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SELECTIONS OF SET-VALUED STOCHASTIC PROCESSES

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We show that $\mathcal{F}_t$-adapted, set-valued stochastic processes satisfying mild continuity conditions admit, $\mathcal{F}_t$-adapted, stochastically continuous selections.

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1. Introduction

In this paper we prove several theorems on the existence of $\mathcal{F}_t$-adapted, continuous selections for $\mathcal{F}_t$-adapted, set-valued stochastic processes, as well as a continuous time version of Hess' result on martingale selection [3]. Such results may be useful in the theory of the set-valued stochastic integral.

2. Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a *probability space* with a filtration $(\mathcal{F}_t)_{t \geq 0}$ (i.e., with a family of σ-fields $\mathcal{F}_t$), such that $0 \leq s \leq t$ implies that $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$. We assume that all P-null sets are in $\mathcal{F}_0$. Let $\mathcal{F}_{t-} = \sigma(\bigcup_{s \geq t} \mathcal{F}_s)$ and $\mathcal{F}_{t+} = \bigcap_{s > t} \mathcal{F}_s$. Obviously, $\mathcal{F}_{t-} \subseteq \mathcal{F}_t \subseteq \mathcal{F}_{t+}$.

For a random variable $\varphi: \Omega \rightarrow R^n$ such that $E(|\varphi|) = \int_{\Omega} |\varphi| dP < +\infty$, by $E(\varphi | \mathcal{F}_t)$ we denote the *conditional expectation* of φ, (i.e., an $\mathcal{F}_t$-*measurable* mapping) such that

$$\int_A E(\varphi | \mathcal{F}_t) dP = \int_A \varphi dP$$

for each $A \in \mathcal{F}_t$.

We say that a set-valued mapping $\Phi: \Omega \rightarrow R^n$ is a set-valued random variable iff Φ is $\mathcal{F}$-*measurable* (*weakly measurable* in the terminology of Himmelberg [5]), i.e.,

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$\{\omega: \Phi(\omega) \cap U \neq \emptyset\} \in \mathcal{F}$ for each open set $U \subseteq R^n$. Equivalently, Φ is *$\mathcal{F}$-measurable* iff the real-valued function $d(z, \Phi): \Omega \to R^n$ defined by

$$d(z, \Phi)(\omega) = d(z, \Phi(\omega)) = \inf_{v \in \Phi(\omega)} \| z - v \|,$$

where $\| w \|$ is the Euclidean norm of $w \in R^n$, is a random variable. Clearly, for a mapping $\varphi: \Omega \to R^n$ identified with the set-valued mapping $\Phi = \{\varphi\}$, this is equivalent to saying that φ is a random variable. Let $(F_t) = (F_t)_{t \geq 0}$ be a *set-valued stochastic process* with closed values in R^n (i.e., a family of $\mathcal{F}$-measurable set-valued mappings $F_t: \Omega \to R^n$, $t \geq 0$, with closed values). We say that (F_t) is *$\mathcal{F}_t$-adapted* iff F_t is $\mathcal{F}_t$-measurable for each $t \geq 0$, and we denote an $\mathcal{F}_t$-adapted process (F_t) such that $E(d(0, F_t)) < +\infty$ for each $t \geq 0$, by $(F_t, \mathcal{F}_t)$. A *selection* of the process (F_t) is a single-valued stochastic process (f_t) such that for every $t \geq 0$, there holds $f_t(\omega) \in F_t(\omega)$ for P-almost all ω. Additionally, if (f_t) is $\mathcal{F}_t$-adapted and satisfies $E(|f_t|) < +\infty$ for each $t \geq 0$, we will denote the process by $(f_t, \mathcal{F}_t)$.

Let us mention that for the unique σ-field $\mathcal{F}$, the result on convergence of measurable selections being extracted from the sequence of measurable set-valued mappings, that converge in the distribution, has been investigated by Salinetti and Wets [9, Theorem 5.1, Corollary 5.2]. On the other hand, Hess has proven the existence of martingale selections for discrete time, set-valued martingales and discussed the convergence of set-valued martingales.

