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OSCILLATION CRITERIA FOR HIGH ORDER DELAY PARTIAL DIFFERENTIAL EQUATIONS[1]

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This paper studies a class of high order delay partial differential equations. Employing high order delay differential inequalities, several oscillation criteria are established for such equations subject to two different boundary conditions. Two examples are also given.

Key words: Oscillation, Higher Order Delay Partial Differential Equations, Differential Inequality, Eventual Positive Solutions.

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1. Introduction

The oscillation theory of delay differential equations has been studied by numerous authors and the number of papers published in this area is enormous. For an excellent exposition of the basic theory, see [5]. In recent years, there has been an increasing interest in oscillation theory of delay partial differential equations, see [6-10] and references therein. However, the corresponding theory is still in its initial stage of development. In this paper, we shall investigate a class of high order delay partial differential equations which will be described in Section 2. In Section 3, we shall establish several oscillation criteria for high order delay partial differential equations subject to two kinds of boundary conditions, employing Green's theorem and high order delay differential inequalities. We then develop, in Section 4, some results on eventual positive and eventual negative solutions of high order differential inequalities, which enable us, in addition to their independent interests, to obtain in Section 5, further oscillation criteria for high order delay partial differential equations. To illustrate our results, two examples are also given.

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2. Preliminaries

We shall consider the following nonlinear high order delay partial differential equation

$$\frac{\partial^m}{\partial t^m}[u+\lambda(t)u(x,t-\tau)]+p(x,t)u+q(x,t)f(u(x,t-\sigma))$$
$$=a(t)\Delta u+\sum_{j=1}^{\ell} a_j(t)\Delta u(x,\sigma_j(t)),(x,t)\in\Omega\times R_+\equiv G, \tag{2.1}$$

where m is an even positive integer, $\tau>0$ and $\sigma>0$ are constants. Let Ω be a bounded domain in R^n with piecewise boundary $\partial\Omega$, Δ is the Laplacian in R^n; $\lambda\in C^m[R_+,R]$; $a,a_j\in C[R_+,R_+]$, $j=1,2,\ldots,\ell$; $p,q\in C[R_+\times\bar{\Omega},R_+]$, $f\in C[R,R]$, $\sigma_j\in C[R_+,R_+$ is nondecreasing in $t,\sigma_j(t)\le t$ and $\lim_{t\to+\infty}\sigma(t)=+\infty$, $j=1,2,\ldots,\ell$.

We shall consider two kinds of boundary conditions

$$\frac{\partial u}{\partial N}+\gamma(x,t)u=0,\quad (x,t)\in\partial\Omega\times R_+ \tag{B_1}$$

and

$$u=0,\quad (x,t)\in\partial\Omega\times R_+, \tag{B_2}$$

where N is the unit exterior normal vector to $\partial\Omega$, $\gamma(x,t)$ is a nonnegative continuous function on $\partial\Omega\times R_+$.

Definition 2.1: The solution $u(x,t)$ of system (2.1) satisfying certain boundary conditions is called oscillatory in the domain G if for each positive number μ, there exists a point $(x_0,t_0)\in\Omega\times[\mu,+\infty)$ such that $u(x_0,t_0)=0$.

3. Oscillation Criteria

In this section we shall establish oscillation criteria for problem (2.1) with boundary condition (B_1) and (B_2) separately. The basic idea of our approach is to reduce the study of high order delay partial differential equations to that of high order delay differential inequalities.

Theorem 3.1: *Assume that the following condition* (H) *holds.*

(H) $f(u)$ *is convex in* R_+ *and* $f(-u)=-f(u)<0$, $u\in R_+$.

If the high order delay differential inequalities

$$\frac{d^m}{dt^m}[U(t)+\lambda(t)U(t-\tau)]+P(t)U(t)+Q(t)f(U(t-\sigma))\le 0 \tag{3.1}$$

has no eventually positive solutions, then all solutions of the problem (2.1) *under* (B_1) *are oscillatory in* G, *where*

$$P(t)=\min_{x\in\bar{\Omega}} p(x,t),\quad Q(t)=\min_{x\in\bar{\Omega}} q(x,t).$$

Proof: Let $u(x,t)$ be a nonoscillatory solution of the problem (2.1) under (B_1). We may assume that $u(x,t)>0$ for $(x,t)\in\Omega\times[\mu,+\infty)$, where μ is a positive number $t_0\ge\mu$, such that

$$u(x,t-\tau)>0,u(x,t-\sigma)>0$$

and

$$u(x,\sigma_j(t))>0,\quad (x,t)\in\Omega\times[t_0,+\infty),\ j=1,2,\ldots,\ell.$$

Integrating both sides of system (2.1) with respect to x over the domain Ω, we obtain

$$\frac{d^m}{dt^m}\left[\int_\Omega u(x,t)dx + \lambda(t)\int_\Omega u(x,t-\tau)dx\right] + \int_\Omega p(x,t)u(x,t)dx$$

$$+ \int_\Omega q(x,t)f(u(x,t-\sigma))dx$$

$$= a(t)\int_\Omega \Delta u(x,t)dx + \sum_{j=1}^{\ell} a_j(t)\int_\Omega \Delta u(x,\sigma_j(t))dx, \quad t \geq t_0. \tag{3.2}$$

