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Research Article

Valuation of Game Options in Jump-Diffusion Model and with Applications to Convertible Bonds

Lei Wang and Zhiming Jin

College of Science, National University of Defense Technology, ChangSha 410073, China

Correspondence should be addressed to Lei Wang, wlnudt@163.com

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Game option is an American-type option with added feature that the writer can exercise the option at any time before maturity. In this paper, we consider some type of game options and obtain explicit expressions through solving Stefan(free boundary) problems under condition that the stock price is driven by some jump-diffusion process. Finally, we give a simple application about convertible bonds.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space hosting a Brownian motion $W = \{W_t : t \geq 0\}$ and an independent Poisson process $N = \{N_t : t \geq 0\}$ with the constant arrival rate λ, both adapted to some filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}$ satisfying usual conditions. Consider the Black-Scholes market. That is, there is only one riskless bond B and a risky asset S. They satisfy, respectively,

$$\begin{gathered} dB_t = rB_t dt, \quad t \geq 0, \\ dS_t = S_{t-}\left[\mu dt + \sigma dW_t - y_0\left(dN_t - \lambda dt\right)\right] \end{gathered} \tag{1.1}$$

for some constants $\mu \in \mathbf{R}$, $r, \sigma > 0$ and $y_0 \in (0,1)$. Note that the absolute value of relative jump sizes is equal to y_0, and jumps are downwards. It can be comprehended as a downward tendency of the risky asset price brought by bad news or default and so on. From Itô formula we can obtain

$$S_t = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2 + \lambda y_0\right)t + \sigma W_t\right\}\left(1 - y_0\right)^{N_t}. \tag{1.2}$$

Suppose that $X = \{X_t : t \leq T\}$ and $Y = \{Y_t : t \leq T\}$ be two continuous stochastic processes defined on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P})$ such that for all $0 \leq t \leq T$, $X_t \leq Y_t$a.s.. The game option is a contract between a holder and writer at time $t = 0$. It is a general American-type option with the added property that the writer has the right to terminate the contract at any time before expiry time T. If the holder exercises first, then he/she may obtain the value of X at the exercise time and if the writer exercise first, then he/she is obliged to pay to the holder the value of Y at the time of exercise. If neither has exercised at time T and $T < \infty$, then the writer pays the holder the value X_T. If both decide to claim at the same time then the lesser of the two claims is paid. In short, if the holder will exercise with strategy τ and the writer with strategy γ, we can conclude that at any moment during the life of the contract, the holder can expect to receive $Z(\tau, \gamma) \triangleq X_\tau 1_{(\tau \leq \gamma)} + Y_\gamma 1_{(\gamma < \tau)}$. For a detailed description and the valuation of game options, we refer the reader to Kifer [1], Kyprianou [2], Ekström [3], Baurdoux and Kyprianou [4], Kühn et al. [5], and so on.

It is well known that in the no-arbitrage pricing framework, the value of a contract contingent on the asset S is the maximum of the expectation of the total discounted payoff of the contract under some equivalent martingale measure. Since the market is incomplete, there are more than one equivalent martingale measure. Following Dayanik and Egami [6], let the restriction to $\mathcal{F}_t$ of every equivalent martingale measure $\mathbf{P}^\alpha$ in a large class admit a Radon-Nikodym derivative in the form of

$$\begin{gathered} \left.\frac{d\mathbf{P}^\alpha}{d\mathbf{P}}\right|_{\mathcal{F}_t} \triangleq \eta_t, \\ d\eta_t = \eta_{t-}\left[\beta dW_t + (\alpha - 1)\left(dN_t - \lambda dt\right)\right], \quad t \geq 0,\ \eta_0 = 1 \end{gathered} \tag{1.3}$$

for some constants $\beta \in \mathbf{R}$ and $\alpha > 0$. The constants β and α are known as the market price of the diffusion risk and the market price of the jump risk, respectively, and satisfy the drift condition

$$\mu - r + \sigma\beta - \lambda y_0(\alpha - 1) = 0. \tag{1.4}$$

Then the discounted value process $\{e^{-rt}S_t : t \geq 0\}$ is a $(\mathbf{P}^\alpha, \mathbf{F})$-martingale. By the Girsanov theorem, the process $\{W_t^\alpha \triangleq W_t - \beta t : t \geq 0\}$ is a Brownian motion under the measure $\mathbf{P}^\alpha$, and $\{N_t : t \geq 0\}$ is a homogeneous Poisson process with the intensity $\lambda^\alpha \triangleq \lambda\alpha$ independent of the Brownian motion W^α under the same measure. The infinitesimal generator of the process S under the probability measure $\mathbf{P}^\alpha$ is given by

$$\mathcal{A}^\alpha f(x) \triangleq (r + \lambda^\alpha y_0) x \frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} + \lambda^\alpha \left[f\left(x(1 - y_0)\right) - f(x)\right], \tag{1.5}$$

on the collection of twice-continuously differentiable functions $f(\cdot)$. It is easily checked that $(\mathcal{A}^\alpha - r)f(x) = 0$ admits two solutions $f(x) = x^{k_1}$ and $f(x) = x^{k_2}$, where $k_1 < 0 < 1 = k_2$ satisfy

$$\frac{1}{2}\sigma^2 k(k-1) + \left(r + \lambda^\alpha y_0\right)k - \left(r + \lambda^\alpha\right) + \lambda^\alpha (1 - y_0)^k = 0. \tag{1.6}$$

Suppose that $\mathbf{P}_x^\alpha$ is the equivalent martingale measure for S under the assumption that $S_0 = x$ for a specified market price $\alpha(\cdot)$ of the jump risk, and denote $\mathbf{E}_x^\alpha$ to be expectation under $\mathbf{P}_x^\alpha$. The following theorem is the Kifer pricing result.