3. Selection Theorem Results

Our first simple result concerns the case when almost all paths $t \mapsto F_t(\omega)$ are continuous, and similar to the results of Salinetti and Wets, are based on the regularity of metric projections. For $z \in R^n$ and the closed, convex set $A \subset R^n$, we denote by $\mathrm{Pr}(z, A)$ *the metric projection of z onto A with respect to Euclidean norm* (i.e., a unique element $\mathrm{Pr}(z, A) \in A$ such that $\| \mathrm{Pr}(z, A) - z \| = d(z, A)$). *The Wijsman topology* for the family $CCl(R^n)$ of all nonempty, closed convex subsets of R^n, is the weakest topology such that for every $y \in R^n$, the function $A \mapsto d(y, A)$ is continuous [10]. We will need the following lemma.

Lemma 1: *The mapping $A \mapsto Pr(z, A)$ of $CCl(R^n)$ into R^n is continuous with respect to the Wijsman topology.*

Proof: For $A, A_0 \in CCl(R^n)$ and $z \in R^n$, let us denote $y_0 = \mathrm{Pr}(y_0, A)$, $y = \mathrm{Pr}(z, A)$. Clearly,

$$\| y - y_0 \| \leq \| y - \mathrm{Pr}(y_0, A) \| + \| \mathrm{Pr}(y_0, A) - y_0 \| = \| y - \mathrm{Pr}(y_0, A) \| + d(y_0, A).$$

By the parallelogram equality, we have

$$\| y - \mathrm{Pr}(y_0, A) \|^2 = 2 \| y - z \|^2 + 2 \| \mathrm{Pr}(y_0, A) - z \|^2 - 4 \left\| \frac{y + \mathrm{Pr}(y_0, A)}{2} - z \right\|^2$$
$$\leq 2 \| \mathrm{Pr}(y_0, A) - z \|^2 - 2d(z, A)^2.$$

But

$$\| \Pr(y_0, A) - z \| \leq \| \Pr(y_0, A) - y_0 \| + \| y_0 - z \| = d(y_0, A) + d(z, A_0).$$

Thus,

$$\| y - \Pr(y_0, A) \|^2$$
$$\leq 2(d(y_0, A) + d(z, A_0) - d(z, A))(d(y_0, A) + d(z, A_0) + d(z, A)).$$

Consequently,

$$\| y - y_0 \| \leq d(y_0 < A)$$
$$+ \sqrt{2}\sqrt{(d(y_0, A) + d(z, A_0) - d(z, A))(d(y_0, A) + d(z, A_0) + d(z, A))}.$$

From this it follows immediately that $A \mapsto \Pr(z, A)$ is continuous with respect to the Wijsman topology.

Theorem 1: *If the stochastic process $(F_t, \mathfrak{F}_t)$ has closed convex values and for every $z \in R^n$, the functions $t \mapsto d(z, F_t)(\omega)$ is continuous for a.e. $\omega \in \Omega$, then for any $y \in R^n$, the process (f_t) defined by $f_t(\omega) = \Pr(y, F_t(\omega))$ is an $\mathfrak{F}_t$-adapted selection of F such that $t \mapsto f_t(\omega)$ is continuous for P-a.e. $\omega \in \Omega$.*

Proof: By virtue of Lemma 1, from the assumption that the functions $t \mapsto d(z, F_t(\omega)), z \in R^n$, and a.e. $\omega \in \Omega$ are continuous, it follows that for every $y \in R^n$, a.e. $\omega \in \Omega$, the mapping $t \mapsto \Pr(y, F_t(\omega))$ is continuous. To see that f_t is $\mathfrak{F}_t$-measurable note that

$$\operatorname{Graph} f_t = \{(\omega, z): \| y - z \| - d(y, F_t(\omega)) = 0\} \cap \operatorname{Graph} F_t.$$

Hence, by virtue of [5, Theorem 3.5 and Corollary 6.3], f_t is $\mathfrak{F}_t$-measurable.

