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EXISTENCE OF SOLUTIONS OF NONLINEAR INTEGRODIFFERENTIAL EQUATION WITH NONLOCAL CONDITION

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In this paper we prove the existence and uniqueness of local and global solutions of a nonlocal Cauchy problem for a class of integrodifferential equation. The method of semigroups and the contraction mapping principle are used to establish the results.

Key words: Existence of Solutions, Integrodifferential Equation, Semigroup Theory.

AMS subject classifications: 34K05, 34K30.

1. Introduction

The problem of existence of solutions of evolution equation with nonlocal conditions in Banach space was first studied by Byszewski [5]. In that paper he established the existence and uniqueness of mild, strong, and classical solutions of the following nonlocal Cauchy problem:

$$\frac{du(t)}{dt} + Au(t) = f(t, u(t)), \quad t \in (t_0, t_0 + a] \tag{1}$$

$$u(t_0) + g(t_1, t_2, \ldots, t_p, u(\cdot)) = u_0, \tag{2}$$

where $-A$ is the infinitesimal generator of a C_0-semigroup $T(t)$, $t \geq 0$, on a Banach space X, $0 \leq t_0 < t_1 < t_2 < \ldots < t_p \leq t_0 + a$, $a > 0$, and $u_0 \in X$ and $f: [t_0, t_0 + a] \times X \rightarrow X$, $g(t_1, \ldots, t_p, \cdot): X \rightarrow X$ are given functions. Subsequently, Byszewski [2-4, 6-8] investigated the same type of problem stated to a different class of evolution equations in Banach space.

The purpose of this paper is to prove the existence and uniqueness of local solution for an integrodifferential equation with nonlocal conditions of the form:

$$\frac{du(t)}{dt} + Au(t) = f(t, u(t)) + \int_0^t h(t, s, u(s), \int_0^s k(s, \tau, u(\tau))d\tau)ds, \tag{3}$$

$$u(0) + g(t_1, t_2, \ldots, t_p, u(\cdot)) = u_0. \tag{4}$$

Here we assume that $-A$ is the infinitesimal generator of a bounded analytic semigroup of linear operator $X(t)$, $t \geq 0$, in a Banach space Z. The operator A^α can be defined for $0 \leq \alpha < 1$, and A^α is a closed linear invertible operator with domain $D(A^\alpha)$ dense in Z. The closedness of A^α implies that $D(A^\alpha)$, endowed with the graph norm of A^α, $|||Z||| = \| Z \| + \| A^\alpha z \|$, is a Banach space. Since A^α is invertible, its graph norm $|||\cdot|||$ is equivalent to the norm $\| Z \|_\alpha = \| A^\alpha Z \|$. Thus, $D(A^\alpha)$ equipped with the norm $\| \cdot \|_\alpha$, is a Banach space, which we denote by Z_α. From this definition, it is clear that $0 < \alpha < \beta$ implies $Z_\alpha \supset Z_\beta$ and that the embedding of Z_β in Z_α is continuous. Take $J = [0, a]$. Let $f: J \times Z_\alpha \to Z$, $h: J \times J \times Z_\alpha \times Z_\alpha \to Z$, $k: J \times J \times Z \to Z_\alpha$, and $g(t_1, \ldots, t_p, \cdot): Z_\alpha \to Z$ be given nonlinear operators. The notation $g(t_1, \ldots, t_p, u(\cdot))$ is used in the sense that in the place of '$\cdot$' we can substitute only an element of the set $\{t_1, \ldots, t_p\}$.

The results obtained in this paper generalize the Theorem 6.3.1 of Pazy [10] about the Cauchy problem. As in [1-3, 6, 7, 9], the nonlocal condition (2) in this paper can be used in physics with a better effect than the classical condition $u(0) = u_0$, since condition (2) is usually more suitable for physical measurements than the classical one.