Theorem 1.1. *Suppose that for all $x > 0$*

$$\mathbf{E}_x^\alpha\left(\sup_{0\le t\le T} e^{-rt} Y_t\right) < \infty \tag{1.7}$$

and if $T = \infty$ that $\mathbf{P}_x^\alpha(\lim_{t\uparrow\infty} e^{-rt} Y_t = 0) = 1$. Let $\mathcal{S}_{t,T}$ be the class of $\mathbf{F}$-stopping times valued in $[t,T]$, and $\mathcal{S} \equiv \mathcal{S}_{0,\infty}$, then the price of the game option is given by

$$V(x) = \inf_{\gamma\in\mathcal{S}_{0,T}} \sup_{\tau\in\mathcal{S}_{0,T}} \mathbf{E}_x^\alpha\left(e^{-r(\tau\wedge\gamma)} Z_{\tau,\gamma}\right) = \sup_{\tau\in\mathcal{S}_{0,T}} \inf_{\gamma\in\mathcal{S}_{0,T}} \mathbf{E}_x^\alpha\left(e^{-r(\tau\wedge\gamma)} Z_{\tau,\gamma}\right). \tag{1.8}$$

Further the optimal stopping strategies for the holder and writer, respectively, are

$$\tau^* = \inf\{t \ge 0 : V(S_t) = X_t\} \wedge T, \qquad \gamma^* = \inf\{t \ge 0 : V(S_t) = Y_t\} \wedge T. \tag{1.9}$$

2. A Game Version of the American Put Option (Perpetual Israeli δ-Penalty Put Option)

In this case, continuous stochastic processes are, respectively, given by

$$X_t = (K - S_t)^+, \qquad Y_t = (K - S_t)^+ + \delta, \tag{2.1}$$

where $K > 0$ is the strike-price of the option, $\delta > 0$ is a constant and can be considered as penalty for terminating contract by the writer. For the computation of the following, let us first consider the case of the perpetual American put option with the same parameter K. From Jin [7] we know that the price of the option is

$$V^A(x) = \sup_{\tau\in\mathcal{S}} \mathbf{E}_x^\alpha\left(e^{-r\tau}(K - S_\tau)^+\right) \tag{2.2}$$

with the superscript A representing American. Through martingale method we have the following.

Theorem 2.1. *The price of the perpetual American option is given by*

$$V^A(x) = \begin{cases} K - x & x \in (0, x^*], \\ (K - x^*)\left(\dfrac{x}{x^*}\right)^{k_1} & x \in (x^*, \infty), \end{cases} \tag{2.3}$$

where $x^* = k_1 K/(k_1 - 1)$, the optimal stopping strategy is

$$\tau_* = \inf\{t \ge 0 : S_t \le x^*\}. \tag{2.4}$$

Proposition 2.2. *$V^A(x)$ is decreasing and convex on $(0,\infty)$, and under equivalent martingale measure $\mathbf{P}_x^\alpha$, one has that $\{e^{-rt}V^A(S_t) : t \geq 0\}$ and $\{e^{-r(t\wedge\tau_{x^*})}V^A(S_{t\wedge\tau_{x^*}}) : t \geq 0\}$ are supermartingale and martingale, respectively.*

Now, let us consider this game option. It is obvious that for the holder, in order to obtain the most profit, he will exercise when S becomes as small as possible. Meanwhile, he must not wait too long for this to happen, otherwise he will be punished by the exponential discounting. Then the compromise is to stop when S is smaller than a given constant. While for the writer, a reasonable strategy is to terminate the contract when the value of the asset S equals to K. Then only the burden of a payment of the form $\delta e^{-r\tau}$ is left. For this case, if the initial value of the risky asset is below K then it would seem rational to terminate the contract as soon as S hits K. On the other hand, if the initial value of the risky asset is above K, it is not optimal to exercise at once although the burden of the payment at this time is only δ. A rational strategy is to wait until the last moment that $S_t \geq K$ in order to prolong the payment. However, it should be noted that the value of the δ must not be too large, otherwise it will be never optimal for the writer to terminate the contract in advance.

Theorem 2.3. *Let $\delta^* \triangleq V^A(K) = (K - x^*)(K/x^*)^{k_1}$, one has the following.*

(1) *If $\delta \geq \delta^*$, then the price of this game option is equal to the price of the perpetual American put option, that is, it is not optimal for the writer to terminate the contract in advance.*

(2) *If $\delta < \delta^*$, then the price of the game option is*

$$V(x) = \begin{cases} K - x & x \in (0, k_*], \\ Ax + Bx^{k_1} & x \in (k_*, K), \\ \delta\left(\dfrac{x}{K}\right)^{k_1} & x \in [K, \infty) \end{cases} \tag{2.5}$$

with

$$A = \frac{\delta k_*^{k_1} - (K - k_*)K^{k_1}}{Kk_*^{k_1} - k_* K^{k_1}}, \qquad B = \frac{K(K - k_*) - \delta k_*}{Kk_*^{k_1} - k_* K^{k_1}}, \tag{2.6}$$

and the optimal stopping strategies for the holder and writer, respectively, are

$$\tau^* = \inf\{t \geq 0 : S_t \leq k_*\}, \qquad \gamma^* = \inf\{t \geq 0 : S_t = K\}, \tag{2.7}$$

where k_* is the (unique) solution in $(0, K)$ to the equation

$$(\delta + K)(1 - k_1)x^{k_1} + K^2 k_1 x^{k_1 - 1} - K^{1+k_1} = 0. \tag{2.8}$$

Before the proof, we will first give two propositions.

Proposition 2.4. *Equation (2.8) has and only has one root in $(0, K)$.*

Remark 2.5. If we denote the root of (2.8) in $(0, K)$ by k_*, then from Proposition 2.4 we know that $K(K - k_*) - \delta k_* > 0$, thus $B > 0$.

Proposition 2.6. *$V(x)$ defined by the right-hand sides of (2.5) is convex and decreasing on $(0, \infty)$.*

Proof. From the expression of $V(x)$ and Remark 2.5 we know that $V(x)$ is convex on $(0, K)$ and (K, ∞). Thus, we only need to prove the convexity of $V(x)$ at the point K, that is, $V'(K+) \geq V'(K-)$. Through elementary calculations we obtain

$$\begin{aligned} V'(K-) &= \frac{1}{Kk_*^{k_1} - k_* K^{k_1}} \left[\delta k_*^{k_1} - (K - k_*)K^{k_1} + (K(K - k_*) - \delta k_*)k_1 K^{k_1 - 1}\right], \\ V'(K+) &= \frac{\delta k_1}{K}. \end{aligned} \tag{2.9}$$

Then if we can prove that

$$\delta k_*^{k_1} - (K - k_*)K^{k_1} \leq 0, \tag{2.10}$$

$V'(K+) \geq V'(K-)$ will hold. From (2.8) we can easily find that when $\delta = \delta^*$, $k_* = x^*$. Further, as δ decreases the solution k_* increases. Especially, when $\delta = 0$, $k_* = K$. So if $0 < \delta < \delta^*$, we have $x^* < k_* < K$.