In the following theorems we dispense completely with the upper semicontinuity assumption for the process $(F_t, \mathfrak{F}_t)$. We do not adopt any lower semicontinuity assumption for the functions $t \mapsto d(y, F_t)(\omega)$; we assume only the stochastic upper semicontinuity of these functions, which means the stochastic lower semicontinuity of the process $(F_t, \mathfrak{F}_t)$. We utilize a well-known theorem on measurable selections due to Kuratowski and Ryll-Nardzewski, as well as theorems on continuous selections of lower semicontinuous, set-valued mappings due to Michael [7] and to Antosiewicz, Cellina (see e.g., [1, Theorem 3]), respectively. We will need the following lemma.

Lemma 2: *Assume that for the stochastic process $(F_t, \mathfrak{F}_t)$, $s \geq 0$ and every $z \in R^n$, $A \in \mathfrak{F}_s$, the real-valued function $t \mapsto E(\chi_A d(z, F_t))$ is right-hand (respectively: left-hand) usc at s. Then for any $\mathfrak{F}_s$-measurable random variable φ with $E(|\varphi|) < +\infty$, the function $t \mapsto E(d(\varphi, F_t))$ is right-hand (respectively: left-hand) usc at s.*

Proof: Let $\epsilon > 0$. By assuming that for any constant function, $\varphi \equiv z$, we have $E(d(\varphi, F_t)) < E(d(\varphi, F_s)) + \frac{\epsilon}{2}$ whenever $t \in [s, s + \delta)$ (respectively, $t \in (s - \delta, s]$) for sufficiently small δ. For a step random variable $\varphi = \sum_{i=1}^{m} z_i \chi_{A_i}, A_i \in \mathfrak{F}_s$, we have

$$E(d(\varphi, F_t)) = \sum_{i=1}^{m} E(\chi_{A_i} d(z_i, F_t)) \leq \sum_{i=1}^{m} (E(\chi_{A_i} d(z_i, F_s)) + \frac{\epsilon}{2^i}) \leq E(d(\varphi, F_s)) + \epsilon,$$

whenever $t \in [s, s + \delta)$ $(t \in (s - \delta, s])$ for sufficiently small δ. For an arbitrary $\mathfrak{F}_s$-measurable φ, first choose a sequence of $\mathfrak{F}_s$-measurable step functions φ_n such that

$E(|\varphi-\varphi_n|)\to 0$. Then choose n such that $E(|\varphi-\varphi_n|)<\frac{\epsilon}{3}$ and let $\delta>0$ be such that $E(d(\varphi_n,F_t))<E(d(\varphi_n,F_s))+\frac{\epsilon}{3}$ for $t\in[s,s+\delta)$ $(t\in(s-\delta,s])$. Then,

$$E(d(\varphi,F_t))\le E(|\varphi-\varphi_n|)+E(d(\varphi_n,F_t))<E(d(\varphi_n,F_s))+\frac{2}{3}\epsilon$$

$$\le E(|\varphi_n-\varphi|)+E(d(\varphi,F_s))+\frac{2}{3}\epsilon<E(d(\varphi,F_s))+\epsilon,$$

whenever $t\in[s,s+\delta)$ $(t\in s-\delta,s])$.

Theorem 2: *Assume that a set-valued stochastic process* $(F_t,\mathfrak{F}_t)$ *has closed convex values and for every* $z\in R^n$, $s\ge 0$, *and* $A\in\mathfrak{F}_s$, *the real-valued function* $t\mapsto E(\chi_A d(z,F_t))$ *is right-hand usc at* s. *Then* $(F_t,\mathfrak{F}_t)$ *has a* L^1*-right-hand continuous selection* $(f_t,\mathfrak{F}_t)$.