2. Existence Theorems

Theorem 2.1: *Assume that*

(i) *$-A$ is the infinitesimal generator of a bounded analytic semigroup of linear operator $X(t)$, $t > 0$, in Z.*

(ii) *For $0 \leq \alpha < 1$, the fractional power A^α satisfies $\| A^\alpha X(t) \| \leq C_\alpha t^{-\alpha}$ for $t > 0$, where C_α is a real constant.*

(iii) *$0 \in \rho(-A)$, the resolvent set.*

(iv) *For an open subset D of $J \times Z_\alpha$, $f: D \to Z$ satisfies the condition, if for every $(t, u) \in D$ there is a neighborhood $V \subset D$ and constants $L \geq 0$, $0 < \Theta \leq 1$, such that*

$$\| f(t_1, u_1) - f(t_2, u_2) \| \leq L(|t_1 - t_2|^\Theta + \| u_1 - u_2 \|_\alpha) \tag{5}$$

for all $(t_i, u_i) \in V$, $i = 1, 2$.

(v) *For an open subset E of $J \times J \times Z_\alpha \times Z_\alpha$, $h: E \to Z$ satisfies the condition, if for every $(t, s, u, v) \in E$ there is a neighborhood $U \subset E$ and constants $L_1 \geq 0$, $0 < \Theta \leq 1$, such that*

$$\| h(t_1, s_1, u_1, v_1) - h(t_2, s_2, u_2, v_2) \|$$

$$\leq L_1(|t_1 - t_2|^\theta + |s_1 - s_2|^\theta) + \| u_1 - u_2 \|_\alpha + \| v_1 - V_2 \|_\alpha) \tag{6}$$

for all $(t_i, s_i, u_i, v_i) \in U$, $i = 1, 2$.

(vi) *For an open subset P of $J \times J \times Z$, $k: P \to Z$ satisfies the condition, if for every $(t, s, u) \in P$ there is a neighborhood $W \subset P$ and constants $L_2 \geq 0$,*

$0 < \Theta \leq 1$, such that

$$\| k(t_1, s_1, u_1) - k(t_2, s_2, u_2) \|$$

$$\leq L_2(| t_1 - t_2 |^{\theta} + | s_1 - s_2 |^{\theta} + \| u_1 - u_2 \|_{\alpha}) \tag{7}$$

for all $(t_i, s_i, u_i) \in W$, $i = 1,2$.

(*vii*) $g: J^p \times Z_{\alpha} \to Z$ *and there exists constants* $B^* > 0$ *and* $L^* > 0$ *such that*

$$\| A^{\alpha} g(t_1, \ldots, t_p, u(\cdot)) \| \leq B^* \text{ for } 0 \leq t < a$$

and

$$\| g(t_1, \ldots, t_p, u_1(\cdot)) - g(t_1, \ldots, t_p, u_2(\cdot)) \| \leq L^* \| u_1 - u_2 \|_{\alpha}.$$

Then the nonlocal Cauchy problem (3), (4) *has a unique local solution* $u \in C([0,a): Z) \cap C^1((0,a): Z)$.

Proof: Choose $t^* > 0$ and $\delta > 0$ such that estimates (5), (6), and (7) hold on the sets

$V = \{(t,u): 0 \leq t \leq t^*, \| u - u_0 \| \leq \delta\}$,
$U = \{(t,s,u,v): 0 \leq t, s \leq t^*, \| u - u_0 \| \leq \delta, \| v - v_0 \| \leq \delta\}$, and
$W = \{(t,s,u): 0 \leq t, s \leq t^*, \| u - u_0 \| \leq \delta\}$, respectively.

Let $B = \max_{0 \leq t < a} \| f(t, u_0) \|$ and

$$H = \max_{0 \leq t, s \leq t^*} \| h(t, s, u_0, \int_0^s k(s, \tau, u_0) d\tau) \|$$

and choose a such that for $0 \leq t < a$,

$$\| X(t) A^{\alpha} u_0 - A^{\alpha} u_0 \| < \delta / 4,$$

$$\| X(t) A^{\alpha} g(t_1, \ldots, t_p, u(\cdot)) - A^{\alpha} g(t_1, \ldots, t_p, u(\cdot)) \| < \delta / 4$$

and

$$0 < a < \min\{t^*, [\delta / 2(1 - \alpha) C_{\alpha}^{-1}(L\delta + B + LB^* + L_1 \delta a + L_1 B^* a + L_1 L_2 \delta a^2 + L_1 L_2 B^* a^2 + Ha)^{-1}]^{(1/(1-\alpha))}\}. \tag{8}$$