From Green's Theorem, it follows that

$$\int_\Omega \Delta u dx = \int_{\partial\Omega} \frac{\partial u}{\partial N} dS = -\int_{\partial\Omega} g(x,t)u(x,t)dS \leq 0, \ t \geq t_0. \tag{3.3}$$

and

$$\int_\Omega \Delta u(x,\sigma_j(t))dx = \int_{\partial\Omega} \frac{\partial}{\partial N} u(x,\sigma_j(t))dS$$

$$= -\int_{\partial\Omega} \gamma(x,\sigma_j(t))u(x,\sigma_j(t))dS \leq 0, \ \ j = 1,2,\ldots,\ell, \ t \geq t_0, \tag{3.4}$$

where dS is the surface integral element on $\partial\Omega$. Since $f(u)$ is convex in R_+, then using Jensen's inequality, we have

$$\int_\Omega f(u(x,t-\sigma)dx \geq |\Omega| f\left(\frac{1}{|\Omega|}\int_\Omega u(x,t-\sigma)dx\right), \tag{3.5}$$

where $|\Omega| = \int_\Omega dx$. Combining (3.2)-(3.5) yields

$$\frac{d^m}{dt^m}\left[\int_\Omega u(x,t)dx + \lambda(t)\int_\Omega u(x,t-\tau)dx\right]$$

$$+ P(t)\int_\Omega u(x,t)dx + Q(t)f\left(\frac{1}{|\Omega|}\int_\Omega u(x,t-\sigma)dx\right)\cdot |\Omega|$$

$$\leq -a(t)\int_{\partial\Omega} \gamma(x,t)u(x,t)dS - \sum_{j=i}^{\ell} a_j(t)\int_{\partial\Omega} \gamma(x,\sigma_j(t))u(x,\sigma_j(t))dS$$

$$\leq 0, \ t \geq t_0.$$

Thus, we see that the function

$$U(t) = \frac{1}{|\Omega|}\int_\Omega u(x,t)dx \tag{3.6}$$

is a positive solution of the inequality (3.1) for $t \geq t_0$, which contradicts the condition of the theorem.

If $u(x,t) < 0$ for $(x,t) \in \Omega \times [\mu, +\infty)$, then set

$$\widetilde{u}(x,t) = -u(x,t), \quad (x,t) \in \Omega \times [\mu, +\infty).$$

Note that since $f(-u) = -f(u)$, $u \in (0, +\infty)$, it is easy to check that $\widetilde{u}(x,t)$ is a positive solution of the problem (2.1) under (B_1), which is impossible. This completes the proof of Theorem 3.1.

The following fact will be used in the proof of Theorem 3.2. Consider the Dirichlet problem

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u\,|_{\partial\Omega} = 0, \end{cases}$$

where $\lambda =$ constant. It is well known that the smallest eigenvalue λ_0 and the corresponding eigenfunction $\Phi(x)$ are positive.

Theorem 3.2: *Assume that the condition* (H) *holds. If the high order delay differential inequality*

$$\frac{d^m}{dt^m}[V(t) + \lambda(t)V(t-\tau)] + (\lambda_0 a(t) + P(t))V(t) + Q(t)f(V(t-\sigma)) \leq 0 \tag{3.7}$$

has no eventually positive solutions, then all solutions of the problem (2.1) *under* (B_2) *are oscillatory in* G.

Proof: Let $u(x,t)$ be a solution of the problem (2.1) under (B_2), having no zeros in the domain $\Omega \times [\mu, +\infty)$, for some $\mu > 0$. If $u(x,t) > 0$ for $(x,t) \in \Omega \times [\mu, +\infty)$, then there exists a $t_0 \geq \mu$ such that

$$u(x,t-\tau) > 0,\ u(x,t-\sigma) > 0 \text{ and } u(x,\sigma_j(t)) > 0,\ (x,t) \in \Omega \times [t_0, +\infty),$$
$$j = 1, 2, \ldots, \ell.$$

Multiplying both sides of (2.1) by the eigenfunction $\Phi(x)$ and integrating with respect to x over the domain Ω, we have

$$\begin{aligned} &\frac{d^m}{dt^m}\left[\int_\Omega u(x,t)\Phi(x)dx + \lambda(t)\int_\Omega u(x,t-\tau)\Phi(x)dx\right] \\ &+ \int_\Omega p(x,t)u(x,t)\Phi(x)dx + \int_\Omega q(x,t)f(u(x,t-\sigma))\Phi(x)dx \\ &= a(t)\int_\Omega \Delta u(x,t)\Phi(x)dx + \sum_{j=1}^{\ell} a_j(t)\int_\Omega \Delta u(x,\sigma_j(t))\Phi(x)dx,\ t \geq t_0. \end{aligned} \tag{3.8}$$

Using Green's Theorem, we obtain

$$\begin{aligned} &\int_\Omega \Delta u(x,t) \cdot \Phi(x)dx \\ &\quad = \int_{\partial\Omega}\Big(\Phi(x)\frac{\partial}{\partial N}u(x,t) - u(x,t)\frac{\partial}{\partial N}\Phi(x)\Big)dS + \int_\Omega u(x,t)\Delta\Phi(x)dx \end{aligned}$$

$$= -\lambda_0 \int_\Omega u(x,t)\Phi(x)dx; \tag{3.9}$$

$$\int_\Omega \Delta u(x,\sigma_j(t)) \cdot \Phi(x)dx$$

$$= \int_{\partial\Omega} \Big(\Phi(x)\frac{\partial}{\partial N}u(x,\sigma_j(t)) - u(x,\sigma_j(t))\frac{\partial}{\partial N}\Phi(x)\Big)dS + \int_\Omega u(x,\sigma_j(t))\Delta\Phi(x)dx$$

$$= -\lambda_0 \int_\Omega u(x,\sigma_j(t))\Phi(x)dx, \quad j = 1,2,\ldots,\ell. \tag{3.10}$$