Now let us verify the correctness of (2.10). If not, that is, $\delta > (K - k_*)(K/k_*)^{k_1}$, then from (2.8) we obtain

$$K^{1+k_1} - K^2 k_1 k_*^{k_1 - 1} - K(1 - k_1)k_*^{k_1} = \delta(1 - k_1)k_*^{k_1} > (K - k_*)(1 - k_1)K^{k_1}, \tag{2.11}$$

rearranging it we have

$$\left(k_* K^{k_1} - K k_*^{k_1}\right)\left(1 - k_1 + \frac{k_1 K}{k_*}\right) > 0. \tag{2.12}$$

Since $k_* > x^*$, so $1 - k_1 + k_1 K/k_* > 0$, whereas $k_* K^{k_1} - K k_*^{k_1} < 0$, which contradicts with (2.12). So the hypothesis is not true, that is, (2.10) holds, which also implies that $A \leq 0$. So $V(x)$ is decreasing on $(0, \infty)$. □

Proof of Theorem 2.3. (1) Suppose that $\delta \geq \delta^*$. From the expression of $V^A(x)$ we can easily find that

$$(K - x)^+ \leq V^A(x) \leq (K - x)^+ + \delta. \tag{2.13}$$

By means of Proposition 2.2 and the Doob Optional Stopping Theorem, we have

$$\begin{aligned} V^A(x) &= \inf_{\gamma\in\mathcal{S}} \mathbf{E}_x^\alpha\left[e^{-r(\tau_*\wedge\gamma)}V^A(S_{\tau_*\wedge\gamma})\right] \\ &\le \inf_{\gamma\in\mathcal{S}} \mathbf{E}_x^\alpha\left\{e^{-r\tau_*}(K-S_{\tau_*})^+1_{(\tau_*\le\gamma)} + e^{-r\gamma}\left[(K-S_\gamma)^+ + \delta\right]1_{(\gamma<\tau_*)}\right\} \\ &\le \inf_{\gamma\in\mathcal{S}}\sup_{\tau\in\mathcal{S}} \mathbf{E}_x^\alpha\left\{e^{-r\tau}(K-S_\tau)^+1_{(\tau\le\gamma)} + e^{-r\gamma}\left[(K-S_\gamma)^+ + \delta\right]1_{(\gamma<\tau)}\right\} \\ &= \sup_{\tau\in\mathcal{S}}\inf_{\gamma\in\mathcal{S}} \mathbf{E}_x^\alpha\left\{e^{-r\tau}(K-S_\tau)^+1_{(\tau\le\gamma)} + e^{-r\gamma}\left[(K-S_\gamma)^+ + \delta\right]1_{(\gamma<\tau)}\right\} \\ &\le \sup_{\tau\in\mathcal{S}} \mathbf{E}_x^\alpha\left[e^{-r\tau}(K-S_\tau)^+\right] \\ &= V^A(x). \end{aligned} \tag{2.14}$$

That is, the price of the game option is equal to the price of the perpetual American put option.

(2) If $\delta < \delta^*$, according to the foregoing discussion and Theorem 1.1, there exists a number k such that the continuation region is

$$C = \left\{x : g_1(x) < V(x) < g_2(x)\right\} = \{x : k < x < \infty,\ x \neq K\} \tag{2.15}$$

with $g_1(x) = (K-x)^+$, $g_2(x) = (K-x)^+ + \delta$, $k \in (0,K)$ a constant to be confirmed, while the stopping area is

$$D = D_1 \cup D_2, \tag{2.16}$$

where $D_1 = \{x : V(x) = g_1(x)\} = \{x : x \le k\}$ is the stopping area of the holder, $D_2 = \{x : V(x) = g_2(x)\} = \{x : x = K\}$ is the stopping area of the writer. For search of the optimal k_* and the value of $V(x)$, we consider the following Stefan(free boundary) problem with unknown number k and $V = V(x)$:

$$\begin{gathered} V(x) = K - x, \quad x \in (0,k], \\ (\mathcal{A}^\alpha - r)V(x) = 0, \quad x \in (k,K) \cup (K,\infty), \end{gathered} \tag{2.17}$$

and additional conditions on the boundary k and K are given by

$$\lim_{x\downarrow k} V(x) = K - k, \qquad \lim_{x\to K} V(x) = \delta, \qquad \lim_{x\downarrow k}\frac{\partial V(x)}{\partial x} = -1, \qquad \lim_{x\uparrow\infty} V(x) = 0. \tag{2.18}$$

By computing Stefan problem we can easily obtain the expression of $V(x)$ (denote it by $\widetilde{V}(x)$) defined by the right-hand sides of (2.5), while from (2.18) we can obtain (2.8). Proposition 2.4 implies that this equation has and only has one root in $(0,K)$, denote it by k_*. Accordingly, we can obtain the expression (2.6) of A and B and optimal stopping strategy τ^* for the holder. Now we must prove that the solution of the Stefan problem gives, in fact, the solution to the optimal stopping problem, that is, $V(x) = \widetilde{V}(x)$. For that it is sufficient to prove that

(a) $\forall \tau \in \mathcal{S}$, $\mathbf{E}_x^{\alpha} e^{-r(\tau\wedge\gamma^*)} Z_{\tau,\gamma^*} \leq \widetilde{V}(x)$;

(b) $\forall \gamma \in \mathcal{S}$, $\mathbf{E}_x^{\alpha} e^{-r(\tau^*\wedge\gamma)} Z_{\tau^*,\gamma} \geq \widetilde{V}(x)$;

(c) $\mathbf{E}_x^{\alpha} e^{-r(\tau^*\wedge\gamma^*)} Z_{\tau^*,\gamma^*} = \widetilde{V}(x)$.

First, from Proposition 2.6 we know that $\widetilde{V}(x)$ is a convex function on $(0,\infty)$ such that

$$(K-x)^+ \leq \widetilde{V}(x) \leq (K-x)^+ + \delta. \tag{2.19}$$

Since $\widetilde{V}(x) \in C^1(0,K) \cap C^2(0,K) \setminus \{k_*\}$, for $x \in (0,K)$, we can apply Itô formula to the process $\{e^{-r(t\wedge\gamma^*)}\widetilde{V}(S_{t\wedge\gamma^*}) : t \geq 0\}$ and have

$$\begin{aligned} e^{-r(t\wedge\gamma^*)}\widetilde{V}(S_{t\wedge\gamma^*}) &= \widetilde{V}(x) + \int_0^{t\wedge\gamma^*} e^{-ru}(\mathcal{A}^{\alpha} - r)\widetilde{V}(S_u)du + \int_0^{t\wedge\gamma^*} e^{-ru}\sigma S_u \widetilde{V}'(S_u)dW_u^{\alpha} \\ &\quad + \int_0^{t\wedge\gamma^*} e^{-ru}\left[\widetilde{V}(S_{u-}(1-y_0)) - \widetilde{V}(S_{u-})\right](dN_u - \lambda^{\alpha}du). \end{aligned} \tag{2.20}$$