Let Y be the Banach space $C((0,a]: Z)$ with usual supremum norm which we denote by $\| \cdot \|_Y$. Define a map $F: Y \to Y$ by

$$Fy(t) = X(t) A^{\alpha} u_0 - X(t) A^{\alpha} g(t_1, \ldots, t_p, A^{-\alpha} y(\cdot)) + \int_0^t A^{\alpha} X(t-s) f(s, A^{-\alpha} y(s)) ds$$
$$+ \int_0^t A^{\alpha} X(t-s) \int_0^s h(s, \tau, A^{-\alpha} y(\tau), \int_0^{\tau} k(\tau, \mu, A^{-\alpha} y(\mu)) d\mu) d\tau ds. \tag{9}$$

Obviously, $Fy(0) = A^{\alpha} u_0 - A^{\alpha} g$. Let S be the nonempty closed and bounded subset of Y defined by

$$S = \{y: y \in Y, y(0) = A^{\alpha} u_0 - A^{\alpha} g, \| y(t) - (A^{\alpha} u_0 - A^{\alpha} g) \| \leq \delta\}.$$

For $y \in S$, we have

$$\| Fy(t) - (A^{\alpha}u_0 - A^{\alpha}g) \| \leq \| X(t)A^{\alpha}u_0 - A^{\alpha}u_0 \|$$

$$+ \| X(t)A^{\alpha}g(t_1,\ldots,t_p, A^{-\alpha}y(\cdot)) - A^{\alpha}g(t_1,\ldots,t_p, A^{-\alpha}y(\cdot)) \|$$

$$+ \int_0^t \| A^{\alpha}X(t-s)[f(s, A^{-\alpha}y(s)) - f(s,u_0)] \| ds + \int_0^t \| A^{\alpha}X(t-s)f(s,u_0) \| ds$$

$$+ \int_0^t \| A^{\alpha}X(t-s)[\int_0^s h(s,\tau, A^{-\alpha}y(\tau), \int_0^{\tau} k(\tau,\mu, A^{-\alpha}y(\mu))d\mu)d\tau$$

$$- \int_0^s h(s,\tau,u_0, \int_0^{\tau} k(\tau,\mu,u_0)d\mu)d\tau] \| ds$$

$$+ \int_0^t \| A^{\alpha}X(t-s) \int_0^s h(s,\tau,u_0, \int_0^{\tau} k(\tau,\mu,u_0)d\mu)d\tau \| ds$$

$$\leq \delta/4 + \delta/4 + \int_0^t \| A^{\alpha}X(t-s)L[A^{-\alpha}y(s) - (u_0 - g) - g] \| ds$$

$$+ B\int_0^t \| A^{\alpha}X(t-s) \| ds + \int_0^t \| A^{\alpha}X(t-s)\{L_1[(A^{-\alpha}y(\tau) - u_0)$$

$$+ L_2(A^{-\alpha}y(\mu) - u_0)a]a\} \| ds + C_{\alpha}a^{(1-\alpha)}(1-\alpha)^{-1}Ha$$

$$\leq \delta/2 + C_{\alpha}(L\delta + B + LB^*)a^{(1-\alpha)}(1-\alpha)^{-1}$$

$$+ C_{\alpha}a^{(1-\alpha)}(1-\alpha)^{-1}\{L_1(\delta + B^* + L_2(\delta + B^*)a)a\} + C_{\alpha}a^{(1-\alpha)}(1-\alpha)^{-1}Ha$$

$$\leq \delta/2 + C_{\alpha}a^{(1-\alpha)}(1-\alpha)^{-1}\{L\delta + B + LB^* + L_1\delta a + L_1B^*a + L_1L_2\delta a^2$$

$$+ L_1L_2B^*a^2 + Ha\} \leq \delta/2 + \delta/2 = \delta.$$

Therefore, F maps S into itself. Moreover, if $y_1, y_2 \in S$, then

$$\| Fy_1(t) - Fy_2(t) \|$$

$$\leq \| X(t)(A^{\alpha}g(t_1,\ldots,t_p, A^{-\alpha}y_1(\cdot)) - A^{\alpha}g(t_1,\ldots,t_p, A^{-\alpha}y_2(\cdot))) \|$$

$$+ \int_0^t \| A^{\alpha}X(t-s)[f(s, A^{-\alpha}y_1(s)) - f(s, A^{-\alpha}y_2(s))] \| ds$$

