# ON THE APPROXIMATION OF AN INTEGRAL BY A SUM OF RANDOM VARIABLES

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We approximate the integral of a smooth function on $[0,1]$, where values are only known at $n$ random points (i.e., a random sample from the uniform-$(0,1)$ distribution), and at 0 and 1. Our approximations are based on the trapezoidal rule and Simpson's rule (generalized to the non-equidistant case), respectively. In the first case, we obtain an $n^2$-rate of convergence with a degenerate limiting distribution; in the second case, the rate of con-vergence is as fast as $n^{3^1/2}$, whereas the limiting distribution is Gaussian then.

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## 1. Introduction and Main Results

Suppose we (can) only observe the values of a smooth function $f:[0,1] \to \mathbb{R}$ at the points $U_0, U_1, \dots, U_n, U_{n+1}$, where $U_1, U_2, \dots, U_n$ are the order statistics $(U_1 \leq U_2 \leq \dots \leq U_n)$ of $n$ independent uniformly-$(0,1)$ distributed random variables and $U_0: = 0$, $U_{n+1}: = 1$. It is our aim to estimate the integral

$$I: = \int_0^1 f(x)dx \tag{1}$$

from these observations, i.e., by only using $(U_i, f(U_i))$, $i = 0, 1, \dots, n+1$. The first

estimator we will employ is constructed by using the 'trapezoidal rule' on each sub-interval $[U_{i-1}, U_i]$, $i = 1, \ldots, n+1$. This rule approximates an integral $\int_a^b g(x)dx$ simply by $\frac{1}{2}(b-a)(g(a)+g(b))$ and it can easily be shown (see, e.g., Isaacson and Keller [2], p. 304) that

$$\frac{1}{2}(b-a)(g(a)+g(b)) - \int_a^b g(x)dx = \frac{1}{12}(b-a)^3 g''(\eta), \tag{2}$$

where $\eta \in (a,b)$. Writing $D_i = U_i - U_{i-1}$, $i = 1, \ldots, n+1$, for the spacings of the $U_i$'s, our estimator of $I$ becomes

$$I_n := \sum_{i=1}^{n+1} \frac{1}{2} D_i (f(U_{i-1}) + f(U_i)). \tag{3}$$

Using (2), we will prove the following limiting result for the standardized difference of $I_n$ and $I$:

**Theorem 1:** *If $|f'''|$ is bounded, then*

$$n^2(I_n - I) \overset{P}{\to} \frac{1}{2}(f'(1) - f'(0)), \quad \text{as } n \to \infty. \tag{4}$$

A much better and probabilistically more interesting estimator is obtained by applying a 3-points formula, i.e., for a given $c \in (a,b)$, we approximate $\int_a^b g(x)dx$ by $w_1 g(a) + w_2 g(c) + w_3 g(b)$ in such a way that the approximation error is zero in the case $g$ is a polynomial of second degree. If the 3 points are equidistant, this approximation is known as Simpson's rule. It is not hard to show that

$$w_1 = \frac{1}{6}(b-a)\left(2 - \frac{b-c}{c-a}\right), \quad w_2 = \frac{1}{6}\frac{(b-a)^3}{(c-a)(b-c)}, \quad w_3 = \frac{1}{6}(b-a)\left(2 - \frac{c-a}{b-c}\right), \tag{5}$$

and it follows (see again Isaacson and Keller [2], p. 304) that

$$w_1 g(a) + w_2 g(c) + w_3 g(b) - \int_a^b g(x)dx = -\frac{1}{6}\int_a^b (x-a)(x-c)(x-b)g^{(3)}(\eta)dx, \tag{6}$$

where $\eta = \eta(x) \in (a,b)$. Hence, our estimator of $I$ in (1), again denoted by $I_n$, becomes

$$I_n = \sum_{i=1}^{\frac{n+1}{2}} \frac{1}{6}(D_{2i-1} + D_{2i})\left\{\left(2 - \frac{D_{2i}}{D_{2i-1}}\right)f(U_{2i-2}) \right.$$
$$\left. + \frac{(D_{2i-1} + D_{2i})^2}{D_{2i-1}D_{2i}}f(U_{2i-1}) + \left(2 - \frac{D_{2i-1}}{D_{2i}}\right)f(U_{2i})\right\}, \tag{7}$$

where, for convenience, $n$ is taken to be odd. Formula (6) will be used to prove our main result:

**Theorem 2:** *Let $n$ be odd. If $|f^{(5)}|$ is bounded, then*

$$n^{3\frac{1}{2}}(I_n - I) \overset{\to}{d} \sqrt{\frac{35}{3}\int_0^1 (f^{(3)}(x))^2 dx} \; Z, \quad \text{as } n \to \infty, \tag{8}$$

*where Z is a standard normal random variable.*

**Remark 1:** The present techniques can be easily adapted to cover the situation where the $U_i$'s are the order statistics of $n$ independent random variables with common distribution function $G$ (on $(0,1)$) having a smooth density $g$. The adaptation is based on the quantile transform, transforming a uniform random variable $V$ into a random variable $G^{-1}(V)$ with distribution function $G$. In this case, under regularity conditions on $g$, we obtain that the weak limit in Theorem 1 becomes $\frac{1}{2}\int_0^1 (f''(x)/g^2(x))dx$ instead of $\frac{1}{2}\int_0^1 f''(x)dx = \frac{1}{2}(f'(1) - f'(0))$. In Theorem 2, the limiting random variable is again centered normal but now the standard deviation becomes

$$\sqrt{\frac{35}{3}\int_0^1 \frac{(f^{(3)}(x))^2}{g^7(x)}dx}.$$

On the other hand, the uniform distribution seems very relevant because of the following. Since $\int_0^1 f(x)dx$ can be considered as the mean 'output', given that the $x$-values are 'equally important', it seems desirable to estimate $\int_0^1 f(x)g(x)dx = \int_0^1 f(G^{-1}(y))dy$ in the case the random variables are distributed according to $G$. But if $G$ is *known*, we can replace the pairs $(U_i, f(U_i))$ (just below (1)), with $U_i$'s being the order statistics from $G$, by $(G(U_i), f(U_i)) = (G(U_i), f(G^{-1}(G(U_i))))$. This brings us back to the 'uniform distribution setup' with $f$ replaced by $f \circ G^{-1}$, but that is just the function whose integral we wanted to estimate as argued above!

This idea leads to possible ways of applying the results. Suppose $U_i$ represents some uncontrollable physical random quantity, like temperature, humidity or light intensity with a known distribution function $G$ having density $g$. Suppose also that we can measure $f$ (the output or yield) only at the $U_i$ and that we are interested in the mean output $I_g = \int_0^1 f(x)g(x)dx$. Then one can use our theorems to obtain rapidly converging estimators of $I_g$. In particular, when measuring the $f$-values is hard or expensive, one can get good estimators based on a few observations.

Also note that for the trapezoidal rule in Theorem 1 and $f''$ being constant, the uniform distribution is optimal, since $\int_0^1 g^{-2}(x)dx \geq \int_0^1 1dx = 1$. (This can be easily seen by using Jensen's inequality:

$$\int_0^1 \frac{1}{g^2(x)}dx = \int_0^1 \frac{1}{g^3(x)}g(x)dx = \mathbb{E}\frac{1}{g^3(X)} \geq \left(\mathbb{E}\frac{1}{g(X)}\right)^3$$

$$= \left(\int_0^1 \frac{1}{g(x)}g(x)dx\right)^3 = 1,$$

where $X$ is a random variable with density $g$.) A similar remark applies to Theorem 2 with $f^{(3)}$ being constant.

**Remark 2:** There are various other ways to extend our results, which we will not pursue here, e.g., applying $m$-points formulas for $m > 3$ (Simpson's rule is 'by far the

most frequently used in obtaining approximate integrals', Davis and Rabinowitz [1], p. 45), combining trapezoidal rules to eliminate the bias $\frac{1}{2}(f'(1) - f'(0))$, proving a 'second order' limit result for $n^2(I_n - I) - \frac{1}{2}(f'(1) - f'(0))$ in Theorem 1, or treating the case $n$ 'even' in Theorem 2. We are not pursuing these extensions because we believe they are not very interesting and/or they do not give good results.