Note that in $(0,K)$, $\mathcal{A}^{\alpha}\widetilde{V}(x) - r\widetilde{V}(x) \leq 0$, while the last two integrals of (2.20) are local martingales, then by choosing localizing sequence and apply the Fatou lemma, we obtain

$$\mathbf{E}_x^{\alpha} e^{-r(\tau\wedge\gamma^*)}\widetilde{V}(S_{\tau\wedge\gamma^*}) \leq \widetilde{V}(x), \tag{2.21}$$

whereas

$$\begin{aligned} Z_{\tau,\gamma^*} &= (K - S_{\tau})^+ 1_{\tau\leq\gamma^*} + \left[(K - S_{\gamma^*})^+ + \delta\right] 1_{\gamma^*<\tau} \\ &= (K - S_{\tau})^+ 1_{\tau\leq\gamma^*} + \delta 1_{\gamma^*<\tau} \\ &\leq \widetilde{V}(S_{\tau\wedge\gamma^*}). \end{aligned} \tag{2.22}$$

For the inequality we have used (2.19), hence from (2.21) we have

$$\mathbf{E}_x^{\alpha} e^{-r(\tau\wedge\gamma^*)} Z_{\tau,\gamma^*} \leq \widetilde{V}(x). \tag{2.23}$$

It is simple for the case that $x \in (K,\infty)$ and the method is the same as before. Thus, we obtain (a).

The proof of (b): apply Itô formula to the process $\{e^{-r(\tau^*\wedge t)}\widetilde{V}(S_{\tau^*\wedge t}) : t \geq 0\}$ and note that $\widetilde{V}$ is only continuous at K, we have

$$\begin{aligned} e^{-r(\tau^*\wedge t)}\widetilde{V}(S_{\tau^*\wedge t}) = \widetilde{V}(x) &+ \int_0^{\tau^*\wedge t} e^{-ru}(\mathcal{A}^\alpha - r)\widetilde{V}(S_u)du + \int_0^{\tau^*\wedge t} e^{-ru}\sigma S_u \widetilde{V}'(S_u)dW_u^\alpha \\ &+ \int_0^{\tau^*\wedge t} e^{-ru}\left[\widetilde{V}(S_{u-}(1-y_0)) - \widetilde{V}(S_{u-})\right](dN_u - \lambda^\alpha du) \\ &+ e^{-r(\tau^*\wedge t)}\left[\widetilde{V}'(K+) - \widetilde{V}'(K-)\right]L^K_{\tau^*\wedge t}, \end{aligned} \tag{2.24}$$

where L^K is the local time at K of S. Since $\widetilde{V}(x)$ is convex on $(0,\infty)$, hence $\widetilde{V}'(K+) - \widetilde{V}'(K-) \geq 0$. While in $(k_*, \infty) \setminus \{K\}$, $(\mathcal{A}^\alpha - r)\widetilde{V}(x) = 0$, then using the same method as before we have

$$\mathbf{E}_x^\alpha e^{-r(\tau^*\wedge\gamma)}\widetilde{V}(S_{\tau^*\wedge\gamma}) \geq \widetilde{V}(x). \tag{2.25}$$

Moreover, since

$$\begin{aligned} \widetilde{V}(S_{\tau^*\wedge\gamma}) &= \widetilde{V}(S_{\tau^*})1_{(\tau^*\leq\gamma)} + \widetilde{V}(S_\gamma)1_{(\gamma<\tau^*)} \\ &\leq (K - S_{\tau^*})^+ 1_{(\tau^*\leq\gamma)} + \left[(K - S_\gamma)^+ + \delta\right]1_{(\gamma<\tau^*)} \\ &= Z_{\tau^*,\gamma}, \end{aligned} \tag{2.26}$$

we can obtain

$$\mathbf{E}_x^\alpha e^{-r(\tau^*\wedge\gamma)} Z_{\tau^*,\gamma} \geq \widetilde{V}(x), \quad \forall\gamma \in \mathcal{M}. \tag{2.27}$$

The proof of (c): taking $\tau = \tau^*$, $\gamma = \gamma^*$, it is sufficient to note that in (k_*, K), we have $\mathcal{A}^\alpha V(x) - rV(x) = 0$ and

$$\begin{aligned} \widetilde{V}(S_{\tau^*\wedge\gamma^*}) &= \widetilde{V}(S_{\tau^*})1_{(\tau^*\leq\gamma^*)} + \widetilde{V}(S_{\gamma^*})1_{(\gamma^*<\tau^*)} \\ &= (K - S_{\tau^*})^+ 1_{(\tau^*\leq\gamma^*)} + \delta 1_{(\gamma^*<\tau^*)} \\ &= Z_{\tau^*,\gamma^*}. \end{aligned} \tag{2.28}$$

The same result is true for the case that $x \in (K, \infty)$. □

3. Game Option with Barrier

Karatzas and Wang [8] obtain closed-form expressions for the prices and optimal hedging strategies of American put options in the presence of an up-and-out barrier by reducing this problem to a variational inequality. Now we will consider the game option connected with this barrier option. Following Karatzas and Wang, the holder may exercise to take the claim of this barrier option

$$X_t = (K - S_t)^+ 1_{(t<\tau_h)}, \quad 0 \leq t < \infty. \tag{3.1}$$

Here $h > 0$ is the barrier, whereas

$$\tau_h = \inf\{t \geq 0 : S_t > h\} \tag{3.2}$$

is the time when the option becomes "knocked-out". The writer is punished by an amount δ for terminating the contract early

$$Y_t = \left[(K - S_t)^+ + \delta\right] 1_{(t<\tau_h)}. \tag{3.3}$$

First, let us consider this type of barrier option. The price is given by

$$V^B(x) = \sup_{\tau \in \mathcal{S}} \mathbf{E}_x e^{-r\tau} (K - S_\tau)^+ 1_{(\tau<\tau_h)} \tag{3.4}$$

with the superscript B representing barrier. Similarly to Karatzas and Wang we can obtain the following.

Theorem 3.1. *The price of American put-option in the presence of an up-and-out barrier is*

$$V^B(x) = \begin{cases} K - x & x \in (0, p_*], \\ Ax + Bx^{k_1} & x \in (p_*, h), \\ 0 & x \in [h, \infty), \end{cases} \tag{3.5}$$

where $A = (p_* - K)h^{k_1}/(hp_*^{k_1} - p_* h^{k_1})$, $B = (K - p_*)h/(hp_*^{k_1} - p_* h^{k_1})$, *and the optimal stopping strategy is*

$$\tau_* = \inf\{t \geq 0 : S_t \leq p_*\}, \tag{3.6}$$

where p_* *is the (unique) solution in* $(0, K)$ *to the equation*

$$h(1 - k_1)x^{k_1} + Kh\, k_1 x^{k_1 - 1} - Kh^{k_1} = 0. \tag{3.7}$$

The proof of the theorem mainly depends on the following propositions and the process will be omitted.