Proof: Define a set-valued mapping $G:[0,+\infty)\to L^1(\Omega,\mathfrak{F},R^n)$ by

$$G(t)=\{\varphi\in L^1(\Omega,\mathfrak{F},R^n):\varphi \text{ is } \mathfrak{F}_t\text{-measurable selection of } F_t\}.$$

Based on the assumption $E(d(z,F_t))<+\infty$ for each $t\ge 0$, the mapping G has nonempty values by virtue of the Kuratowski and Ryll-Nardzewski measurable selection theorem (see e.g., [5, Theorem 5.1]). Moreover, the sets $G(t)$ are closed and convex because the set-valued random variables F_t have closed, convex values. If we equip $[0,+\infty)$ with the arrow topology $\tau_{\rightarrow}$ (i.e., the topology generated by the intervals $[s,t), 0\le s<t$), then it follows from the assumptions that $G:[0,+\infty)\to L(\Omega,\mathfrak{F},R^n)$ is a lower semicontinuous, set-valued mapping. Indeed, it suffices to show that $d(\varphi,G(t))=\inf_{\psi\in G(t)}E(|\varphi-\psi|)\to 0$ as $t\downarrow s$ for any $\varphi\in G(s)$, $s\ge 0$. Since φ is $\mathfrak{F}_t$-measurable for $t\ge s$, as a consequence of Kuratowski and Ryll-Nardzewski selection theorem, we have that

$$d(\varphi,G(t))=E(d(\varphi,F_t))$$

for $t\ge s$, (see Hiai and Umegaki [4, Theorem 2.2] and Rybiński [8, Lemma 6]). But by virtue of Lemma 2 we have that $E(d(\varphi,F_t))\to 0$ as $t\downarrow s$. This shows that G is lower semicontinuous on $([0,+\infty),\tau_{\rightarrow})$. Since $([0,+\infty),\tau_{\rightarrow})$ is a Lindelöff space, hence paracompact (see Engelking [2]), we can then apply the Michael continuous selection theorem to G ([7, Theorem 3.2″]), and get a continuous mapping $g:[0,+\infty)\to L^1(\Omega,\mathfrak{F},R^n)$ such that $g(t)\in G(t)$ for all $t\ge 0$. Obviously, continuity with respect to $\tau_{\rightarrow}$ means the right-hand continuity of g. We can then define the stochastic process $(f_t)_{t\ge 0}$ by $f_t(\omega)=g(t)(\omega)$. Clearly, a selection (f_t) is $\mathfrak{F}_t$-adapted. Since $E(|f_t-f_s|)=E(|g(t)-g(s)|)\to 0$ as $t\downarrow s$, then by the Chebyshev inequality, $P(|f_t-f_s|>\epsilon)\to 0$ as $t\to s$. Thus, $(f_t,\mathfrak{F}_t)$ is stochastically right-hand continuous.

For the proof of the next selection theorem, we will need also the following consequence of Levy's martingale convergence theorem.

Proposition 1: $\mathfrak{F}_t=\mathfrak{F}_{t-}$ *if and only if the function* $s\mapsto E(\varphi\,|\,\mathfrak{F}_s)$ *is* P*-almost everywhere left-hand continuous at* t *for each* $\mathfrak{F}_t$*-measurable* φ *such that* $E(|\varphi|)<+\infty$. *Analogously,* $\mathfrak{F}_t=\mathfrak{F}_{t+}$ *if and only if the function* $s\mapsto E(\varphi\,|\,\mathfrak{F}_s)$ *is* P*-almost everywhere right-hand continuous at* t *for each* $\mathfrak{F}$*-measurable* φ *such that* $E(|\varphi|)<+\infty$.