**Remark 3:** We briefly compare our results with the deterministic, equidistant case, i.e., $U_i = \frac{i}{n+1}$, $i = 0, 1, \ldots, n+1$. It is well-known that the limit in Theorem 1 is $\frac{1}{12}(f'(1) - f'(0))$ in that case, which means that we loose a factor of 6 by having random $U_i$'s. (Essentially, this 6 is coming from the third moment of a standard exponential random variable.) From Theorem 2, it is well-known that in the equidistant case (Simpson's rule), the rate is $n^4$. So, there our loss is of order $n^{1/2}$. Nevertheless, from statistical point of view, $n^{3\frac{1}{2}}$ is a remarkably fast rate of convergence.

## 2. Proofs

The following well-known lemma will be used frequently; it can be found in, e.g., Shorack and Wellner [3], p. 721.

**Lemma 1:** *Let* $E_1, \ldots, E_{n+1}$ *be independent exponential random variables with mean 1 and* $S_{n+1}$ *be their sum. With* $D_i$, $i = 1, \ldots, n+1$, *as before, we have*

$$(D_1, \ldots, D_{n+1}) \stackrel{d}{=} \left( \frac{E_1}{S_{n+1}}, \ldots, \frac{E_{n+1}}{S_{n+1}} \right).$$

**Proof of Theorem 1:** Using (3), (1) and (2) we see that

$$n^2(I_n - I) = \frac{n^2}{12} \sum_{i=1}^{n+1} D_i^3 f''(\widetilde{U}_i)$$

for some $\widetilde{U}_i \in (U_{i-1}, U_i)$, and hence,

$$n^2(I_n - I) = \frac{n^2}{12} \sum_{i=1}^{n+1} D_i^3 f''\left(\frac{i}{n+1}\right) + \frac{n^2}{12} \sum_{i=1}^{n+1} D_i^3 (\widetilde{U}_i - \frac{i}{n+1}) f'''(\widetilde{\widetilde{U}}_i), \qquad (9)$$

with $\widetilde{\widetilde{U}}_i$ between $\widetilde{U}_i$ and $\frac{i}{n+1}$. From the boundedness of $|f'''|$ (by $M$, say) and the weak convergence (to a Brownian bridge) of the uniform quantile process (see, e.g., Shorack and Wellner [3]), it is readily seen that

$$\left| \frac{n^2}{12} \sum_{i=1}^{n+1} D_i^3 (\widetilde{U}_i - \frac{i}{n+1}) f'''(\widetilde{\widetilde{U}}_i) \right| \qquad (10)$$

$$\leq \frac{n^2}{12} M \sup_{i \in \{1, \ldots, n+1\}} \left| \widetilde{U}_i - \frac{i}{n+1} \right| \sum_{i=1}^{n+1} D_i^3 = O_p(n^{1\frac{1}{2}}) \sum_{i=1}^{n+1} D_i^3.$$

But

$$\sum_{i=1}^{n+1} D_i^3 \stackrel{d}{=} \frac{1}{S_{n+1}^3} \sum_{i=1}^{n+1} E_i^3, \qquad (11)$$

by Lemma 1, and by two applications of the weak law of large numbers, this last expression is $O_p(n^{-2})$. Combining this with (10) and (11) yields that the second term on the right in (9) converges to zero in probability. Hence, it remains to consider the first term

$$\frac{n^2}{12}\sum_{i=1}^{n+1}D_i^3 f''\left(\frac{i}{n+1}\right)\overset{d}{=}\frac{1}{12n}\left(\frac{n}{S_{n+1}}\right)^3\sum_{i=1}^{n+1}f''\left(\frac{i}{n+1}\right)E_i^3,$$

or, since $(n/S_{n+1})^3\overset{P}{\to}1$,

$$\frac{1}{12n}\sum_{i=1}^{n+1}f''\left(\frac{i}{n+1}\right)E_i^3.$$

By Chebysev's inequality, it follows that

$$\frac{1}{12n}\sum_{i=1}^{n+1}f''\left(\frac{i}{n+1}\right)E_i^3-\frac{1}{2n}\sum_{i=1}^{n+1}f''\left(\frac{i}{n+1}\right)\overset{P}{\to}0.$$

The proof is complete by noting that

$$\frac{1}{2n}\sum_{i=1}^{n+1}f''\left(\frac{i}{n+1}\right)\to\frac{1}{2}\int_0^1 f''(x)dx=\tfrac{1}{2}(f'(1)-f'(0)).\qquad\square$$

The proof of Theorem 2 is heavily based on the following two lemmas.