$$+\int_0^t \| A^\alpha X(t-s)[\int_0^s h(s,\tau,A^{-\alpha}y_1(\tau),\int_0^\tau k(\tau,\mu,A^{-\alpha}y_1(\mu))d\mu)$$

$$-\int_0^s h(s,\tau,A^{-\alpha}y_2(\tau),\int_0^\tau k(\tau,\mu,A^{-\alpha}y_2(\mu))d\mu)d\tau]\| ds$$

$$\leq C_\alpha a^{(1-\alpha)}(1-\alpha)^{-1}L^* \| y_1-y_2\|_Y + C_\alpha a^{(1-\alpha)}(1-\alpha)^{-1}L\| y_1-y_2\|_Y$$

$$+C_\alpha a^{(1-\alpha)}(1-\alpha)^{-1}L_1[(\| y_1-y_2\|_Y + L_2\| y_1-y_2\|_Y a)a]$$

$$\leq C_\alpha a^{(1-\alpha)}(1-\alpha)^{-1}[L^*+L+L_1(1+L_2a)a]\| y_1-y_2\|_Y$$

$$\leq (1/2)\| y_1-y_2\|_Y,$$

which implies that

$$\| Fy_1-Fy_2\|_Y \leq (1/2)\| y_1-y_2\|_Y.$$

By the contraction mapping theorem, mapping F has a unique fixed point $y\in S$. This fixed point satisfies the integral equation

$$y(t)=X(t)A^\alpha u_0 - X(t)A^\alpha g(t_1,\ldots,t_p,A^{-\alpha}y(\cdot))$$

$$+\int_0^t A^\alpha X(t-s)f(s,A^{-\alpha}y(s))ds$$

$$+\int_0^t A^\alpha X(t-s)\int_0^s h(s,\tau,A^{-\alpha}y(\tau),\int_0^\tau k(\tau,\mu,A^{-\alpha}y(\mu))d\mu)d\tau ds. \tag{10}$$

From (5), (6) and the continuity of y it follows that

$$t\rightarrow f(t,A^{-\alpha}y(t)) \text{ and } t\rightarrow h(t,s,A^{-\alpha}y(s),\int_0^s k(s,\tau,A^{-\alpha}y(\tau))d\tau$$

are continuous on $[0,a]$, and, hence, there exist constants N and H^* such that

$$\| f(t,A^{-\alpha}y(t))\| \leq N \tag{11}$$

and

$$\| h(t,s,A^{-\alpha}y(s),\int_0^s k(s,\tau,A^{-\alpha}y(\tau))d\tau\| \leq H^*. \tag{12}$$

Note that for every β satisfying $0<\beta<1-\alpha$ and every $0<h<1$, we have by Theorem 2.6.13 of [10] that

$$\| (X(h)-I)A^\alpha X(t-s)\|$$

$$\leq C_\beta h^\beta \| A^{\alpha+\beta} X(t-s) \| \leq C h^\beta (t-s)^{-(\alpha-\beta)} \text{ for some } C > 0. \tag{13}$$

If $0 < t < t+h \leq a$, then

$$\| y(t+h) - y(t) \| \leq \| (X(h)-I) A^\alpha X(t) u_0 \|$$

$$+ \| (X(h)-I) A^\alpha X(t) g(t_1, \ldots, t_p, A^{-\alpha} y(\cdot)) \|$$

$$+ \int_0^t \| (X(h)-I) A^\alpha X(t-s) f(s, A^{-\alpha} y(s)) \| ds$$

$$+ \int_t^{t+h} \| A^\alpha X(t+h-s) f(s, A^{-\alpha} y(s)) \| ds$$

$$+ \int_0^t \| (X(h)-I) A^\alpha X(t-s) \int_0^s h(s, \tau, A^{-\alpha} y(\tau), \int_0^\tau k(\tau, \mu, A^{-\alpha} y(\mu)) d\mu) d\tau \| ds$$

$$+ \int_t^{t+h} \| A^\alpha X(t+h-s) \int_0^s h(s, \tau, A^{-\alpha} y(\tau), \int_0^\tau k(\tau, \mu, A^{-\alpha} y(\mu)) d\mu) d\tau \| ds$$

