psi_j^2}. \tag{3.32}$$

Moreover, for the normal mixture GARCH, $K_c = 3$.

Corollary 3.12. *The kurtosis of power* GARCH(p,q) *process having k component mixture distribution for* Z_t *is given by*

(a)

$$K^{(y)} = \frac{E(Z_t^4)}{E(Z_t^4) - [E(Z_t^4) - 1]\sum_{j=o}^{\infty}\psi_j^2}, \tag{3.33}$$

where $Z_t = (Z - \mu)/\sigma$ *with* μ *is the mean of the mixture distribution,* σ *is the standard deviation of the mixture distribution, and* $E(Z_t^4) = K_c \sum_j \lambda_j \sigma_j^4$, *where* K_c *stands for the kurtosis of the component distribution,*

(b) (i) *the variance of the* $|y_t|^\delta$ *process is* $\gamma_0^{y^\delta} = \sum_{j=-\infty}^{\infty}\psi_j^2\sigma_u^2$, *where*

$$\sigma_u^2 = \frac{(\omega/(1-\phi_1-\phi_2-\cdots))^2}{1-(E(Z_t^2)-1)\sum_{j=1}^{\infty}\psi_j^2}[E(Z_t^4)-1], \tag{3.34}$$

(ii) *the l-lag autocovariance of the* $|y_t|^\delta$ *process is*

$$\gamma_l^{y^\delta} = \sigma_u^2 \sum_{j=-\infty}^{\infty}\psi_{l+j}\psi_j \quad \text{for } l \geq 1, \tag{3.35}$$

(c) *for a k-component mixture power* GARCH *process,*

$$\mathrm{KM}^{(y)} = \frac{K_c\sum_j\lambda_j\sigma_j^4}{K_c\sum_j\lambda_j\sigma_j^4 - [K_c\sum_j\lambda_j\sigma_j^4 - 1]\sum_{j=0}^{\infty}\psi_j^2}. \tag{3.36}$$

Theorem 3.13. *Suppose* y_t *be a second order linear stationary process having a* MA *(moving average) representation of the form* $y_t - \mu = \sum_{j=0}^{\infty}\psi_j a_{t-j}$, *where* a_t *is a* GARCH *process given by*

$$\begin{aligned} a_t &= \sqrt{h_t}Z_t, \\ h_t &= \omega + \sum_{i=1}^{p}\alpha_i a_{t-i}^2 + \sum_{j=1}^{q}\beta_j h_{t-j} \end{aligned} \tag{3.37}$$

with mean zero, variance σ_a^2, *and kurtosis* $K^{(a)}$, *and* Z_t *is a sequence of independent, identically distributed random variables with zero mean, unit variance. Then the variance and the kurtosis of* y_t *are*

$$\begin{aligned} &\text{(i) } \operatorname{Var}(y_t) = \sigma_a^2\sum_{j=0}^{\infty}\psi_j^2, \\ &\text{(ii) } K^{(y)} = \frac{K^{(a)}\left[\sum_{j=0}^{\infty}\psi_j^4\right] + 6\sum_{i<j}^{\infty}\psi_i^2\psi_j^2}{\left(\sum_{j=0}^{\infty}\psi_j^2\right)^2} \end{aligned} \tag{3.38}$$

provided $\sum_{j=0}^{\infty}\psi_j^4 < \infty$.

Proof of the above theorem is given in Ghahramani and Thavaneswaran [11].

4. Applications

In Thavaneswaran et al. [21], we have studied the volatility forecasting for zero mean GARCH processes and derived the forecast error variance in terms of GARCH kurtosis. In this section, we give recent application of GARCH kurtosis in forecasting and in Black-Scholes model-based option pricing.

4.1. GARCH forecasts. Let $y_n^2(l)$ be the forecast of y_{n+l}^2 based on n observations $y_1, y_2, \ldots, y_n$. The following theorem gives the formula for forecast error variance in terms of the kurtosis and the ψ weights.

