Journal of Applied Mathematics and Stochastic Analysis, **10**:1 (1997), 67-70.

MEAN NUMBER OF REAL ZEROS OF A RANDOM TRIGONOMETRIC POLYNOMIAL IV

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(Received September, 1995; Revised May, 1996)

If a_j $(j=1,2,\ldots,n)$ are independent, normally distributed random variables with mean 0 and variance 1, if p is one half of any odd positive integer except one, and if ν_{np} is the mean number of zeros on $(0,2\pi)$ of the trigonometric polynomial $a_1\cos x + 2^p a_2 \cos 2x + \ldots + n^p a_n \cos nx$, then $\nu_{np} = \mu_p\{(2n+1) + D_{1p} + (2n+1)^{-1}D_{2p} + (2n+1)^{-2}D_{3p}\} + O\{(2n+1)^{-3}\}$, in which $\mu_p = \{(2p+1)/(2p+3)\}^{1/2}$, and D_{1p}, D_{2p} and D_{3p} are explicitly stated constants.

Key words: Random Polynomials, Real Zeros.
AMS subject classifications: 60G99.

1. Introduction

Suppose that a_j $(j=1,2,\ldots,n)$ are independent, normally distributed random variables with mean 0 and variance 1, and that ν_{np} is the mean value of the number of zeros on the interval $(0,2\pi)$ of the random trigonometric polynomial

$$\sum_{j=1}^{n} j^p a_j \cos jx, \tag{1.1}$$

in which p is a nonnegative real number. Das [1] has shown that, for large n,

$$\nu_{np} = 2\mu_p n + O(n^{1/2}), \mu_p = \{(2p+1)/(2p+3)\}^{1/2}. \tag{1.2}$$

When $p=0$, the author [2] has shown that the error term $O(n^{1/2})$ is actually $O(1)$. Moreover, the error term is also $O(1)$ when p is a positive integer [3]. In fact, if p is a nonnegative integer, there exist constants $D_{0p}=1$, D_{1p}, D_{2p} and D_{3p} such that

$$\nu_{np} = (2n+1)\mu_p \sum_{r=0}^{3} (2n+1)^{-r} D_{rp} + O\{(2n+1)^{-3}\}. \tag{1.3}$$

The author and Souter [4] have also derived a relation of the form (1.3) when $p=1/2$.

In this paper we show that (1.3) remains valid when $p=(2s+1)/2$, in which s is a positive integer. In combination with the earlier results, this implies that a relation of the form (1.3) is valid when $2p$ is any nonnegative integer, although we have not been able to construct a unified derivation that covers the various cases in [2], [3], [4] and this paper. We have also not been able to extend our techniques to the case in which $2p$ is not a nonnegative integer.

The logical organization of our analysis is identical to that in [3], although the algebraic details are occasionally different. For this reason we rely heavily on the contents of [3], recording in Section 2 only those portions of the analysis that are essentially new.

2. Derivation of (1.3)

Exactly as in [3], we find that

$$\nu_{np} = 4\pi^{-1}\int_0^{\pi/2} F_{np}(x)dx. \tag{2.1}$$

when $n \geq 2$, in which

$$F_{np} = A_{np}^{-1}(A_{np}C_{np} - B_{np}^2)^{1/2}, \tag{2.2}$$

$$A_{np} = \sum_{j=1}^{n} j^{2p}\cos^2 jx, \tag{2.3}$$

$$B_{np} = \sum_{j=1}^{n} j^{2p+1}\sin jx \cos jx, \tag{2.4}$$

$$C_{np} = \sum_{j=1}^{n} j^{2p+2}\sin^2 jx. \tag{2.5}$$

If we define the constant S_{np} so that

$$2S_{np} = \sum_{j=1}^{n} j^{2p}, \tag{2.6}$$

and assume that $2p = 2s+1$, in which s is a positive integer, it is then clear that

$$A_{np} = S_{np} + (-4)^{-s}d^{2s}A_{n,1/2}/dx^{2s}. \tag{2.7}$$

It is known [4, Eqs. (2.4), (2.7), (2.8) and (2.9)] that

$$8A_{n,1/2} = (2n+1)^2 g_0(z) + (2n+1)g_1 + g_2, \tag{2.8}$$

in which

$$z = (2n+1x, f(x) = csc\ x - x^{-1}, \varphi(x) = f^2(x) + 2x^{-1}f(x) = csc^2 x - x^{-2}, \tag{2.9}$$

$$g_0(z) = (1/2) + z^{-1}\sin z - z^{-2}(1-\cos z), \quad g_1 = f(x)\sin z, \tag{2.10}$$

$$g_2 = -\{(1/2) + \varphi(x) + f'(x)\cos z\}. \tag{2.11}$$

Lemma 1: *If the constants* γ_{rp} $(p - 1/2 = 0, 1, \ldots; r = 0, 1, \ldots, p + 1/2)$ *are defined so that*

$$\gamma_{0p} = (2p+1)^{-1}, \gamma_{rp} = {}_{2p}C_{2r-1}f^{(2r-1)}(0) \quad (r = 1, 2, \ldots, s), \tag{2.12}$$

$$\gamma_{s+1,p} = \varphi^{(2s)}(0) + f^{(2s+1)}(0),$$

in which ${}_hC_k$ *is the binomial coefficient* $h!/\{k!(h-k)!\}$*, then*

$$4^{p+1}S_{np} = \sum_{r=0}^{s+1}(-1)^r\gamma_{rp}(2n+1)^{2p+1-2r}. \tag{2.13}$$