Proposition 3.2. *The expression of* $V^B(x)$ *defined by* (3.5) *is convex and decreasing on* $(0, \infty)$*, and under risk-neutral measure* $\mathbf{P}_x^\alpha$*, one has that* $\{e^{-rt} V^B(S_t) : t \geq 0\}$ *and* $\{e^{-r(t\wedge\tau_*)} V^B(S_{t\wedge\tau_*}) : t \geq 0\}$ *are supermartingale and martingale, respectively.*

Proposition 3.3. *Equation* (3.7) *has and only has one root in* $(0, K)$.

Now let us consider the game option with barrier h. The price is given by

$$V(x) = \sup_{\tau \in \mathcal{S}} \inf_{\gamma \in \mathcal{S}} \mathbf{E}_x \Big(e^{-r\tau} (K - S_\tau)^+ 1_{(\tau\leq\gamma)} \cdot 1_{(\tau<\tau_h)} + e^{-r\gamma} \left[(K - S_\gamma)^+ + \delta\right] 1_{(\gamma<\tau)} \cdot 1_{(\gamma<\tau_h)} \Big). \tag{3.8}$$

For this game option, the logic of its solution is similar to the former, and based on this consideration, we have the following theorem.

Theorem 3.4. *Let* $\delta^* \triangleq V^B(K) = (K - p_*)(hK^{k_1} - Kh^{k_1})/(hp_*^{k_1} - p_* h^{k_1})$*, one has the following.*

(1) *If* $\delta \geq \delta^*$*, then the price of this game option is equal to the price of American put options in the presence of an up-and-out barrier, that is, it is not optimal for the writer to exercise early.*

(2) *If* $\delta < \delta^*$*, then the price of the game option is given by*

$$V(x) = \begin{cases} K - x & x \in (0, b_*], \\ C_1 x + C_2 x^{k_1} & x \in (b_*, K), \\ D_1 x + D_2 x^{k_1} & x \in [K, h), \\ 0 & x \in [h, \infty), \end{cases} \tag{3.9}$$

where

$$\begin{aligned} C_1 &= \frac{\delta b_*^{k_1} - (K - b_*) K^{k_1}}{K b_*^{k_1} - b_* K^{k_1}}, \qquad C_2 = \frac{K(K - b_*) - \delta b_*}{K b_*^{k_1} - b_* K^{k_1}}, \\ D_1 &= \frac{-\delta h^{k_1}}{hK^{k_1} - Kh^{k_1}}, \qquad D_2 = \frac{\delta h}{hK^{k_1} - Kh^{k_1}}, \end{aligned} \tag{3.10}$$

and b_* *is the (unique) solution in* $(0, K)$ *to the equation*

$$(\delta + K)(1 - k_1) x^{k_1} + K^2 k_1 x^{k_1 - 1} - K^{1 + k_1} = 0, \tag{3.11}$$

and the optimal stopping strategies for the holder and writer, respectively, are

$$\tau^* = \inf\{t \geq 0 : S_t \leq b_*\}, \qquad \gamma^* = \inf\{t \geq 0 : S_t = K\}. \tag{3.12}$$

Proposition 3.5. *The function* $V(x)$ *defined by* (3.9) *is convex and decreasing on* $(0, \infty)$*.*

Proof. Similar to Proposition 2.6, we only need to prove the convexity of $V(x)$ at the point K, that is,

$$V'(K+) - V'(K-) = (D_2 - C_2) k_1 K^{k_1 - 1} + (D_1 - C_1) \geq 0. \tag{3.13}$$

Through lengthy calculations we know that it is sufficient to show that

$$\delta(hb_*^{k_1} - b_* h^{k_1}) \leq (K - b_*)(hK^{k_1} - Kh^{k_1}). \tag{3.14}$$

Suppose that (3.14) does not hold, that is, $\delta > (K-b_*)(hK^{k_1}-Kh^{k_1})/(hb_*^{k_1}-b_*h^{k_1})$, then from (3.11) we find that

$$\begin{aligned}K^{1+k_1}-K^2k_1x^{k_1-1}-K(1-k_1)b_*^{k_1} &= \delta(1-k_1)b_*^{k_1}\\ &> (1-k_1)b_*^{k_1}\frac{(K-b_*)(hK^{k_1}-Kh^{k_1})}{(hb_*^{k_1}-b_*h^{k_1})},\end{aligned}\tag{3.15}$$

rearranging it we have

$$h(1-k_1)b_*^{k_1}+Kh\,k_1b_*^{k_1-1}-Kh^{k_1}<0.\tag{3.16}$$

From (3.11), through complex verification we get that when $\delta=\delta^*$, $b_*=p_*$. Furthermore, as δ decreases the solution b_* increases, especially when $\delta=0$, $b_*=K$. So if $0<\delta<\delta^*$, we have $p_*<b_*<K$. Thus from the property of (3.7) we know that $h(1-k_1)b_*^{k_1}+Kh\,k_1b_*^{k_1-1}-Kh^{k_1}>0$, which contradicts with (3.16). So the hypothesis is not true, that is, (3.14) holds. It is evident that $V(x)$ is decreasing. □

Remark 3.6. It is obvious that (2.8) is the same as (3.11), however, their roots not always be equal to each other. Because of these two cases, the scope of δ is different. Penalty with barrier is usually smaller than the other, that is, $V^B(K)<V^A(K)$.

Proof of Theorem 3.4. (1) Suppose that $\delta\geq\delta^*$. From Proposition 3.2 we know that

$$(K-x)^+\leq V^B(x)\leq(K-x)^++\delta.\tag{3.17}$$

By means of the Doob optional stopping theorem and (3.17), we have

$$\begin{aligned}V^B(x)&=\inf_{\gamma\in\mathcal{S}}\mathbf{E}_x^\alpha\left[e^{-r(\tau^*\wedge\gamma\wedge\tau_h)}V^B(S_{\tau^*\wedge\gamma\wedge\tau_h})\right]\\ &\leq\inf_{\gamma\in\mathcal{S}}\mathbf{E}_x^\alpha\left\{e^{-r\tau^*}(K-S_{\tau^*})^+1_{(\tau^*\leq\gamma)}1_{(\tau^*<\tau_h)}+e^{-r\gamma}\left[(K-S_\gamma)^++\delta\right]1_{(\gamma<\tau^*)}1_{(\gamma<\tau_h)}\right\}\\ &\leq\inf_{\gamma\in\mathcal{S}}\sup_{\tau\in\mathcal{S}}\mathbf{E}_x^\alpha\left\{e^{-r\tau}(K-S_\tau)^+1_{(\tau\leq\gamma)}1_{(\tau<\tau_h)}+e^{-r\gamma}\left[(K-S_\gamma)^++\delta\right]1_{(\gamma<\tau)}1_{(\gamma<\tau_h)}\right\}\\ &=\sup_{\tau\in\mathcal{S}}\inf_{\gamma\in\mathcal{S}}\mathbf{E}_x^\alpha\left\{e^{-r\tau}(K-S_\tau)^+1_{(\tau\leq\gamma)}1_{(\tau<\tau_h)}+e^{-r\gamma}\left[(K-S_\gamma)^++\delta\right]1_{(\gamma<\tau)}1_{(\gamma<\tau_h)}\right\}\\ &\leq\sup_{\tau\in\mathcal{S}}\mathbf{E}_x^\alpha\left[e^{-r\tau}(K-S_\tau)^+1_{(\tau<\tau_h)}\right]\\ &=V^B(x).\end{aligned}\tag{3.18}$$

That is, the price of the game option is equal to the price of American put-options in the presence of an up-and-out barrier.