Using Jensen's inequality, we have

$$\int_\Omega f(u(x,t-\sigma))\Phi(x)dx$$

$$\geq \int_\Omega \Phi(x)dx \cdot f\left(\frac{1}{\int_\Omega \Phi(x)dx}\int_\Omega u(x,t-\sigma)\Phi(x)dx\right). \tag{3.11}$$

Combining (3.8)-(3.10) yields

$$\frac{d^m}{dt^m}\left[\int_\Omega u(x,t)\Phi(x)dx + \lambda(t)\int_\Omega u(x,t-\tau)\Phi(x)dx\right]$$

$$+ P(t)\int_\Omega u(x,t)\Phi(x)dx + \int_\Omega \Phi(x)dx \cdot f\left(\frac{1}{\int_\Omega \Phi(x)dx}\int_\Omega u(x,t-\sigma)\Phi(x)dx\right)$$

$$\leq -\lambda_0 a(t)\int_\Omega u(x,t)\Phi(x)dx - \lambda_0\sum_{j=1}^{\ell} a_j(t)\int_\Omega u(x,\sigma_j(t))\Phi(x)dx$$

$$\leq -\lambda_0 a(t)\int_\Omega u(x,t)\Phi(x)dx, \quad t \geq t_0,$$

i.e., the inequality (3.7) has positive solution

$$V(t) = \frac{1}{\int_\Omega \Phi(x)dx}\int_\Omega u(x,t)\Phi(x)dx, \quad t \geq t_0,$$

which contradicts the condition of the theorem.

If $u(x,t) < 0$ for $(x,t) \in \Omega \times [\mu, +\infty)$, then $\widetilde{u} \equiv -u$ is a positive solution of the problem (2.1) under (B_2), which also provides a contradiction. The proof of Theorem 3.2 is complete.

4. High Order Delay Differential Inequalities

From the discussion in Section 3 it follows that the problem of establishing oscillation criteria for the system (2.1) can be reduced to the investigation of the properties of the solution of high order delay differential inequalities for the form

$$\frac{d^m}{dt^m}[y(t)+\lambda(t)y(t-\tau)]+Q(t)f(y(t-\sigma))\leq 0,\ t\geq t_0, \tag{4.1}$$

and

$$\frac{d^m}{dt^m}[y(t)+\lambda(t)y(t-\tau)]+Q(t)f(y(t-\sigma))=0,\quad t\geq t_0. \tag{4.2}$$

Along with (4.1) and (4.2), we consider the high order delay differential equation

$$\frac{d^m}{dt^m}[y(t)+\lambda(t)y(t-\tau)]+Q(t)f(y(t-\sigma))=0,\quad t\geq t_0, \tag{4.3}$$

where m is an even positive integer, $\tau>0$ and $\sigma>0$ are constants; $\lambda\in C^m[[t_0,+\infty),R]$, $Q\in C[[t_0,+\infty),R_+]$ for some $t_0>0$, $f\in C[R,R]$. We shall first consider the case $\lambda(t)\geq 0$.

Assume that $y(t)$ is a nonoscillatory solution of equation (4.3). Let

$$z(t)=y(t)+\lambda(t)y(t-\tau).$$

We shall use the following lemma.

Lemma 4.1: *If $z(t)$ is of definite sign and not identically zero for all sufficiently large t; there exist a $T\geq t_0$ and an integer k, $0\leq k\leq m$, with $m+k$ even for $z(t)z^{(m)}(t)\geq 0$, or $m+k$ odd for $z(t)z^{(m)}(t)\leq 0$, then*

$$z(t)z^{(i)}(t)>0 \text{ on } [\tau,+\infty) \text{ for } 0\leq i\leq k,$$

$$(-1)^{i-k}z(t)z^{(i)}(t)>0 \text{ on } [\tau,+\infty) \text{ for } k\leq i\leq m.$$

Theorem 4.1: *Assume that $f(-y)=-f(y)$ for $y\in R_+$, and that*

$$0\leq\lambda(t)\leq 1,\ Q(t)\geq 0,\ t\geq t_0; \tag{4.4}$$

$$\frac{f(y)}{y}\geq\epsilon=\text{constant}>0,\ y\in(0,+\infty). \tag{4.5}$$

If

$$\int^{+\infty}Q(s)[1-\lambda(s-\sigma)]ds=+\infty, \tag{4.6}$$

then

(*i*) *the inequality* (4.1) *has no eventually positive solutions;*
(*ii*) *the inequality* (4.2) *has no eventually negative solutions; and*
(*iii*) *all solutions of the equation* (4.3) *are oscillatory.*

Proof: Let $y(t)$ be an eventually positive solution of the inequality (4.1). Then, there exists a $t_1\geq t_0$, such that

$$y(t)>0,\ y(t-\tau)>0 \text{ and } y(t-\sigma)>0 \text{ for all } t\geq t_1.$$

Setting

$$z(t)=y(t)+\lambda(t)y(t-\tau),\quad t\geq t_1, \tag{4.7}$$

we have

$$z(t)>0,\ t\geq t_1.$$

From (4.1), (4.4) and (4.5) it follows that

$$z^{(m)}(t) \leq -Q(t)f(y(t-\sigma)) \leq -\epsilon Q(t)y(t-\sigma) \leq 0, \quad t \geq t_1.$$