Theorem 4.1. *For the* GARCH(p,q) *process specified by (3.4), under the stationarity assumptions and finite fourth moment, the kurtosis* $K^{(y)}$ *of the process is given by*

(a)

$$K^{(y)} = \frac{E(Z_t^4)}{E(Z_t^4) - [E(Z_t^4) - 1]\sum_{j=0}^{\infty}\psi_j^2}, \tag{4.1}$$

(b) *the variance of the* y_t^2 *process is* $\gamma_0^{y^2} = \sum_{j=0}^{\infty}\psi_j^2\sigma_u^2$, *where* $\sigma_u^2 = \mu^2(K^y - 1)/\sum_{j=0}^{\infty}\psi_j^2$ *and* $\mu = E(y_t^2) = \omega/(1 - \phi_1 - \phi_2 - \cdots - \phi_r)$,

(c)

$$\operatorname{Var}(e_n(l)) = \frac{(\omega/(1-\phi_1-\phi_2-\cdots-\phi_r))^2}{\sum_{j=0}^{\infty}\psi_j^2}[K^{(y)} - 1]\left[1 + \sum_{j=1}^{l-1}\psi_j^2\right]. \tag{4.2}$$

Proof of part (a) follows from Theorem 3.1. For the proof of part (b), from (3.4), $\mu = E(y_t^2) = \omega/(1-\phi_1-\phi_2-\cdots-\phi_r)$, and $K^{(y)} = E(y_t^4)/(E(y_t^2))^2 = E(y_t^4)/\mu^2$.

Hence $\sigma_u^2 = \mu^2(K^{(y)} - 1)/\sum_{j=0}^{\infty}\psi_j^2$. Parts (c) and (d) follow from the fact that for a stationary ARMA process with error variance σ_u^2 the $\operatorname{Var}(e_n(l)) = \sigma_u^2(1 + \psi_1^2 + \cdots + \psi_{l-1}^2)$ and by part (b). It has been shown that the l steps ahead forecast error variance depends on the ψ weights which vary from one model to another.

Example 4.2. Now, we show that the results for normal GARCH(1,1) and ARCH(1) are special cases.

(a) For the GARCH(1,1) model of the form given in (3.4), $y_t = \sqrt{h_t}Z_t$, $h_t = \omega + \alpha_1 y_{t-1}^2 + \beta_1 h_{t-1}$, $u_t = y_t^2 - h_t$, $y_t^2 - u_t = \omega + \alpha_1 y_{t-1}^2 + \beta_1(y_{t-1}^2 - u_{t-1})$, $y_t^2 - \alpha_1 y_{t-1}^2 - \beta_1 y_{t-1}^2 = \omega + u_t - \beta_1 u_{t-1}$, $(1-\phi_1 B)y_t^2 = \omega + (1-\theta B)u_t$, where $\psi_1 = \alpha_1$, $\psi_2 = \alpha_1(\phi_1+\beta_1)$, $\psi_3 = \alpha_1(\phi_1+\beta_1)^2, \ldots, \psi_j = \alpha_1(\phi_1+\beta_1)^{(j-1)}$, $\sum_{j=1}^{\infty}\psi_j^2 = \alpha_1^2 + \alpha_1^2(\alpha_1+\beta_1)^2 + \cdots = \alpha_1^2/(1-(\alpha_1+\beta_1)^2)$.

By part (c) of the theorem $K^{(y)} = 3/(1 - 2\sum_{j=1}^{\infty}\psi_j^2) = 3/(1 - 2\alpha_1^2/(1-(\alpha_1+\beta_1)^2)) = 3[1-(\alpha_1+\beta_1)^2]/(1-(\alpha_1+\beta_1)^2 - 2\alpha_1^2)$, and it turns out to be the same as the one given in Bollerslev [4]. Moreover,

$$\sigma_u^2 = \frac{(K^{(y)} - 1)\mu^2(1-\alpha_1^2)}{1 + \beta_1^2 - 2\phi_1\beta_1}. \tag{4.3}$$

(b) For the ARCH(1) model of the form $y_t = \sqrt{h_t}Z_t$, $h_t = \omega + \alpha_1 y_{t-1}^2$, $u_t = y_t^2 - h_t$, if we set $\beta_1 = 0$ in (a), then $K^{(y)}$ turns out to be $3(1-\alpha_1^2)^2/(1-3\alpha_1^2)$ and $\sigma_u^2 = (K^{(y)}-1)\mu^2(1-\alpha_1^2)$.

For the GARCH(1,1) models, the variance for l steps ahead forecast is given by $\mathrm{Var}(e_n(l)) = [(K^{(y)}-1)\mu^2(1-\alpha_1^2)/(1+\beta_1^2-2\alpha_1\beta_1)](1+\psi_1^2+\cdots+\psi_{l-1}^2)$ and for the ARCH(1) models, the variance for l-steps ahead forecast is given by $\mathrm{Var}(e_n(l)) = [(K^{(y)}-1)\mu^2(1-\alpha_1^2)](1+\psi_1^2+\cdots+\psi_l^2)$.

Example 4.3. In this example, we show that the results for normal sign GARCH(1,1) model and ARCH(1) are special cases.