Proof: If $\mathfrak{F}_t=\mathfrak{F}_{t-}$, then by Levy's theorem (see Liptser and Shiraev [6, p. 24]) we have that $E(\varphi \mid \mathcal{F}_{s_n}) \to E(\varphi \mid \mathcal{F}_t)$ whenever $s_n \uparrow t$. Conversely, observe that for $A \in \mathcal{F}_t$, $E(\chi_A \mid \mathcal{F}_{s_n}) \to E(\chi_A \mid \mathcal{F}_{t-})$ by Levy's theorem whenever $s_n \uparrow t$. On the other hand, by assumption $E(\chi_A \mid \mathcal{F}_{s_n}) \to E(\chi_A \mid \mathcal{F}_t) = \chi_A$, thus $\chi_A = E(\chi_A \mid \mathcal{F}_{t-})$ P-almost everywhere. Therefore, for $B = (E_{\chi_A} \mid \mathcal{F}_{t-})^{-1}(1) \in F_{t-}$, $P((A \backslash B) \cup (B \backslash A)) = 0$. Since all P-null sets are in $\mathcal{F}_{t-}$, we conclude that $A \in \mathcal{F}_{t-}$. The analogous statement regarding $\mathcal{F}_t = \mathcal{F}_{t+}$ can be verified in the same way.

Theorem 3: *Let $\mathcal{F}_t = \mathcal{F}_{t-}$ for each $t > 0$. Assume that a set-valued stochastic process $(F_t, \mathcal{F}_t)$ has closed values and for every $z \in R^n$, $s \geq 0$, $A \in \mathcal{F}_s$, the real-valued function $t \mapsto E(\chi_A d(z, F_t))$ is usc at s. Assume also that P is nonatomic or $(F_t, \mathcal{F}_t)$ has convex values. Then $(F_t, \mathcal{F}_t)$ has an L^1-continuous selection $(f_t, \mathcal{F}_t)$.*

Proof: We consider $[0, +\infty)$ with the usual topology and will show that G (defined in the proof of Theorem 2) is lower semicontinuous. The right-hand lower semicontinuity can be proved exactly in the same way as in Theorem 2, so it suffices to show that for fixed $s > 0$, $\varphi \in G(s)$, we have $d(\varphi, G(t)) \to 0$ as $t \uparrow s$. But for $t < s$, we have

$$d(\varphi, G(t)) \leq E(\mid \varphi - E(\varphi \mid \mathcal{F}_t) \mid) + d(E(\varphi \mid \mathcal{F}_t), G(t))$$

$$= E(\mid \varphi - E(\varphi \mid \mathcal{F}_t) \mid) + E(d(E(\varphi \mid \mathcal{F}_t), F_t)))$$

$$\leq E(\mid \varphi - E(\varphi \mid \mathcal{F}_t) \mid) + E(\mid E(\varphi \mid \mathcal{F}_t) - \varphi \mid) + E(d(\varphi, F_t)).$$

By Proposition 1 we have $E(\mid \varphi - E(\varphi \mid \mathcal{F}_t) \mid) \to 0$ as $t \uparrow s$, and by Lemma 2 we have $E(d(\varphi, F_t)) \to 0$ as $t \uparrow s$. Therefore G is a lower semicontinuous set-valued mapping with closed values. Suppose now that P is nonatomic. Clearly, the sets $G(t)$ are decomposable (i.e., $\varphi \chi_A + \psi \chi_{\Omega \backslash A} \in G(t)$ whenever $\varphi, \psi \in G(t)$ and $A \in \mathcal{F}_t$). We can apply the Antosiewicz-Cellina continuous selection theorem (see Bressan and Colombo [1, Theorem 3]) to G, and get a continuous mapping $g: [0, +\infty) \to L^1(\Omega, \mathcal{F}, R^n)$ such that $g(t) \in G(t)$ for all $t \geq 0$. If $(F_t, \mathcal{F}_t)$ has convex values, as in the proof of Theorem 2, we get a continuous selection g applying Michael's theorem. Thus, the stochastic process (f_t) defined by $f_t(\omega) = g(t)(\omega)$ has desired properties.