Thus, it follows from Lemma 4.1, that there exists an odd number k and a $t_2 \geq t_1$ such that

$$z^{(i)}(t) > 0, \ 0 \leq i \leq k, \ t \geq t_2$$

and

$$(-1)^{i-k} z^{(i)}(t) > 0, \quad k \leq i \leq m, \quad t \geq t_2.$$

It is easy to see that

$$z'(t) > 0, \ z^{(m-1)}(t) > 0, \ t \geq t_2. \tag{4.8}$$

Using (4.5) and (4.7), we have

$$\begin{aligned} 0 &\geq z^{(m)}(t) + Q(t)f(y(t-\sigma)) \\ &\geq z^{(m)}(t) + Q(t) \cdot \epsilon y(t-\sigma) \\ &= z^{(m)}(t) + \epsilon Q(t)[z(t-\sigma) - \lambda(t-\sigma)y(t-\tau-\sigma)], \ t \geq t_2. \end{aligned}$$

Note $z(t) \geq y(t)$ for $t \geq t_2$, thus we obtain

$$0 \geq z^{(m)}(t) + \epsilon Q(t)[z(t-\sigma) - \lambda(t-\sigma)z(t-\tau-\sigma)], \ t \geq t_2.$$

Since $z(t)$ is increasing for $t \geq t_2$, we have

$$z^{(m)}(t) + \epsilon Q(t)[1 - \lambda(t-\sigma)]z(t-\sigma) \leq 0, \quad t \geq t_2. \tag{4.9}$$

Integrating both sides of (4.9) from t_2 to $t(t > t_2)$, we get

$$z^{(m-1)}(t) \leq z^{(m-1)}(t_2) - \epsilon z(t_2 - \sigma) \int_{t_2}^{t} Q(s)[1 - \lambda(s-\sigma)]ds.$$

Since $z^{(m-1)}(t) > 0$ for $t \geq t_2$, the above inequality leads to a contradiction in view of (4.6). This proves assertion (i).

Assertion (ii) follows from the fact that if $y(t)$ is an eventually negative solution of (4.2), then $-y(t)$ is an eventually positive solution of (4.1). The proof of the assertion (iii) is obvious.

Theorem 4.2: *Assume that condition* (4.4) *holds;* $f(-y) = -f(y) > 0$, $y \in R_+$, *and that* $f(y)$ *is a monotone increasing function in* R_+. *If for any* $c > 0$,

$$\int^{+\infty} Q(s)f([1-\lambda(s-\sigma)]c)ds = +\infty, \tag{4.10}$$

then conclusions (i)-(iii) *of Theorem* 4.1 *remain true.*

Proof: Let $y(t)$ be an eventually positive solution of inequality (4.1). Then, there exists a $t_1 \geq t_0$ such that

$$y(t) > 0, \ y(t-\tau) > 0 \text{ and } y(t-\sigma) > 0 \text{ for all } t \geq t_1.$$

The following inequalities can be proved by the analogous arguments as in the proof of Theorem 4.1:

$$z^{(m)}(t) \leq 0, \; t \geq t_1;$$

$$z'(t) > 0, \; z^{(m-1)} > 0, \; t \geq t_2 \geq t_1,$$

with $z(t)$ defined by (4.7). We have $z(t) > 0$ for $t \geq t_1$ and

$$z(t-\tau) \leq z(t) \leq y(t) + \lambda(t)z(t-\tau), \; t \geq t_2,$$

i.e.,

$$[1-\lambda(t)]z(t-\tau) \leq y(t), \quad t \geq t_2.$$

Choose a $t^* > t_2$ such that

$$z(t-\sigma) > 0, \; t \geq t^*.$$

Since $f(y)$ is increasing, we obtain

$$0 \geq z^{(m)}(t) + Q(t)f(y(t-\sigma 0)$$
$$\geq z^{(m)}(t) + Q(t)f[(1-\lambda(t-\sigma)]z(t-\tau-\sigma)), \; t \geq t^*.$$

Note that since $z(t^*-\tau-\sigma) < z(t-t-\sigma)$ for $t > t^*$, we have

$$z^{(m)}(t) + Q(t)f([1-\lambda(t-\sigma)]c) \leq 0, \quad t \geq t^*,$$

where $c = z(t^*-\tau-\sigma) > 0$. Integrating the above inequality from t^* to $t(t > t^*)$, we get

$$z^{(m-1)}(t) - z^{(m-1)}(t^*) + \int_{t^*}^{t} Q(s)f([1-\lambda(s-\sigma)]c)ds \leq 0.$$

This leads to a contradiction in view of (4.10), since $z^{(m-1)}(t) > 0$ for $t \geq t_2$. This proves the assertion (i).

We can prove assertion (ii) and (iii) by the same arguments as in the proof of Theorem 4.1. This completes the proof.