(2) Suppose that $\delta<\delta^*$. Then we may conclude that the holder should search optimal stopping strategy in the class of the stopping times of the form $\tau_b=\inf\{t\geq0:S_t\leq b\}$ with

$b \in (0, K)$ to be confirmed. While the optimal stopping strategy for the writer is $\gamma^* = \inf\{t \geq 0 : S_t = K\}$. Considering the following Stefan problem:

$$V(x) = K - x, \quad x \in (0, b], \tag{3.19}$$

$$(\mathcal{A} - r)V(x) = 0, \quad x \in (b, K) \cup (K, h), \tag{3.20}$$

$$V(x) = 0, \quad x \in [h, \infty), \tag{3.21}$$

$$\lim_{x \downarrow b} V(x) = K - b, \qquad \lim_{x \to K} V(x) = \delta, \qquad \lim_{x \uparrow h} V(x) = 0, \qquad \lim_{x \downarrow b} \frac{\partial V(x)}{\partial x} = -1. \tag{3.22}$$

Through straightforward calculations we can obtain the expression of $V(x)$ (denote it by $\widetilde{V}(x)$) defined by the right-hand sides of (3.9). From condition (3.22) we can obtain (3.11). Proposition 2.4 implies that the root of this equation is unique in $(0, K)$, denote it by b_* and consequently τ_{b_*} by τ^*. Now we only need to prove that $V(x) = \widetilde{V}(x)$. For that it is sufficient to prove that

$$\text{(a)} \quad \forall \tau \in \mathcal{S}, \quad \mathbf{E}_x^\alpha e^{-r(\tau \wedge \gamma^*)} Z_{\tau,\gamma^*} 1_{(\tau \wedge \gamma^* < \tau_h)} \leq \widetilde{V}(x); \tag{3.23}$$

$$\text{(b)} \quad \forall \gamma \in \mathcal{S}, \quad \mathbf{E}_x^\alpha e^{-r(\tau^* \wedge \gamma)} Z_{\tau^*,\gamma} 1_{(\tau^* \wedge \gamma < \tau_h)} \geq \widetilde{V}(x). \tag{3.24}$$

(c) Taking stopping time $\tau = \tau^*$, $\gamma = \gamma^*$, we have

$$\mathbf{E}_x^\alpha e^{-r(\tau^* \wedge \gamma^*)} Z_{\tau^*,\gamma^*} 1_{(\tau^* \wedge \gamma^* < \tau_h)} = \widetilde{V}(x). \tag{3.25}$$

First, from Proposition 3.5 we know that $\widetilde{V}(x)$ is convex in $(0, \infty)$ and further

$$(K - x)^+ \leq \widetilde{V}(x) \leq (K - x)^+ + \delta. \tag{3.26}$$

Applying Itô formula to the process $\{e^{-r(t \wedge \gamma^* \wedge \tau_h)} \widetilde{V}(S_{t \wedge \gamma^* \wedge \tau_h}) : t \geq 0\}$, we have

$$\begin{aligned} e^{-r(t \wedge \gamma^* \wedge \tau_h)} \widetilde{V}(S_{t \wedge \gamma^* \wedge \tau_h}) = \widetilde{V}(x) &+ \int_0^{t \wedge \gamma^* \wedge \tau_h} e^{-ru} (\mathcal{A}^\alpha - r) \widetilde{V}(S_u) du + \int_0^{t \wedge \gamma^* \wedge \tau_h} e^{-ru} \sigma S_u \widetilde{V}'(S_u) dW_u^\alpha \\ &+ \int_0^{t \wedge \gamma^* \wedge \tau_h} e^{-ru} \left[\widetilde{V}(S_{u-}(1 - y_0)) - \widetilde{V}(S_{u-}) \right] (dN_u - \lambda^\alpha du). \end{aligned} \tag{3.27}$$

It is obvious that when $x \in (0, K) \cup (K, h)$, we have $(\mathcal{A} - r)\widetilde{V}(x) \leq 0$. Since the second and the third integrals of the right-hand sides of (3.27) are local martingales, so

$$\mathbf{E}_x^\alpha e^{-r(\tau \wedge \gamma^* \wedge \tau_h)} \widetilde{V}(S_{\tau \wedge \gamma^* \wedge \tau_h}) \leq \widetilde{V}(x), \tag{3.28}$$

while

$$\mathbf{E}_x^{\alpha} e^{-r(\tau\wedge\gamma^*\wedge\tau_h)} \widetilde{V}\left(S_{\tau\wedge\gamma^*\wedge\tau_h}\right) = \mathbf{E}_x^{\alpha} e^{-r(\tau\wedge\gamma^*)} \widetilde{V}\left(S_{\tau\wedge\gamma^*}\right) 1_{(\tau\wedge\gamma^*<\tau_h)} \geq \mathbf{E}_x^{\alpha} e^{-r(\tau\wedge\gamma^*)} Z_{\tau,\gamma^*} 1_{(\tau\wedge\gamma^*<\tau_h)}. \tag{3.29}$$

The inequality is obtained from (3.26), and combining (3.28) we obtain (3.23), that is, (a) holds.