$$= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 \tag{14}$$

Using (*vii*), (11), (12), and (13) we find that

$$I_1 \leq C t^{-(\alpha+\beta)} h^\beta \leq M_1 h^\beta$$

$$I_2 \leq C B^* t^{-(\alpha+\beta)} h^\beta \leq M_2 h^\beta$$

$$I_3 \leq C N h^\beta \int_0^t (t-s)^{-(\alpha+\beta)} ds \leq M_3 h^\beta$$

$$I_4 \leq N C_\alpha \int_t^{t+h} (t+h-s)^{-\alpha} ds = N C_\alpha (1-\alpha)^{-1} h^{(1-\alpha)} \leq M_4 h^\beta$$

$$I_5 \leq C h^\beta H^* a \int_0^t (t-s)^{-(\alpha+\beta)} ds \leq M_5 h^\beta$$

$$I_6 \leq H^* C_\alpha a \int_t^{t+h} (t+h-s)^{-\alpha} ds = H^* a C_\alpha (1-\alpha)^{-1} h^{(1-\alpha)} \leq M_6 h^\beta.$$

Here, M_1 and M_2 depend on t and vanish at $t \to 0$, but M_3, M_4, M_5, and M_6 can be selected to be independent of $t \in J$. Combining (14) with these estimates it follows that for every $t' > 0$ there is a constant C such that $\| y(t) - y(s) \| \leq C \mid t - s \mid^{\beta}$ for $0 \leq t' \leq t,\ s \leq a$ and therefore, y is locally Hölder continuous on $(0, a]$. The local Hölder continuity of $t \to f(t, A^{-\alpha} y(t))$ follows from

$$\| f(t, A^{-\alpha} y(t)) - f(s, A^{-\alpha} y(s)) \| \leq L(\mid t - s \mid^{\Theta} + \| y(t) - y(s) \|)$$

$$\leq C_1(\mid t - s \mid^{\Theta} + \mid t - s \mid^{\beta}) \text{ for some } C_1 > 0$$

and the local Hölder continuity of

$$t \to h(t, s, A^{-\alpha} y(s), \int_0^s k(s, \tau, A^{-\alpha} y(\tau) d\tau)$$

follows from

$$\| h(t, s, A^{-\alpha} y(s), \int_0^s k(s, \tau, A^{-\alpha} y(\tau)) d\tau)$$

$$- h(t, \mu, A^{-\alpha} y(\mu), \int_0^s k(\mu, \phi, A^{-\alpha} y(\phi)) d\phi) \|$$

$$\leq L_1\{ \mid s - \mu \mid^{\Theta} + \| y(s) - y(\mu) \| + L_2(\mid s - \mu \mid^{\Theta}$$

$$+ \mid \tau - \phi \mid^{\Theta} + \| y(\tau) - y(\phi) \|) a\}$$

$$\leq L_1\{ \mid s - \mu \mid^{\Theta} + \mid s - \mu \mid^{\beta} + L_3(\mid s - \mu \mid^{\Theta} + \mid \tau - \phi \mid^{\Theta} + \mid \tau - \phi \mid^{\beta}) a\}$$

for some $L_3 > 0$. Let y be a solution of (10). Consider the inhomogeneous initial value problem

$$\frac{du(t)}{dt} + Au(t) = f(t, A^{-\alpha} y(t)) + \int_0^t h(t, s, A^{-\alpha} y(s), \int_0^s k(s, \tau, A^{-\alpha} y(\tau)) d\tau) ds \tag{15}$$

$$u(0) + g(t_1, \ldots, t_p, A^{-\alpha} y(\cdot)) = u_0. \tag{16}$$

This problem has a unique solution $u \in C^1((0, a]: Z)$ [10], which is given by

$$u(t) = X(t) u_0 - X(t) g(t_1, \ldots, t_p, A^{-\alpha} y(\cdot)) + \int_0^t X(t - s) f(s, A^{-\alpha} y(s)) ds + \int_0^t X(t - s) \int_0^s h(s, \tau, A^{-\alpha} y(\tau), \int_0^{\tau} k(\tau, \mu, A^{-\alpha} y(\mu)) d\mu) d\tau ds. \tag{17}$$

For $t > 0$, each term of (17) is in $D(A)$ and *a fortiori* in $D(A^{\alpha})$. Operating on both sides of (17) with A^{α} we find that

$$A^{\alpha}u(t) = X(t)A^{\alpha}u_0 - X(t)A^{\alpha}g(t_1,\ldots,t_p,A^{-\alpha}y(\cdot))$$

$$+\int_0^t A^{\alpha}X(t-s)f(s,A^{-\alpha}y(s))ds$$

$$+\int_0^t A^{\alpha}X(t-s)\int_0^s h(s,\tau,A^{-\alpha}y(\tau),\int_0^{\tau} k(\tau,\mu,A^{-\alpha}y(\mu)d\mu)d\tau ds. \qquad (18)$$