Theorem 4.3: *Assume that* $f(-y) = -f(y)$ *for* $y \in R_+$ *and that* (4.4) *and* (4.5) *hold. If there exists a monotonically increasing function* $\xi \in C^1[[t_0, +\infty), (0, +\infty)]$ *such that*

$$\int^{+\infty} [\epsilon\xi(s)Q(s)(1-\lambda(s-\sigma)) - c\xi'(s)]ds = +\infty \tag{4.11}$$

for any number $c > 0$*, then conclusions* (i)*-*(iii) *of Theorem* 4.1 *remain true.*

Proof: Let $y(t)$ be an eventually positive solution of the inequality (4.1). Then, there exists a $t_1 \geq t_0$ such that

$$y(t) > 0, \; y(t-\tau) > 0 \text{ and } y(t-\sigma) > 0 \text{ for all } t \geq t_1.$$

The following inequalities can be proved by the analogous arguments as in the proof of Theorem 4.1:

$$z(t) > 0, \ \ z^{(m)}(t) \leq 0, \quad t \geq t_1;$$

$$z'(t) > 0, \ \ z^{(m-1)}(t) > 0, \quad t \geq t_2 \leq t_1;$$

$$z^{(m)}(t) + \epsilon Q(t)[1 - \lambda(t-\sigma)]z(t-\sigma) \leq 0, \quad t \geq t_2.$$

Thus, there exists $T \geq t_2$ such that $z(T-\sigma) > 0$ and

$$z^{(m-1)}(t) \leq z^{(m-1)}(T), \ \ t \geq T; \tag{4.12}$$

$$z^{(m)}(t) + \epsilon z(T-\sigma)Q(t)[1 - \lambda(t-\sigma)] \leq 0, t \geq t. \tag{4.13}$$

Set

$$\Psi(t) = \frac{\xi(t) \cdot z^{(m-1)}(t)}{z(T-\sigma)};$$

then we obviously have

$$\dot{\Psi}(t) > 0 \text{ for all } t \geq T.$$

Note that $\xi(t)$ is a montonically increasing functions and using (4.12) and (4.13), we obtain

$$\Psi'(t) = \frac{\xi'(t)z^{(m-1)}(t)}{z(T-\sigma)} + \frac{\xi(t)z^{(m)}(t)}{z(T-\sigma)}$$

$$\leq \frac{z^{(m-1)}(T)}{z(T-\sigma)}\xi'(t) + \xi(t)\frac{-\epsilon z(T-\sigma)Q(t)[1-\lambda(t-\sigma)]}{z(T-\sigma)}, \quad t \geq T.$$

Set

$$\frac{z^{(m-1)}(T)}{z(T-\sigma)} = c > 0;$$

we have

$$\Psi'(t) \leq -[\epsilon\xi(t)Q(t)(1-\lambda(t-\sigma)) - c\xi'(t)], \ t \geq T.$$

Integrating both sides to the above inequality from T to $t(t > T)$, we get

$$\Psi(t) \leq \Psi(T) - \int_T^t [\epsilon\xi(s)Q(s)(1-\lambda(s-\sigma)) - c\xi'(s)]ds,$$

which is impossible in view of assumption (4.11). This proves assertion (i).

We can prove assertion (ii) and (iii) by the same arguments as in the proof of Theorem 4.1. The proof of Theorem 4.3 is complete.

Theorem 4.4: *Assume that* $\lambda(t) \equiv \lambda = constant > 0$, $f(-y) = -f(y) > 0$ *for* $y \in R_+$ *and that* $f(y)$ *is an increasing function and satisfies:*

$$f(x+y) \leq f(x) + f(y), \ f(kx) \leq kf(x) \ for \ x > 0, \ y > 0, \ k > 0. \tag{4.14}$$

If $Q(t)$ *is periodic with period* τ *and satisfies*

$$\int^{+\infty} Q(s)ds = +\infty, \tag{4.15}$$

then conclusions (i)-(iii) of Theorem 4.1 remain true.

Proof: Let $y(t)$ be an eventually positive solution of the inequality (4.1). Then, there exists a $t_1 \geq t_0$ such that

$$y(t) > 0,\ y(t-\tau) > 0 \text{ and } y(t-\sigma) > 0 \text{ for all } t \geq t_1$$

and for

$$z(t) = y(t) + \lambda y(t-\tau),$$

we have

$$z(t) > 0,\ z^{(m)}(t) \leq 0, t \geq t_1;$$

$$z'(t) > 0,\ z^{(m-1)}(t) > 0,\ t \geq t_2 \geq t_1.$$

Set

$$\alpha(t) = z(t) + \lambda z(t-\tau) = y(t) + 2\lambda y(t-\tau) + \lambda^2 y(t-2\tau),\ t \geq t_2. \tag{4.16}$$

Then, there exists a $t_3 > t_1$ such that

$$\alpha(t) > 0,\ \alpha(t-\sigma) > 0,\ \alpha'(t) > 0,\ t \geq t_3$$

and

$$\alpha^{(m-1)}(t) > 0,\ \alpha^{(m-1)}(t-\tau) > 0,\ t \geq t_3.$$

From (4.1) and (4.16) it follows that

$$\begin{aligned} \alpha^{(m)}(t) &= y^{(m)}(t) + \lambda y^{(m)}(t-\tau) + \lambda[y^{(m)}(t-\tau) + \lambda y^{(m)}(t-2\tau)] \\ &\leq -Q(t)f(y(t-\sigma)) - \lambda Q(t-\tau)f(y(t-\tau-\sigma)). \end{aligned} \tag{4.17}$$