**Lemma 2:** *Let $E_1,\ldots,E_{n+1}$, $n$ odd, be independent exponential random variables with mean 1. Write*

$$X_i=(E_{2i-1}+E_{2i})^3(E_{2i}-E_{2i-1}),\quad i=1,2,\ldots,\frac{n+1}{2},$$

$$Y_i=X_i\sum_{j=1}^{2i-2}(E_j-1),\quad i=2,3,\ldots,\frac{n+1}{2}.$$

*Then,*

$$\mathbb{E}X_i=0,\ \ \mathrm{Var}\,X_i=120960,\ \ \mathbb{E}Y_i=0,\ \mathrm{Var}\,Y_i=120960(2i-2),$$

$$\mathrm{Cov}\,(Y_i,Y_k)=0,\ \text{for } i\neq k.$$

**Proof:** By symmetry, we see that $\mathbb{E}X_i=0$; a straightforward computation yields $\mathrm{Var}\,X_i=\mathbb{E}X_i^2=120960$. For the $Y_i$'s we have

$$\mathbb{E}Y_i=\mathbb{E}X_i\mathbb{E}\sum_{j=1}^{2i-2}(E_j-1)=0,$$

$$\mathrm{Var}\,Y_i=\mathbb{E}Y_i^2=\mathbb{E}X_i^2\mathbb{E}\left(\sum_{j=1}^{2i-2}(E_j-1)\right)^2$$

$$=\mathrm{Var}\,X_i\,\mathrm{Var}\left(\sum_{j=1}^{2i-2}E_j\right)=120960(2i-2),$$

and for $i<k$,

$$\mathrm{Cov}\,(Y_i,Y_k)=\mathbb{E}Y_iY_k$$

$$=\mathbb{E}X_k\left(\sum_{j=1}^{2k-2}(E_j-1)\right)X_i\left(\sum_{j=1}^{2i-2}(E_j-1)\right)$$

$$=\mathbb{E}X_k\,\mathbb{E}\left(\sum_{j=1}^{2k-2}(E_j-1)\right)X_i\left(\sum_{j=1}^{2i-2}(E_j-1)\right)$$

$$=0.\qquad\square$$

**Lemma 3:** *Under the conditions of Theorem 2, we have, as* $n \to \infty$,

$$\left| n^{3\frac{1}{2}}(I_n - I) - \frac{n^{\frac{3}{2}}}{72} \sum_{i=1}^{\frac{n+1}{2}} f^{(3)}\left(\frac{2i-2}{n+1}\right)(D_{2i-1} + D_{2i})^3(D_{2i} - D_{2i-1}) \right| = o_p(1).$$

**Proof:** By (7), (1) and (6) we have

$$n^{3\frac{1}{2}}(I_n - I)$$

$$= - \frac{n^{\frac{3}{2}}}{6} \sum_{i=1}^{\frac{n+1}{2}} \int_{U_{2i-2}}^{U_{2i}} (x - U_{2i-2})(x - U_{2i-1})(x - U_{2i})f^{(3)}(\widetilde{U}_{2i})dx, \qquad (12)$$

for some $\widetilde{U}_{2i} = \widetilde{U}_{2i}(x) \in (U_{2i-2}, U_{2i})$ and hence for some $\widetilde{\widetilde{U}}_{2i} = \widetilde{\widetilde{U}}_{2i}(x) \in (U_{2i-2}, \widetilde{U}_{2i})$, the right-hand side of (12) is equal to

$$- \frac{n^{\frac{3}{2}}}{6} \sum_{i=1}^{\frac{n+1}{2}} \int_{U_{2i-2}}^{U_{2i}} (x - U_{2i-2})(x - U_{2i-1})(x - U_{2i})$$

$$\times (f^{(3)}(U_{2i-2}) + (\widetilde{U}_{2i} - U_{2i-2})f^{(4)}(\widetilde{\widetilde{U}}_{2i}))dx$$

$$= \frac{n^{\frac{3}{2}}}{72} \sum_{i=1}^{\frac{n+1}{2}} f^{(3)}(U_{2i-2})(D_{2i-1} + D_{2i})^3(D_{2i} - D_{2i-1})$$

$$- \frac{n^{\frac{3}{2}}}{6} \sum_{i=1}^{\frac{n+1}{2}} \int_{U_{2i-2}}^{U_{2i}} (x - U_{2i-2})(x - U_{2i-1})(x - U_{2i})(\widetilde{U}_{2i} - U_{2i-2})f^{(4)}(\widetilde{\widetilde{U}}_{2i})dx.$$