(a) For the GARCH(1,1) model, $y_t = \sqrt{h_t}Z_t$, $h_t = \omega + \alpha_1 y_{t-1}^2 + \beta_1 h_{t-1} + \Phi s_t$, $u_t = y_t^2 - h_t$, $y_t^2 - u_t = \omega + \alpha_1 y_{t-1}^2 + \beta_1(y_{t-1}^2 - u_{t-1}) + \Phi s_t$, $y_t^2 - \alpha_1 y_{t-1}^2 - \beta_1 y_{t-1}^2 = \omega + u_t - \beta_1 u_{t-1} + \Phi s_t$, $(1-\phi_1 B)y_t^2 = \omega + (1-\beta_1 B)u_t + \Phi s_t$, where $\psi_1 = \alpha_1$, $\psi_2 = \alpha_1(\alpha_1+\beta_1)$, $\psi_3 = \alpha_1(\alpha_1+\beta_1)^2, \dots, \psi_j = \alpha_1(\alpha_1+\beta_1)^{(j-1)}$, $\sum_{j=1}^{\infty}\psi_j^2 = \alpha_1^2 + \alpha_1^2(\alpha_1+\beta_1)^2 + \cdots = \alpha_1^2/(1-(\alpha_1+\beta_1)^2)$. By part (c) of the theorem,

$$\begin{aligned} K^{(y)} &= \left[\frac{(E[h_t])^2 + \Phi^2}{(E[h_t])^2}\right]\left[\frac{E(Z_t^4)}{1-[E(Z_t^4)-1]\sum_{j=1}^{\infty}\psi_j^2}\right] \\ &= \left[\frac{\omega^2 + \Phi^2(1-(\alpha+\beta))^2}{\omega^2}\right]\left[\frac{3[1-(\alpha+\beta)^2]}{1-(\alpha+\beta)^2-2\alpha^2}\right], \end{aligned} \tag{4.4}$$

and it turns out to be the same kurtosis formula reported by Fornari and Mele [10].

(b) For the ARCH(1) model of the form $y_t = \sqrt{h_t}Z_t$, $h_t = \omega + \alpha_1 y_{t-1}^2$, $u_t = y_t^2 - h_t$, if we set $\beta_1 = 0$ in (a), then $K^{(y)}$ turns out to be

$$\begin{aligned} K^{(y)} &= \left[\frac{\omega^2 + \Phi^2(1-\alpha)^2}{\omega^2}\right]\left[\frac{3(1-\alpha^2)}{1-3\alpha^2}\right], \\ \mathrm{Var}(e_n(l)) &= \frac{(\omega/(1-\phi_1-\phi_2-\cdots-\phi_r))^2}{\sum_{j=0}^{\infty}\psi_j^2}[K^{(y)}-1]\left[1+\sum_{j=1}^{l-1}\psi_j^2\right]. \end{aligned} \tag{4.5}$$

4.2. Option pricing. The Black-Scholes (BS) option pricing model is the cornerstone for option pricing (a geometric Brownian motion model). Black and Scholes used the following model for stock price:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{4.6}$$

where the process W_t is a standard Brownian motion. Generally, a call (resp., put) option is the right to buy (resp., sell) a particular asset for a specified amount, the strike price K at a specified time in the future, the expiration time T. If the option is of such a type that it can be exercised only on the expiration date itself, then it is called a European option. Let S_T be the price of the underlying asset at expiration time T. Then the payoff g of a European call option at time T is given by

$$g(S_T) = \mathrm{Max}(S_T - K, 0) = (S_T - K)^+. \tag{4.7}$$

This means that the option is exercised if $S_T > K$ and abandoned otherwise. Let r be the risk-free interest rate. Then a probability measure Q is called an equivalent martingale measure to the probability measure P for the discounted price process $\widetilde{S}_t = e^{-rt}S_t$ if

$$E_Q[\widetilde{S}_t \mid F_s] = \widetilde{S}_s \tag{4.8}$$

for each $s \leq t \leq T$ and $Q \sim P$, where F_t is the history of the process up to time t. That is, the discounted price process $(\widetilde{S}_t)$ is a martingale under the probability measure Q. According to the fundamental theorem of asset pricing, an arbitrage-free price C_t of an option at time t is given by the conditional expectation of the *discounted payoff* under an equivalent martingale measure Q,

$$\begin{aligned} C_t &= E_Q[e^{-r(T-t)}g(S_T) \mid F_t], \\ X_t &= \log S_t - \log S_{t-1} \end{aligned} \tag{4.9}$$

are normally distributed. Now (4.6) becomes

$$S_t = S_0 e^{\sigma W_t + (\mu - \sigma^2/2)t} \tag{4.10}$$

with $0 \leq t \leq T$, where (W_t) is a standard Brownian motion, g is the drift, and m the volatility of the underlying stock. Using Ito's formula, the model can equivalently be described by