Choose $T \geq t_3$ such that

$$y(t-2\tau-\sigma) > 0,\ t \geq T.$$

Since $Q(t)$ is periodic with period τ, we get by (4.14), (4.16) and (4.17):

$$\begin{aligned} &\alpha^{(m)}(t) + \lambda\alpha^{(m)}(t-\tau) + Q(t)f(\alpha(t-\sigma)) \\ &\leq -Q(t)f(y(t-\sigma)) - 2\lambda Q(t-\tau)f(y(t-\tau-\sigma)) - \lambda^2 Q(t-2\tau)f(y(t-2\tau-\sigma)) \\ &\quad + Q(t)f(y(t-\sigma) + 2\lambda y(t-\tau-\sigma) + \lambda^2 y(t-2\tau-\sigma)) \\ &\leq -Q(t)f(y(t-\sigma g)) - 2\lambda Q(t)f(y(t-\tau-\sigma)) - \lambda^2 Q(t)f(y(t-2\tau-\sigma)) \\ &\quad + Q(t)f(y(t-\sigma)) - 2\lambda Q(t)f(y(t-\tau-\sigma)) + \lambda^2 Q(t)f(y(t-2\tau-\sigma)) = 0, t \geq T. \end{aligned} \tag{4.18}$$

Since α and f are increasing, we have

$$0 < \alpha(T-\sigma) \leq \alpha(s-\sigma),\ s \geq T$$

and

$$f(\alpha(T-\sigma)) \leq f(\alpha(s-\sigma)), s \geq T.$$

Integrating both sides of (4.18) from T to $t(t > T)$, we get

$$0 \geq \alpha^{(m-1)}(t) - \alpha^{(m-1)}(T) + \lambda\alpha^{(m-1)}(t-\tau) - \lambda\alpha^{(m-1)}(T-\tau) + \int_T^t Q(s)f(\alpha(s-\sigma))ds$$

$$\geq \alpha^{(m-1)}(t) - \alpha^{(m-1)}(T) + \lambda\alpha^{(m-1)}(t-\tau) - \lambda\alpha^{(m-1)}(T-\tau) + f(\alpha(T-\sigma))\int_T^t Q(s)ds.$$

This leads to a contradiction in view of (4.15), since $\alpha^{(m-1)}(t) > 0$ and $\alpha^{(m-1)}(t-\tau) > 0$ for $t \geq t_3$. This proves assertion (i).

We can proves assertion (ii) and (iii) by the same arguments as in the proof of Theorem 4.1. This completes the proof of Theorem 4.4.

We shall consider next the case of $\lambda(t) < 0$. The following lemma is a special case of Theorem 2 in [3].

Lemma 4.2: [3] *Assume that* $\beta \in C[[t_0, +\infty), R_+]$ *such that*

$$\liminf_{t \to +\infty} \int_{t-\delta}^{t} \beta(s)ds > \frac{1}{\epsilon} \tag{4.19}$$

and

$$\liminf_{t \to +\infty} \int_{t-\frac{\delta}{2}}^{t} \beta(s)ds > 0. \tag{4.20}$$

Then, the inequality

$$x^{(m)}(t) - \beta^m(t)x(t-m\delta) \leq 0$$

has no eventually negative bounded solutions.

We introduce the following notations:

$$\beta^m(t) = -\frac{\epsilon Q(t)}{\lambda(t-\sigma+\tau)} > 0$$

and

$$\delta = \frac{\sigma - \tau}{m} > 0.$$

Theorem 4.5: *Assume that the condition* (4.5) *holds,* $\sigma > \tau$, $f(-y) = -f(y)$ *for* $y \in \mathbb{R}_+$, *and that there exist constants* λ_1, λ_2 *and* M *such that*

$$-1 \leq \lambda_1 \leq \lambda(t) \leq \lambda_2 < 0, \;\; t \geq t_0 \tag{4.21}$$

and

$$Q(t) \geq M > 0, \; t \geq t_0.$$

If

$$\liminf_{t\to+\infty} \int_{t-\delta}^{t} \beta(s)ds > \frac{1}{\epsilon}, \tag{4.22}$$

then conclusions (*i*) *and* (*iii*) *of Theorem* 4.1 *remain true.*

Proof: Let $y(t)$ be an eventually positive solution of the inequality (4.1). Then, there exists a $t_1 \geq t_0$ such that

$$y(t) > 0,\ y(t-\tau) > 0 \text{ and } y(t-\sigma) > 0 \text{ for all } t \geq t_1.$$

Set

$$z(t) = y(t) + \lambda(t)y(t-\tau).$$

We have

$$z^{(m)}(t) \leq -Q(t)f(y(t-\sigma)) \leq -\epsilon q(t)y(t-\sigma) \leq 0,\ t \geq t_1.$$

We claim that

$$z(t) < 0, \quad t \geq t_1. \tag{4.23}$$

If true, from (4.1) it follows that

$$z^{(m)}(t) \leq -\epsilon Q(t)y(t-\sigma) \leq -\epsilon My(t-\sigma), t \leq t_1. \tag{4.24}$$

Thus, we see that $z^{(m-1)}(t)$ is strictly decreasing on $(t_1, +\infty)$ and $z^{(i)}(t)$ are strictly monotonically functions on $[t, +\infty)$, $i = 0, 1, \ldots, m-2$. Then, we have

$$\lim_{t\to+\infty} z^{(m-1)}(t) = -\infty \tag{4.25}$$

or

$$\lim_{t\to+\infty} z^{(m-1)}(t) = \eta < +\infty. \tag{4.26}$$

If (4.25) holds, then we have

$$\lim_{t\to\infty} z^{(i)}(t) = -\infty, \quad i = 0, 1, \ldots, m-1.$$

Hence (4.23) is true.