Let $M$ be a bound on $|f^{(5)}|$ and all lower order derivatives of $f$. Then the absolute value of this last term is bounded from above by

$$M\frac{n^{\frac{3}{2}}}{6} \sum_{i=1}^{\frac{n+1}{2}} \int_{U_{2i-2}}^{U_{2i}} (x - U_{2i-2})\,|\,x - U_{2i-1}\,|\,(U_{2i} - x)(\widetilde{U}_{2i} - U_{2i-2})dx$$

$$\leq \frac{M}{6}n^{\frac{3}{2}} \sum_{i=1}^{\frac{n+1}{2}} (D_{2i-1} + D_{2i})^5 \overset{d}{=} \frac{M}{6}n^{\frac{3}{2}}\frac{1}{S_{n+1}^5} \sum_{i=1}^{\frac{n+1}{2}} (E_{2i-1} + E_{2i})^5 = o_p(1),$$

due to Lemma 1 and two applications of the weak law of large numbers.

So, it suffices to show the convergence to zero in probability of

$$\frac{n^{\frac{3}{2}}}{72} \sum_{i=1}^{\frac{n+1}{2}} \left(f^{(3)}(U_{2i-2}) - f^{(3)}\left(\frac{2i-2}{n+1}\right)\right)(D_{2i-1} + D_{2i})^3(D_{2i} - D_{2i-1})$$

$$= \frac{n^{\frac{3}{2}}}{72} \sum_{i=2}^{\frac{n+1}{2}} \left(U_{2i-2} - \frac{2i-2}{n+1}\right)f^{(4)}\left(\frac{2i-2}{n+1}\right)(D_{2i-1} + D_{2i})^3(D_{2i} - D_{2i-1})$$

$$+ \frac{n^{3\frac{1}{2}}}{144} \sum_{i=2}^{\frac{n+1}{2}} \left(U_{2i-2} - \frac{2i-2}{n+1}\right)^2 f^{(5)}(\bar{U}_{2i-2})(D_{2i-1} + D_{2i})^3 (D_{2i} - D_{2i-1})$$

$$=: T_{1,n} + T_{2,n},$$

for some $\bar{U}_{2i-2}$ between $U_{2i-2}$ and $\frac{2i-2}{n+1}$. By the weak convergence of the uniform quantile process,

$$|T_{2,n}| \le \frac{n^{3\frac{1}{2}}}{144} M \sup_{i \in \{2,3,\dots,\frac{n+1}{2}\}} \left(U_{2i-2} - \frac{2i-2}{n+1}\right)^2 \sum_{i=2}^{\frac{n+1}{2}} (D_{2i-1} + D_{2i})^4$$

$$= O_p(n^{2\frac{1}{2}}) \sum_{i=2}^{\frac{n+1}{2}} (D_{2i-1} + D_{2i})^4.$$

By Lemma 1 and twice the weak law of large numbers, this last expression is easily seen to be $o_p(1)$. Hence, the proof of Lemma 3 is complete if we show $T_{1,n} = o_p(1)$.

From Lemma 1 we obtain

$$T_{1,n} \overset{d}{=} \frac{n^{3\frac{1}{2}}}{72} \sum_{i=2}^{\frac{n+1}{2}} \left( \frac{\sum_{j=1}^{2i-2} E_j}{S_{n+1}} - \frac{2i-2}{n+1} \right) f^{(4)}\left(\frac{2i-2}{n+1}\right) \left(\frac{E_{2i-1} + E_{2i}}{S_{n+1}}\right)^3 \left(\frac{E_{2i} - E_{2i-1}}{S_{n+1}}\right)$$

$$= \frac{n^{3\frac{1}{2}}}{72} S_{n+1}^{-5} \sum_{i=2}^{\frac{n+1}{2}} \left( \sum_{j=1}^{2i-2} (E_j - 1) \right) f^{(4)}\left(\frac{2i-2}{n+1}\right) (E_{2i-1} + E_{2i})^3 (E_{2i} - E_{2i-1})$$