Applying Itô formula to the process $\{e^{-r(\tau^*\wedge t\wedge\tau_h)} \widetilde{V}(S_{\tau^*\wedge t\wedge\tau_h}) : t \geq 0\}$, we have

$$\begin{aligned} e^{-r(\tau^*\wedge t\wedge\tau_h)} \widetilde{V}\left(S_{\tau^*\wedge t\wedge\tau_h}\right) = \widetilde{V}(x) &+ \int_0^{\tau^*\wedge t\wedge\tau_h} e^{-ru} (\mathcal{A}-r) \widetilde{V}(S_u) du + \int_0^{\tau^*\wedge t\wedge\tau_h} e^{-ru} \sigma(S_u) \widetilde{V}'(S_u) dW_u^{\alpha} \\ &+ e^{-r(\tau^*\wedge t\wedge\tau_h)} \left[\widetilde{V}'(K+) - \widetilde{V}'(K-)\right] L_{\tau^*\wedge t\wedge\tau_h}^K \\ &+ \int_0^{\tau^*\wedge t\wedge\tau_h} e^{-ru} \left[\widetilde{V}\left(S_{u-}(1-y_0)\right) - \widetilde{V}(S_{u-})\right] (dN_u - \lambda^{\alpha} du). \end{aligned} \tag{3.30}$$

The definition of L^K is the same as Theorem 2.3. From the convexity of $\widetilde{V}(x)$ we know that $\widetilde{V}'(K+) - \widetilde{V}'(K-) \geq 0$. Since when $x \in (b_*, K) \cup (K, h)$, $(\mathcal{A} - r)\widetilde{V}(x) = 0$, so from above expression we have

$$\mathbf{E}_x^{\alpha} e^{-r(\tau^*\wedge\gamma\wedge\tau_h)} \widetilde{V}\left(S_{\tau^*\wedge\gamma\wedge\tau_h}\right) \geq \widetilde{V}(x). \tag{3.31}$$

Similarly we have

$$\mathbf{E}_x^{\alpha} e^{-r(\tau^*\wedge\gamma\wedge\tau_h)} \widetilde{V}\left(S_{\tau^*\wedge\gamma\wedge\tau_h}\right) = \mathbf{E}_x^{\alpha} e^{-r(\tau^*\wedge\gamma)} \widetilde{V}\left(S_{\tau^*\wedge\gamma}\right) 1_{(\tau^*\wedge\gamma<\tau_h)} \leq \mathbf{E}_x^{\alpha} e^{-r(\tau^*\wedge\gamma)} Z_{\tau^*,\gamma} 1_{(\tau^*\wedge\gamma<\tau_h)}. \tag{3.32}$$

From (3.31) and (3.32) we know that (b) holds. Combining (a) and (b) we can easily obtain (c). □

4. A Simple Example: Application to Convertible Bonds

To raise capital on financial markets, companies may choose among three major asset classes: equity, bonds, and hybrid instruments, such as convertible bonds. As hybrid instruments, convertible bonds has been investigated rather extensively during the recent years. It entitles its owner to receive coupons plus the return of the principle at maturity. However, the holder can convert it into a preset number of shares of stock prior to maturity. Then the price of the bond is dependent on the price of the firm stock. Finally, prior to maturity, the firm may call the bond, forcing the bondholder to either surrender it to the firm for a previously agreed price or else convert it for stock as above. Therefore, the pricing problem has also a game-theoretic aspect. For more detailed information and research about convertible bonds, one is referred to Gapeev and Kühn [9], Sîrbu et al. [10, 11], and so on.

Now, we will give a simple example of pricing convertible bonds, as the application of pricing game options. Consider the stock process which pays dividends at a certain fixed rate $d \in (0, r)$, that is,

$$dS_t = S_{t-}\left[(\mu - d)dt + \sigma dW_t - y_0\left(dN_t - \lambda dt\right)\right]. \tag{4.1}$$

Then the infinitesimal generator of S becomes

$$\mathcal{A}^{\alpha} f(x) \triangleq \left(r - d + \lambda^{\alpha} y_0\right) x \frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} + \lambda^{\alpha}\left[f\left(x\left(1 - y_0\right)\right) - f(x)\right], \tag{4.2}$$

and $(\mathcal{A}^{\alpha} - r) f(x) = 0$ admits two solutions $f(x) = x^{k_1}$ and $f(x) = x^{k_2}$ with $k_1 < 0 < 1 < k_2$ satisfying

$$\frac{1}{2}\sigma^2 k(k-1) + \left(r - d + \lambda^{\alpha} y_0\right) k - \left(r + \lambda^{\alpha}\right) + \lambda^{\alpha}\left(1 - y_0\right)^k = 0. \tag{4.3}$$

At any time, the bondholder can convert it into a predetermined number $\eta > 0$ of stocks, or continue to hold the bond and collecting coupons at the fixed rate $c > 0$. On the other hand, at any time the firm can call the bond, which requires the bondholder to either immediately surrender it for the fixed conversion price $K > 0$ or else immediately convert it as described above. In short, the firm can terminate the contract by paying the amount $\max\{K, \eta S\}$ to the holder. Then, if the holder terminates the contract first by converting the bond into η stocks, he/she can expect to (discounted) receive

$$L_t = \int_0^t c \cdot e^{-ru} du + e^{-rt} \eta S_t, \tag{4.4}$$

while if the firm terminate the contract first, he/she will pay the holder

$$U_t = \int_0^t c \cdot e^{-ru} du + e^{-rt}\left(K \vee \eta S_t\right). \tag{4.5}$$

Then, according to Theorem 1.1, the price of the convertible bonds is given by

$$V^{\mathrm{CB}}(x) = \inf_{\gamma \in \mathcal{S}} \sup_{\tau \in \mathcal{S}} \mathbf{E}_x^{\alpha}\left(L_\tau 1_{(\tau \le \gamma)} + U_\gamma 1_{(\gamma < \tau)}\right) = \sup_{\tau \in \mathcal{S}} \inf_{\gamma \in \mathcal{S}} \mathbf{E}_x^{\alpha}\left(L_\tau 1_{(\tau \le \gamma)} + U_\gamma 1_{(\gamma < \tau)}\right). \tag{4.6}$$

Note that when $c \ge rK$, the solution of (4.6) is trivial and the firm should call the bond immediately. This implies that the bigger the coupon rate c, the more the payoff of the issuer, then they will choose to terminate the contract immediately. So we will assume that $c < rK$ in the following.

Now, let us first consider the logic of solving this problem. It is obvious that $\eta x \le V^{\mathrm{CB}}(x) \le K \vee \eta x$ for all $x > 0$ (choose $\tau = 0$ and $\gamma = 0$, resp.). Note when $S_t \ge K/\eta$, $L_t = U_t$, then $V^{\mathrm{CB}}(x) = \eta x$ for all $x \geq K/\eta$. Hence the issuer and the holder should search optimal stopping in the class of stopping times of the form

$$\gamma_a = \inf\{t \geq 0 : S_t \geq a\}, \qquad \tau_b = \inf\{t \geq 0 : S_t \geq b\}, \tag{4.7}$$

respectively, with numbers $0 < a, b \leq K/\eta$ to be determined. Note when the process S fluctuates in the interval $(0, K/\eta)$, it is not optimal to terminate the contract simultaneously by both issuer and holder. For example, if the issuer chooses to terminate the contract at the first time that S exceeds some point $a \in (0, K/\eta)$, then $\eta a < K$, and the holder will choose the payoff of coupon rather than converting the bond into the stock, which is a contradiction. Similarly, one can explain another case. Then only the following situation can occur: either $a < b = K/\eta$, $b < a = K/\eta$, or $b = a = K/\eta$.