If (4.26) holds, then integrating both sides of (4.24) from t_1 to t and letting $t \to +\infty$, we get

$$\int_{t_1}^{+\infty} \epsilon My(s-\sigma)ds \leq z^{(m-1)}(t_1) - \eta, \tag{4.27}$$

which implies that $y \in L^1[t_1, +\infty)$. In view of (4.2), we obtain

$$z \in L^1[t_1, +\infty).$$

Note that $z(t)$ is montonically function, we see that

$$\lim_{t\to+\infty} z(t) = 0. \tag{4.28}$$

Thus $\eta = 0$. From (4.28), it follows that

$$z^{(i)}(t)z^{(i+1)}(t) < 0, \quad i = 0, 1, \ldots, m-1, \quad t \geq t_1. \tag{4.29}$$

Equations (4.28) and (4.29) imply that (4.23) is true.

Now we have

$$y(t) < -\lambda(t)y(t-\tau) \leq -\lambda_1 y(t-\tau) \leq y(t-\tau),$$

which implies that $y(t)$ is a bounded function. Thus $z(t)$ is bounded. Since

$$\begin{aligned} z(t-\sigma+\tau) &= \lambda(t-\sigma+\tau)y(t-\sigma) + y(t-\sigma+\tau) \\ &\geq \lambda(t-\sigma+\tau)y(t-\sigma) \text{ for } t \geq t_1, \end{aligned}$$

we have

$$\frac{Q(t)}{\lambda(t-\sigma+\tau)} z(t-\sigma+\tau) \leq Q(t)y(t-\sigma), \quad t \geq t_1. \tag{4.30}$$

From (4.24) and (4.30), it follows that

$$z^{(m)}(t) = \left(\frac{-\epsilon Q(t)}{\lambda(t-\sigma+\tau)} \right) z(t-(\sigma-\tau)) \leq 0, \quad t \geq t_1,$$

i.e.,

$$z^{(m)}(t) - \beta^m(t)z(t-m\delta) \leq 0, \quad t \geq t_1. \tag{4.31}$$

In view of (4.22), by Lemma 4.2 we see that the inequality (4.31) has no eventually negative bounded solutions, which contradicts the fact that $z(t) < 0$ and $z(t)$ is bounded. This proves assertion (i). We can prove assertion (ii) and (iii) by the same arguments as in the proof of Theorem 4.1. The proof is therefore complete.

5. Further Oscillation Criteria

In this section we shall establish some further oscillation criteria for the higher order delay hyperbolic boundary value problem (2.1) under (B_1) and (2.1) under (B_2) using the results obtained in the last two sections.

Theorem 5.1: *Assume that conditions* (H) *and* (4.5) *hold, and that* $0 \leq \lambda(t) \leq 1$. *If*

$$\int^{+\infty} \min_{x \in \bar{\Omega}} q(x,s)[1-\lambda(s-\sigma)]ds = +\infty, \tag{5.1}$$

then

(i) *all solutions of the problem* (2.1) *under* (B_1) *are oscillatory in* G *and*

(ii) *all solutions of the problem* (2.1) *under* (B_2) *are oscillatory in* G.

Proof: Let $u(x,t)$ be a nonoscillatory solution of the problem (2.1) under (B_1). We may assume that $u(x,t) > 0$ for $(x,t) \in \Omega \times [\mu, +\infty)$, where μ is a positive number. By the analogous arguments as in the proof of Theorem 3.1, we can see that the function $U(t)$ defined by (3.6) is a positive solution of the inequality (3.1) for $t \geq t_0 \geq \mu$, which implies that the function $U(t)$ defined by (3.6) also is a positive solution of the inequality

$$\frac{d^m}{dt^m}[U(t)+\lambda(t)U(t-\tau)]+\min_{z\in\overline{\Omega}} q(x,t)f(U(t-\sigma))\le 0. \tag{5.2}$$

However, by Theorem 4.1, we see that the inequality (5.2) has no eventually positive solutions. Thus, we obtain a contradiction.

If $u(x,t)<0$ for $(x,t)\in\Omega\times[\mu,+\infty)$, then$^{\sim} = -u$ is an eventually positive solution of the problem (2.1) under (B_1) which is impossible. This proves assertion (i).

The assertion (ii) can be proved by the analogous arguments as in the proof of assertion (i). The proof of Theorem 5.1 is complete.

Using Theorem 4.2-4.5, respectively, it is easy to obtain the corresponding results for problem (2.1) under (B_1) or (2.1) under (B_2) also. We merely state them below.

Theorem 5.2: *Assume that the condition (H) holds, and that $0\le\lambda(t)\le 1$, $f(y)$ is a monotone increasing function in R_+. If for any $c>0$,*

$$\int^{+\infty}\min_{x\in\overline{\Omega}} q(x,s)f([1-\lambda(s-\sigma)]c)ds = +\infty, \tag{5.3}$$

then

(i) *all solutions of the problem* (2.1) *under* (B_1) *are oscillatory in* G *and*

(ii) *all solutions of the problem* (2.1) *under* (B_2) *are oscillatory in* G.