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \tag{4.11}$$

For this model, there exists a unique martingale measure Q which is given by Girsanov's theorem

$$\frac{dQ}{dP} = \exp\left(\frac{r-\mu}{\sigma} W_T - \frac{(r-\mu)^2}{2\sigma^2} T\right). \tag{4.12}$$

Some calculations, yield the Black-Scholes formula

$$C_{\mathrm{BS}} = S_0 \Phi\left(\frac{\log(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\log(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right), \tag{4.13}$$

where Φ denotes the cumulative distribution function of a standard normal variable. In formula (4.13), only the volatility parameter σ appears and the drift term μ vanishes. In the literature two different ways of calculating volatility has been discussed. The first is the empirical estimation from historical data. The second method is to calculate the implied volatility by equating the theoretical call price from the Black-Scholes formula and equate with the market price.

The implied volatility of the underlying stock which, when substituted into the Black-Scholes formula, gives a theoretical price equal to the market price. This equation can be solved numerically. However, in practice, if we calculate the implied volatility for different strikes and expiration times on the same underlying asset, then we find that the *volatility is not constant*. The received shape of the implied volatility versus the strike curve is called the smile. This effect is also a consequence of the fact that the constant volatility model is not adequate for the log-returns. In the literature, nonconstant volatility had been modeled by GARCH processes. However, for calculation purposes volatility σ_t in the Black-Scholes formula (4.13) had been replaced by $E(h_t) = E\sigma_t^2$ (a constant) as in the following example. Consider the results reported by Gouriéroux [12] on the implicit price index associated with the GDP. This variable, denoted by GD_t, is first transformed to obtained stationarity,

$$y_t = 100\log\left(\frac{\mathrm{GD}_t}{\mathrm{GD}_{t-1}}\right) \tag{4.14}$$

(see Gouriéroux [12]) has been fitted from the quarterly data covering the period 1948 to 1983. For an autoregressive model with GARCH(1,1) errors, the results are summarized below,

$$\begin{aligned} y_t &= \underset{(0.060)}{0.141} + \underset{(0.081)}{0.433}y_{t-1} + \underset{(0.110)}{0.229}y_{t-2} + \underset{(0.077)}{0.349}y_{t-3} - \underset{(0.104),}{0.162}y_{t-4} + \varepsilon_t \\ h_t &= \underset{(0.060)}{0.007} + \underset{(0.081)}{0.135}\varepsilon_{t-1}^2 + \underset{(0.110).}{0.829}h_{t-1} \end{aligned} \tag{4.15}$$

The unconditional variance resulting from the AR-GARCH estimation is

$$\begin{aligned} E[h_t] &\approx 0.007 + 0.135E(\varepsilon_{t-1}^2) + 0.829E(\sigma_{t-1}^2) \\ \Longleftrightarrow E[h_t] &\approx \frac{0.007}{1-0.135-0.829} = 0.199. \end{aligned} \tag{4.16}$$

In (4.13), the value of 0.199 for σ^2 has been used to calculate the price of an option.

4.3. Analytical approximation in option pricing using GARCH kurtosis. We start by assuming that the asset return dynamics, under the physical measure P, is

$$\ln\left(\frac{S_{t+1}}{S_t}\right) = r + \lambda\sqrt{h_{t+1}} - \frac{1}{2}h_{t+1} + \sqrt{h_{t+1}}\varepsilon_{t+1}, \tag{4.17}$$

where $\varepsilon_t \overset{P}{\sim} N(0,1)$. For the conditional variance, h_{t+1}, the following three models have been used to obtain the approximate value of a European option in Heston and Nandi [17], Duan et al. [5], and Duan and Wei [6]:

$$\begin{aligned} h_{t+1} &= \beta_0 + \beta_1 h_t + \beta_2 h_t\left(\varepsilon_t^2 - \theta^2\right)^2, \\ h_{t+1} &= \beta_0 + \beta_1 h_t + \beta_2 h_t \varepsilon_t^2 + \beta_3 h_t \operatorname{Max}(0, -\varepsilon_t^2), \\ \ln(h_{t+1}) &= \beta_0 + \beta_1 \ln(h_t) + \beta_4\left(|\varepsilon_t| + \gamma\varepsilon_t\right). \end{aligned} \tag{4.18}$$