For search of the optimal a_*, b_* and the value of $V^{\mathrm{CB}}(x)$, we consider an auxiliary Stefan problem with unknown numbers a, b, and $V(x)$

$$\begin{gathered}(\mathcal{A}^{\alpha} - r)V(x) = -c, \quad 0 < x < a \wedge b, \\ \eta x < V(x) < \eta x \vee K, \quad 0 < x < a \wedge b\end{gathered} \tag{4.8}$$

with continuous fit boundary conditions

$$V(b-) = \eta b, \qquad V(x) = \eta x \tag{4.9}$$

for all $x > b$, $b \leq a = K/\eta$, and

$$V(a-) = K, \qquad V(x) = \eta x \vee K \tag{4.10}$$

for all $x > a$, $a \leq b = K/\eta$, and smooth fit boundary conditions

$$V'(b-) = \eta \quad \text{if } b < a = \frac{K}{\eta}, \qquad V'(a-) = 0 \quad \text{if } a < b = \frac{K}{\eta}. \tag{4.11}$$

By computing the Stefan problem we can obtain that if

$$K > \frac{k_2}{k_2 - 1}\frac{c}{r}, \tag{4.12}$$

then $b_* < a_* = K/\eta$, and the expression of $V(x)$ is given by

$$V(x) = \frac{\eta b_*}{k_2}\left(\frac{x}{b_*}\right)^{k_2} + \frac{c}{r} \tag{4.13}$$

for all $0 < x < b_*$, with

$$b_* = \frac{k_2}{\eta(k_2 - 1)} \frac{c}{r}, \tag{4.14}$$

and if

$$\frac{c}{r} < K \leq \frac{k_2}{k_2 - 1} \frac{c}{r}, \tag{4.15}$$

then $a_* = b_* = K/\eta$, and the value of $V(x)$ is

$$V(x) = \left(K - \frac{c}{r}\right) \left(\frac{\eta x}{K}\right)^{k_2} + \frac{c}{r} \tag{4.16}$$

for all $0 < x < K/\eta$.

From the result we can observe that there are only two regions for K, and the situation $a_* < b_* = K/\eta$ fails to hold. This implies that in this case, when S fluctuates in the interval $(0, K/\eta)$, the issuer will never recall the bond. Now, we only need to prove that $V(x) = V^{\mathrm{CB}}(x)$, and the stopping times γ^* and τ^* defined by (4.7) with boundaries a_* and b_* are optimal.

Applying Itô formula to the process $\{e^{-rt} V(S_t) : t \geq 0\}$, we have

$$\begin{aligned} e^{-rt} V(S_t) = V(x) &+ \int_0^t e^{-ru} (\mathcal{A}^\alpha - r) V(S_u) 1_{(S_u \neq a_*, S_u \neq b_*, S_u \neq K/\eta)} du \\ &+ \int_0^t e^{-ru} \sigma S_u V'(S_u) 1_{(S_u \neq a_*, S_u \neq b_*, S_u \neq K/\eta)} dW_u^\alpha \\ &+ \int_0^t e^{-ru} [V(S_{u-}(1 - y_0)) - V(S_{u-})] (dN_u - \lambda^\alpha du) \\ &+ e^{-rt} \left[V'\left(\frac{K}{\eta} +\right) - V'\left(\frac{K}{\eta} -\right)\right] L_t^{K/\eta}. \end{aligned} \tag{4.17}$$

Let

$$\begin{aligned} M_t = &\int_0^t e^{-ru} \sigma S_u V'(S_u) 1_{(S_u \neq a_*, S_u \neq b_*, S_u \neq K/\eta)} dW_u^\alpha \\ &+ \int_0^t e^{-ru} [V(S_{u-}(1 - y_0)) - V(S_{u-})] (dN_u - \lambda^\alpha du). \end{aligned} \tag{4.18}$$

Note for all $0 < x < a_*$, $(\mathcal{A} - r)V(x) \leq -c$, while for all $0 < x < b_*$, $(\mathcal{A} - r)V(x) = -c$. Since $\eta x \leq V(x) \leq \eta x \vee K$, so for $0 < a_* \leq K/\eta$, $0 < b_* \leq K/\eta$, we have

$$\begin{aligned}
L_{\tau\wedge\gamma^*} &= \int_0^{\tau\wedge\gamma^*} c \cdot e^{-ru} du + e^{-r(\tau\wedge\gamma^*)} \eta S_{\tau\wedge\gamma^*} \\
&\leq \int_0^{\tau\wedge\gamma^*} c \cdot e^{-ru} du + e^{-r(\tau\wedge\gamma^*)} V\left(S_{\tau\wedge\gamma^*}\right) \\
&\leq V(x) + M_{\tau\wedge\gamma^*}, \\
U_{\tau^*\wedge\gamma} &= \int_0^{\tau^*\wedge\gamma} c \cdot e^{-ru} du + e^{-r(\tau^*\wedge\gamma)} \left(\eta S_{\tau^*\wedge\gamma} \vee K\right) \\
&\geq \int_0^{\tau^*\wedge\gamma} c \cdot e^{-ru} du + e^{-r(\tau^*\wedge\gamma)} V\left(S_{\tau^*\wedge\gamma}\right) \\
&= V(x) + M_{\tau^*\wedge\gamma}.
\end{aligned} \tag{4.19}$$

Because $V(S_{\gamma^*}) = K \vee \eta S_{\gamma^*}$, $V(S_{\tau^*}) = \eta S_{\tau^*}$, then by choosing localizing sequence and apply the Fatou lemma, we obtain

$$\mathbf{E}_x^{\alpha}\left[L_{\tau} 1_{(\tau\leq\gamma^*)} + U_{\gamma^*} 1_{(\gamma^*<\tau)}\right] \leq V(x) \leq \mathbf{E}_x^{\alpha}\left[L_{\tau^*} 1_{(\tau^*\leq\gamma)} + U_{\gamma} 1_{(\gamma<\tau^*)}\right]. \tag{4.20}$$

Taking supremum and infimum for τ and γ of both sides, respectively, we can obtain the result. While for (4.20), taking $\tau = \tau^*$, $\gamma = \gamma^*$, we have

$$V(x) = \mathbf{E}_x^{\alpha}\left[L_{\tau^*} 1_{(\tau^*\leq\gamma^*)} + U_{\gamma^*} 1_{(\gamma^*<\tau^*)}\right]. \tag{4.21}$$

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