If we assure the continuity of the conditional expectation operator $t \mapsto E(\varphi \mid \mathcal{F}_t)$, then we can extend Hess' result [3, Theorem 3.2] on the martingale selection of discrete time set-valued martingale and obtain a continuous martingale selection result. A set-valued process $(F_t, \mathcal{F}_t)$ is a *set-valued martingale* if

$$\{\varphi \in L^1(\Omega, \mathcal{F}, \mathcal{P}): \varphi \text{ is } \mathcal{F}_s\text{-measurable selection of } F_s\}$$

$$= \text{cl}\{E(\varphi \mid \mathcal{F}_s): \varphi \text{ is } \mathcal{F}_t\text{-measurable selection of } \mathcal{F}_t\}$$

for any $0 \leq s \leq t$, (see Hiai and Umegaki [4], Hess [3]). We propose the following continuous time version of Hess' theorem.

Proposition 2: *Let $(F_t, \mathcal{F}_t)$ be a set-valued martingale. If for every $t \geq 0$ we have $\mathcal{F}_t = \mathcal{F}_{t-}$, then $(F_t, \mathcal{F}_t)$ admits a martingale selection $(f_t, \mathcal{F}_t)$ with P-almost all paths left-hand continuous. If for every $t \geq 0$ we have $\mathcal{F}_t = \mathcal{F}_{t+}$, then $(F_t, \mathcal{F}_t)$ admits a martingale selection $(f_t, \mathcal{F}_t)$ with P-almost all paths right-hand continuous.*

Proof: Consider the discrete time set-valued martingale $(F_n)_{n = 0, \ldots}$ obtained from $(F_t, \mathcal{F}_t)$ by taking $t = 0, 1, \ldots$. By the Hess result, (F_n) has a martingale selection (f_n) (i.e., there exists a sequence of $\mathcal{F}_n$-measurable mappings $f_n: \Omega \rightarrow R^n$ such that f_n is a selection of F_n and $f_n = E(f_{n+1} \mid \mathcal{F}_n)$ for $n = 0, 1, \ldots$). For $t \in [0, +\infty) \backslash \{0, 1, 2, \ldots\}$ we define $f_t: \Omega \rightarrow R^n$ by $f_t = E(f_n \mid \mathcal{F}_t)$ where $n - 1 < t < n$. Clearly, (f_t) is a martingale selection of F. By Proposition 1, (f_t) has P-almost all paths left-hand (respectively, right-hand) continuous.

References

[1] Bressan, A. and Colombo, G., Extensions and selections of maps with decomposable values, *Studia Math* **90** (1988), 69-85.

[2] Engelking, R., *General Topology*, Monografie Matematyczne, PWN Warszawa, Poland 1976.

[3] Hess, C., On multivalued martingales whose values may be unbounded: Martingale selectors and Mosco convergence, *J. Multivar. Anal.* **39** (1991), 175-201.

[4] Hiai, F. and Umegaki, H., Integrals, conditional expectations, and martingales of multivalued functions, *J. Multivar. Anal.* **7** (1977), 149-182.

[5] Himmelberg, C.J., Measurable relations, *Fund. Math.* **87** (1975), 53-72.

[6] Liptser, R.Sh. and Shiraev, A.N., *Stochastics of Random Processes*, Springer-Verlag, Berlin 1976.

[7] Michael, E., Continuous selections I, *Ann. of Math.* **63** (1956), 361-382.

[8] Rybiński, L.E., Random fixed points and viable random solutions of functional-differential inclusions, *J. Math. Anal. Appl.* **142** (1989), 53-61.

[9] Salinetti, G. and Wets, R.J.-B., On the convergence in distribution of measurable multifunctions (random sets), normal integrands, stochastic processes and stochastic infima, *Math. Oper. Res.* **11** (1986), 385-419.

[10] Wijsman, R.A., Convergence of sequences of convex sets, cones and functions, *Bull. Amer. Math. Soc.* **70** (1964), 186-188.