Note that r is the one-period continuously compounded risk-free rate, λ is a constant unit risk premium, h_{t+1} is the conditional variance of the asset return, and $\{\varepsilon_t,\ t = 0,1,2,\ldots\}$ forms a sequence of independent standard normal random variables with respect to the measure P. We may also model the volatility by a class of RCA GARCH models proposed in Thavaneswaran et al. [22] and by models discussed in Section 3. It is of interest to note that under the physical measure P neither the log return process $\ln(S_{t+1}/S_t)$ nor the return process (S_{t+1}/S_t) are martingales. In order to calculate the option price, we have to find the locally risk-neutralized measure Q under which the return process (S_{t+1}/S_t) is a martingale. Using the basic properties of log normal distribution, one can easily show, under the probability measure Q, the asset return process

$$\begin{gathered}\ln\frac{S_{t+1}}{S_t} = r - \frac{h_{t+1}}{2} + \sqrt{h_{t+1}}\varepsilon^*_{t+1},\\ h_{t+1} = \beta_0 + \beta_1 h_t + \beta_2 h_t(\varepsilon^*_t - \theta - \lambda)^2,\\ \varepsilon^*_{t+1} \mid F_t \overset{Q}{\sim} N(0,1).\end{gathered} \tag{4.19}$$

This implies

$$S_T = S_0 \exp\left[rT - \frac{1}{2}\sum_{s=1}^{T} h_t + \sum_{s=1}^{T} h_t\varepsilon^*_t\right]. \tag{4.20}$$

For a European call option with a payoff at time T, $\operatorname{Max}(S_T - K, 0)$ its time-0 value is $C(S_0,\sigma_1;T,r,\beta_0,\beta_1,\beta_2,\theta+\lambda) = e^{-rT}E_0^Q\{\operatorname{Max}(S_T - K,0)\}$.

The approximate closed-form solution using edgeworth expansion is given by

$$C_{\text{app}} = C + k_3 A_3 + (k_4 - 3)A_4, \tag{4.21}$$

where

$$\begin{gathered}C = S_0 N(\tilde{d}) - Ke^{-rT}N(\tilde{d} - \sigma_{\rho_T}),\\ \tilde{d} = d + \delta,\\ d = \frac{\ln(S_0/K) + rT + (1/2)\sigma^2_{\rho_T}}{\sigma_{\rho_T}},\\ \delta = \frac{\mu_{\rho_T} - rT + (1/2)\sigma^2_{\rho_T}}{\sigma_{\rho_T}},\\ A_3 = \frac{1}{3!}S_0\sigma_{\rho_T}\left[(2\sigma_{\rho_T} - \tilde{d})n(\tilde{d}) - \sigma^2_{\rho_T}N(\tilde{d})\right],\\ A_4 = \frac{1}{4!}S_0\sigma_{\rho_T}\left[(\tilde{d}^2 - 1 - 3\sigma_{\rho_T}(\tilde{d} - \sigma_{\rho_T}))n(\tilde{d}) + \sigma^3_{\rho_T}N(\tilde{d})\right].\end{gathered} \tag{4.22}$$

Note. μ_{ρ_T} and σ_{ρ_T} are the mean and standard deviation of the cumulative return, that is, $\ln(S_T/S_0)$, conditional on time 0 information; k_3 and k_4 are the skewness and kurtosis coefficients of the standardized cumulative return, conditional on time-0 information. Similarly for the models (4.18) and (4.19) under the locally risk-neutralized probability measure Q, the asset return dynamic becomes

$$\ln\left(\frac{S_{t+1}}{S_t}\right) = r - \frac{1}{2}h_{t+1} + \sqrt{h_{t+1}}\varepsilon_{t+1}, \tag{4.23}$$

where $\varepsilon_t \overset{Q}{\sim} N(0,1)$

$$\begin{aligned} h_{t+1} &= \beta_0 + h_t\left[\beta_1 + \beta_2(\varepsilon_t - \lambda)^2 + \beta_3 \max(0, -\varepsilon_t + \lambda)^2\right], \\ \ln(h_{t+1}) &= \beta_0 + \beta_1 \ln(h_t) + \beta_4(|\varepsilon_t - \lambda| + \gamma(\varepsilon_t - \lambda)), \end{aligned} \tag{4.24}$$

respectively.

5. Conclusions

Granger [13], a Nobel Prize winner (2003), had cited the first authors' work (Abraham and Thavaneswaran [1]) in his Berkeley Symposium. In this paper, some results in [1] are extended to volatility models. Some new volatility models are introduced and their moment properties are discussed. Kurtosis of these models is expressed in terms of the model parameters. Application to volatility forecasting and analytical approximation to option pricing are also discussed in some detail.

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