From (10) the right-hand side of (18) equals $y(t)$ and therefore $u(t) = A^{-\alpha}y(t)$ and by (17), u is a $C^1((0,a]:Z)$ solution of (3), (4). The uniqueness of u follows from the uniqueness of the solutions of (10) and (15), (16). Hence, the theorem is proved.

Next, we shall prove the existence of global solutions of (3), (4).

Theorem 2.2: *Let $0 \in \rho(-A)$ and let $-A$ be the infinitesimal generator of an analytic semigroup $X(t)$ satisfying $\| X(t) \| \leq M$ for $t \geq 0$. Let $f: I \times Z_{\alpha} \rightarrow Z$, $h: I \times I \times Z_{\alpha} \times Z_{\alpha} \rightarrow Z$, and $g(t_1,\ldots,t_p,u(\cdot)): I^p \times Z_{\alpha} \rightarrow Z_{\alpha}$ satisfy (5), (6), and (7), respectively, with $I = [0,\infty)$. If there are continuous nondecreasing real valued functions $q_1(t)$ and $q_2(t)$ such that*

$$\| f(t,u) \| \leq q_1(t)(1 + \| u \|_{\alpha})$$

and

$$\| h(t,s,u,\int_0^s k(s,\tau,u)d\tau) \| \leq q_2(t)(1 + \| u \|_{\alpha})$$

for $t \geq 0$, $u \in Z_{\alpha}$, then for every $u_0 \in Z_{\alpha}$, equations (3), (4) have a unique solution u.

Proof: As in the proof of Theorem 2.1, the solution of (3) can be continued as long as $\| u(t) \|_{\alpha}$ remains bounded. It is enough to prove that if u exists on $[0,a)$ then $\| u(t) \|_{\alpha}$ is bounded as $t \rightarrow a$.

Since

$$A^{\alpha}u(t) = X(t)A^{\alpha}u_0 - X(t)A^{\alpha}g(t_1,\ldots,t_p,u(\cdot)) + \int_0^t A^{\alpha}X(t-s)f(s,u(s))ds$$

$$+\int_0^t A^{\alpha}X(t-s)\int_0^s h(s,\tau,u(\tau),\int_0^{\tau} k(\tau,\mu,u(\mu))d\mu)d\tau ds,$$

then

$$\| u(t) \|_{\alpha} \leq M \| A^{\alpha}u_0 \| + M \| A^{\alpha}g(t_1,\ldots,t_p,u(\cdot)) \|$$

$$+\int_0^t \| A^{\alpha}X(t-s)f(s,u(s)) \| ds$$

$$+\int_0^t \| A^{\alpha}X(t-s)\int_0^s h(s,\tau,u(\tau),\int_0^{\tau} k(\tau,\mu,u(\mu))d\mu)d\tau \| ds$$

$$\leq M[\, \| A^{\alpha}u_0 \| + \| A^{\alpha}g(t_1,\ldots,t_p,u(\cdot)) \| \,]$$

$$+q_1(t)a^{(1-\alpha)}(1-\alpha)^{-1}+q_1(t)\int_0^t (t-s)^{-\alpha}\|u(s)\|_\alpha ds$$

$$+q_2(t)a^{(1-\alpha)}(1-\alpha)^{-1}+q_2(t)\int_0^t (t-s)^{-\alpha}\|u(s)\|_\alpha ds$$

$$\leq M[\|A^\alpha u_0\| + \|A^\alpha g(t_1,\ldots,t_p,u(\cdot))\|]$$

$$+q_1(t)a^{(1-\alpha)}(1-\alpha)^{-1}+q_2(t)a^{(1-\alpha)}(1-\alpha)^{-1}$$

$$+(q_1(t)+q_2(t))\int_0^t (t-s)^{-\alpha}\|u(s)\|_\alpha ds.$$

By Gronwall's inequality, we get

$$\|u(t)\|_\alpha \leq C \text{ on } [0,a).$$

Hence, the proof.

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