We start the proof of Lemma 1 with the inference from (2.7), (2.8), (2.9), (2.10) and (2.11) that

$$\begin{aligned}2^{2s+3}A_{np} &= 2^{2s+3}S_{np} + (-1)^s(2n+1)^{2s+2}g_0^{(2s)}(z) - (-1)^s\varphi^{(2s)}(x) \\ &+ (-1)^s(2n+1)\sum_{r=0}^{2s} {}_{2s}C_r f^{(r)}(x)(2n+1)^{2s-r}(\sin z)^{(2s-r)} \\ &- (-1)^s\sum_{r=0}^{2s} {}_{2s}C_r f^{(r+1)}(x)(2n+1)^{2s-r}(\cos z)^{(2s-r)}.\end{aligned} \tag{2.14}$$

If we replace x by 0 in (2.14) and note that $g_0^{(2s)}(0) = (-1)^s(2p+1)^{-1}$, that $A_{np}(0) = 2S_{np}$, and that $f^{(2r)}(0) = 0$ if r is a nonnegative integer, some simple manipulations suffice to prove the lemma.

We will need the explicit representations of A_{np}, B_{np} and C_{np} stated in the following lemma, whose proof is essentially the same as that of Lemma 2 in [3].

Lemma 2: *It is true that*

$$2^{2p+2}A_{np} = (2n+1)^{2p+1}\sum_{r=0}^{2p+1} g_{rp}(2n+1)^{-r}, \tag{2.15}$$

$$2^{2p+3}B_{np} = (2n+1)^{2p+2}\sum_{r=0}^{2p+2} h_{rp}(2n+1)^{-r}, \tag{2.16}$$

$$2^{2p+4}C_{np} = (2n+1)^{2p+3}\sum_{r=0}^{2p+3} k_{rp}(2n+1)^{-r}, \tag{2.17}$$

if the coefficients g_{rp}, h_{rp} and k_{rp} are defined as they were in Lemma 2 in [3], *with the following exceptions:*

a. *When in* [3, *Eqs.* (2.19a)-(2.21d)] *the letter p occurs as a superscript, or in a range of values of r, it should be replaced by the letter s.*

b. *The coefficients not defined in* [3] *are defined as follows:*

$$g_{2p+1,p} = (-1)^{s+1}\{\gamma_{s+1,p} + f^{(2s+1)}(x)\cos z + \varphi^{(2s)}(s)\}, \tag{2.18}$$

$$h_{2p+2,p} = (-1)^s\{f^{(2s+2)}(x)\cos z + \varphi^{(2s+1)}(x)\}, \tag{2.19}$$

$$\begin{aligned}k_{2p+3,p} = (-1)^{s+1}\{&\varphi^{(2s+2)}(x) - \varphi^{(2s+2)}(0) + f^{(2s+3)}(x)\cos z \\ &- f^{(2s+3)}(0)\}.\end{aligned} \tag{2.20}$$

If we start from(2.15), (2.16) and (2.17), we can reproduce the statements and proofs of Lemmas 3 through 7 of [3] almost verbatim. (The quantity $O(1)$ in line 8, p. 587 of [3] should have been $o(1)$. The quantity g in line 9, p. 587 of [3] could have been, and in this paper should be, g_0.) The last Lemma 7 exhibits quantities v_{rp} such that

$$\nu_{np} = (2n+1)\sum_{r=0}^{\infty}(2n+1)^{-r}v_{rp} \tag{2.21}$$

when n is sufficiently large. The definition v_{rp} is the same as in [3], although the underlying functions g_0, g_1 and g_2 are different, and the quantities γ_{mp}, g_{mp}, h_{mp} and k_{mp} have been modified as described in Lemmas 1 and 2 above.

In a similar manner, the constants $S_{rmp}(0 \leq r+m \leq 3)$ and $S_{rp}(r=0,1,2,3)$ exhibited in Lemmas 8 through 11 of [3] can be shown by arguments essentially the same as those in [3] to be such that

$$\mu_p^{-1} v_{rp} = \sum_{m=0}^{3-r} (2n+1)^{-m} S_{rmp} + (-1)^n (2n+1)^{r-3} S_{rp} + O\{(2n+1)^{r-4}\} \tag{2.22}$$

when $r=0,1,2,3$. (The coefficient $8p^2+6p+3$ of $\cos 2z$ in Eq. (3.6) of [3] should have been $8p^2+12p+3$.) The desired result (1.3) now follows from (2.21) and (2.22) if the coefficients D_{rp} are defined so that

$$D_{rp} = \sum_{m=0}^{r} S_{r-m,mp} \quad (r=0,1,2,3). \tag{2.23}$$

Just as in [3] we then find the following explicit formulas for D_{rp}

$$D_{op} = 1, D_{1p} = 2\pi^{-1} \int_0^\infty \{\mu_p^{-1} G_p(z) - 1\} dz, \; D_{2p} = -(4p+3)/6, \tag{2.24}$$

$$D_{3p} = (3\pi)^{-1} \int_0^\infty \{\mu_p^{-1} J_p(z) + 4p + 3 - (4p^2+6p+3)\cos z\} dz - (3\pi)^{-1}$$

$$\cdot \int_0^\infty [\mu_p^{-1} H_p(z) - 2(p+1)\sin z - (2z)^{-1}\{4p+3+(8p^2+12p+3)\cos 2z\}] z dz.$$

These formulas for the coefficients D_{rp} are identical with Equations (3.28) and (3.29) in [3], apart from three inexplicable typographical errors in (3.29). They are, therefore, valid when $2p$ is any nonnegative integer except 0 and 1. On the other hand, the results of [2] and [4] show that $D_{20} = -0.25973$ and $D_{2,1/2} = -1/2$ instead of the values $-1/2$ and $-5/6$ predicted by (2.24).

References

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