Theorem 5.3: *Assume that conditions (H) and (4.5) hold, and that $0\le\lambda(t)\le 1$. If there exists a monotonically increasing function $\xi\in C^1[R_+,(-,+\infty)]$ such that*

$$\int^{+\infty}[\epsilon\xi(s)\min_{x\in\overline{\Omega}} a(x,s)(1-\lambda(s-\sigma))-c\xi'(s)]ds = +\infty, \tag{5.4}$$

for any number $c>0$, then

(i) *all solutions of the problem* (2.1) *under* (B_1) *are oscillatory in* G *and*

(ii) *all solutions of the problem* (2.1) *under* (B_2) *are oscillatory in* G.

Theorem 5.4: *Assume that condition (H) holds, $\lambda(t)\equiv\lambda=constant>0$, and that $f(y)$ is an increasing function and satisfies* (4.14). *If $q(x,t)$ is periodic in t with period τ and satisfies*

$$\int^{+\infty}\min_{x\in\overline{\Omega}} q(x,s)ds = +\infty, \tag{5.5}$$

then

(i) *all solutions of the problem* (2.1) *under* (B_1) *are oscillatory in* G *and*

(ii) *all solutions of the problem* (2.1) *under* (B_2) *are oscillatory in* G.

Theorem 5.5: *Assume that conditions (H) and (4.5) hold, $\sigma>\tau$, and that there exist constants λ_1,λ_2, and M such that*

$$-1\le\lambda_1\le\lambda(t)\le\lambda_2<0,\ t\in R_+$$

and

$$\min_{x\in\Omega} q(x,t)\ge M>0,\ \ t\in R_+.$$

If

$$\liminf_{t\to+\infty}\int_{t-\sigma}^{t}\widetilde{\beta}(s)ds>\frac{1}{e}, \tag{5.6}$$

then

(*i*) *all solutions of the problem* (2.1) *under* (B_1) *are oscillatory in* G *and*

(*ii*) *all solutions of the problem* (2.1) *under* (B_2) *are oscillatory in* G,

where

$$\widetilde{\beta}^m(t) = -\frac{\epsilon \min\limits_{x \in \overline{\Omega}} q(x,t)}{\lambda(t-\sigma+\tau)}$$

and

$$\delta = \frac{\sigma-\tau}{m}.$$

Finally, we discuss two examples.

Example 5.1: Consider the equation

$$\frac{\partial t^6}{\partial t^6}[u+(1-e^{-t})u(x,t-\pi)]+3u+2u(x,t-\tfrac{\pi}{2})\exp[3t+x+u^2(x,t-\tfrac{\pi}{2})]$$
$$=\tfrac{4}{5}\Delta u+(2+\cos t)\Delta u(x,t-\tfrac{\pi}{3}), \quad (x,t)\in(0,\pi)\times(0,+\infty) \tag{5.7}$$

and a boundary condition of type (B_1)

$$-u_x(0,t)+u(0,t)=0,\ u_x(\pi,t)+u(\pi,t)=0, \quad t>0. \tag{5.8}$$

Here, $m=6; n=1;, \ell=1; \Omega=(0,\pi); \tau=\pi; \sigma=\frac{\pi}{2}; \gamma(x,t)\equiv 1$ for $x=0$, $t>0$; $\lambda(t)=1-e^{-t}$;

$$q(x,t)=2e^{x+3t}, \min_{x\in[0,\pi]} q(t)=2e^{3t}; \quad f(u)=ue^{u^2}.$$

It is easy to see that the function $f(u)$ satisfies condition (H) and

$$\int^{+\infty} \min_{x\in[0,\pi]} q(x,s)[1-\lambda(s-\sigma)]ds = \int^{+\infty} 2e^{3s}\cdot e^{-s+\frac{\pi}{2}}ds = +\infty.$$

Then, all conditions of Theorem 5.1 are fulfilled. Hence, all solutions of problems (5.7) and (5.8) are oscillatory in$(0,\pi)\times(0,+\infty)$.

Example 5.2: Consider the equation

$$\begin{aligned}&\frac{\partial^4}{\partial t^4}[u-u(x,t-2\pi)]+2u+\pi^4(2-\sin x)u(x,t-4\pi)\\&=e^t\Delta u+3\Delta u(x,t-\tfrac{\pi}{2}), \quad (x,t)i(0,\pi)\times(0,+\infty)\end{aligned} \tag{5.9}$$

and a boundary condition of the type (B_2)

$$u(0,t)=u(\pi,t)=0,\ t>0. \tag{5.10}$$

Here, $m=4; n=1; \ell=1; \Omega=(0,\pi); \lambda(t)=-1; \tau=2\pi; \sigma=4\pi; q(x,t)=\pi^4(2-\sin x)$; $f(u)=u$.

In this case,

$$\delta=\frac{\sigma-\tau}{m}=\frac{\pi}{2}$$

and

$$\widetilde{\beta}^4(t)=\frac{-\epsilon\min\limits_{x\in[0,\pi]} q(x,t)}{\lambda(t-\sigma+\tau)}=\pi^4,$$

where $\epsilon = 1$. It is easy to see that

$$\liminf_{t \to +\infty} \int_{t-\sigma}^{t} \widetilde{\beta}(s)ds = \liminf_{t \to +\infty} \int_{t-\frac{\pi}{2}}^{t} \pi ds = \frac{\pi^2}{2} > \frac{1}{e}.$$

The hypotheses of Theorem 5.5 are satisfied and hence all solutions of the problem (5.9) and (5.10) are oscillatory in $(0,\pi) \times (0, +\infty)$.

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