$$+ \frac{n^{3\frac{1}{2}}}{72} S_{n+1}^{-5} \left( 1 - \frac{S_{n+1}}{n+1} \right) \sum_{i=2}^{\frac{n+1}{2}} (2i-2) f^{(4)}\left(\frac{2i-2}{n+1}\right) (E_{2i-1} + E_{2i})^3 (E_{2i} - E_{2i-1})$$

$$=: T_{3,n} + T_{4,n}.$$

It is immediate from the central limit theorem for $S_{n+1}/(n+1)$ that

$$T_{4,n} = O_p(n^{-2}) \sum_{i=2}^{\frac{n+1}{2}} (2i-2) f^{(4)}\left(\frac{2i-2}{n+1}\right) X_i,$$

where the $X_i$'s are as in Lemma 2. Now using that lemma in conjunction with Chebysev's inequality, it readily follows that $T_{4,n} = o_p(1)$. Finally, in the notation of Lemma 2,

$$T_{3,n} = \frac{n^{3\frac{1}{2}}}{72} S_{n+1}^{-5} \sum_{i=2}^{\frac{n+1}{2}} f^{(4)}\left(\frac{2i-2}{n+1}\right) Y_i = O_p(n^{-1\frac{1}{2}}) \sum_{i=2}^{\frac{n+1}{2}} f^{(4)}\left(\frac{2i-2}{n+1}\right) Y_i.$$

From Lemma 2, we have

$$\mathbb{E} \sum_{i=2}^{\frac{n+1}{2}} f^{(4)}\left(\frac{2i-2}{n+1}\right) Y_i = 0,$$

$$\text{Var} \sum_{i=2}^{\frac{n+1}{2}} f^{(4)}\left(\frac{2i-2}{n+1}\right) Y_i = \sum_{i=2}^{\frac{n+1}{2}} \left( f^{(4)}\left(\frac{2i-2}{n+1}\right) \right)^2 \text{Var}\, Y_i = O(n^2).$$

Now, Chebysev's inequality yields $T_{3,n} = o_p(1)$ and hence $T_{1,n} = o_p(1)$.   □

**Proof of Theorem 2:** Given the lemmas, especially Lemma 3, the proof of Theorem 2 is rather easy. If $\int_0^1 (f^{(3)}(x))^2 dx = 0$, then $f^{(3)}(x) = 0$ for all $x \in [0,1]$ and hence trivially $I_n = I$, because $f$ is a polynomial of second degree. Therefore, we assume now $\int_0^1 (f^{(3)}(x))^2 dx > 0$. Using Lemma 1 we have

$$\frac{n^{\frac{3}{2}}}{72} \sum_{i=1}^{\frac{n+1}{2}} f^{(3)}\left(\frac{2i-2}{n+1}\right)(D_{2i-1} + D_{2i})^3 (D_{2i} - D_{2i-1})$$

$$\stackrel{d}{=} \frac{1}{72\sqrt{2}}\left(\frac{n}{S_{n+1}}\right)^4 \frac{1}{(n/2)^{1/2}} \sum_{i=1}^{\frac{n+1}{2}} f^{(3)}\left(\frac{2i-2}{n+1}\right)(E_{2i-1} + E_{2i})^3 (E_{2i} - E_{2i-1})$$

$$=: \left(\frac{n}{S_{n+1}}\right)^4 W_n.$$

By the weak law of large numbers and Lemma 3, it now remains to show Theorem 2 with $n^{\frac{3}{2}}(I_n - I)$ replaced by $W_n$. By Lemma 2, we see that $\mathbb{E}W_n = 0$ and

$$\text{Var}\, W_n = \frac{1}{2(72)^2} \frac{2}{n} \sum_{i=1}^{\frac{n+1}{2}} \left(f^{(3)}\left(\frac{2i-2}{n+1}\right)\right)^2 120960 \to \frac{35}{3} \int_0^1 \left(f^{(3)}(x)\right)^2 dx.$$

Now, the Lindeberg central limit theorem applies, because of the boundedness of $|f^{(3)}|$, and it yields the result.   □

## Acknowledgements

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