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GENERIC STABILITY OF THE SPECTRA FOR BOUNDED LINEAR OPERATORS ON BANACH SPACES

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In this paper, we study the stability of the spectra of bounded linear operators $B(X)$ in a Banach space X, and obtain that their spectra are stable on a dense residual subset of $B(X)$.

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1. Introduction

Spectral theory is an important part of functional analysis, which attracted many authors, e.g. [1, 3]. It is known (see Kreyszig [3]) that the spectra of a bounded linear operator is a nonempty compact subset of complex plane C. When the operator is perturbed, how does its spectrum change? After Rayleigh and Schrödinger created perturbation theory, stability of spectra has been intensively developed. In a finite-dimensional space, the eigenvalues of a linear operator T depend on T continuous [1], but it does not apply to a general Banach space. Kato [1, pp. 210] gives an example, in which he shows that the set of spectrum of a bounded linear operator in Banach space is not stable.

In this paper, by using Lemma 2.3 of K.K. Tan, J. Yu, and X.Z. Yuan [4], we obtain that the spectra of a bounded linear operator is stable on a dense residual subset of $B(X)$.

2. Preliminaries

If X and Y are two topological spaces, we shall denote by $K(X)$ and $P_0(Y)$ the space of all nonempty compact subsets of X and the space all nonempty subsets of Y, respectively, both endowed with the Vietoris topology (see Klein and Thompson [2]). Then a mapping $T: X \rightarrow P_0()$ is said to be (i) *upper* (resp. *lower*) *semicontinuous at*

$x \in X$ if, for each open set G in Y with $G \supset T(x)$ (resp. $G \cap T(x) \neq \emptyset$), there exists an open neighborhood $O(x)$ of x in X such that $G \supset T(x')$ (resp. $G \cap T(x') \neq \emptyset$) for each $x' \in O(x)$; *(ii)* T is *upper* (resp. *lower*) *semicontinuous on* X if, T is upper (resp. lower) semicontinuous at each $x \in X$; *(iii)* T is an *usco mapping* if, T is upper semicontinuous with nonempty compact values.

The following result is due to K.K. Tan, J. Yu, and X.Z. Yuan [4, Lemma 2.3].

Theorem 2.1: *If X is (completely) metrizable, Y is a Baire space and T: $Y \rightarrow K(X)$ is a usco mapping, then the set of points where T is lower semicontinuous is a (dense) residual set in Y.*

Let C denotes the whole complex plane, X denotes a complex Banach space, T: $X \rightarrow X$ linear operator. Then, $\sigma(T) = \{\lambda \in C: T - \lambda I \text{ is not invertible}\}$ is called the *spectra of* T, the complementary set $\rho(T) = C \backslash \sigma(T)$ is called the *resolvent set of* T. Here I is the identity mapping.

The following theorems are from Kreyszig [3].

Theorem 2.2: *Let X and Y be complex (or real) topological vector spaces, T: $D(T) \rightarrow Y$ be a linear operator, and $D(T) \subset X$, $R(T) \subset Y$. Then*

(1) *$T^{-1}: R(T) \rightarrow D(T)$ exists if and only if $Tx = 0$ implies that $x = 0$;*

(2) *if T^{-1} exists, then T^{-1} is a linear operator.*

Theorem 2.3: *The spectra $\sigma(T)$ of a bounded linear operator T in Banach space is a nonempty compact subset of C.*

Let $B(X,Y)$ be the set of all bounded linear operators from X to Y and let $CO(X,Y)$ be the set of closed linear operators from X to Y.

The following theorem is due to T. Kato [1, Theorem 2.23].

Theorem 2.4: *Let $T, T_n \in CO(X,Y)$, $n = 1, 2, \ldots$,*

(1) *if $T \in B(X,Y)$, then $T_n \rightarrow T$ in the generalized sense if and only if $T_n \in B(X,Y)$ for sufficiently large n and $\| T_n - T \| \rightarrow 0$;*

(2) *if T^{-1} exists and belongs to $B(X,Y)$, then $T_n \rightarrow T$ in the generalized sense if and only if T_n^{-1} exists and belongs to $B(X,Y)$ for sufficiently large n and $\| T_n^{-1} - T^{-1} \| \rightarrow 0$.*

3. Main Results

Let $X \neq \{0\}$ be a complex Banach space, and let $B(X)$ denote the set of all bounded linear operators in X. Then $B(X)$ is a Banach space.

Theorem 3.1: *$\sigma: B(X) \rightarrow K(C)$ is an usco mapping.*

Proof: By Theorem 2.3, $\sigma(T)$ is a nonempty compact subset of C for each $T \in B(X)$. Suppose that σ is not upper semicontinuous at some $T_0 \in B(X)$, i.e., that for any $\varepsilon_0 > 0$ there is a $\delta > 0$, such that for all $S \in B(X)$ with $\| X - T_0 \| < \delta$,

$$H_+(\sigma(T_0), \sigma(S)) = \sup_{\lambda \in (S)} \{\text{dist}(\lambda, \sigma(T_0))\} \geq \varepsilon_0$$

where H is the Hausdorff metric and $H(\cdot,\cdot) = \max\{H_+(\cdot,\cdot),\ H_-(\cdot,\cdot)\}$. Then there exists a $\lambda_0 \in \sigma(S)$ such that $\text{dist}(\lambda_0, \sigma(T_0)) \geq \varepsilon_0 > 0$. Thus

$$\lambda_0 \notin \sigma(T_0), \quad \lambda_0 \in \rho(T_0), \quad (T_0 - \lambda_0 I)^{-1} \in B(X),$$

and, by Theorem 2.4,

$$(S-\lambda_0 I)^{-1} \in B(X),\ \lambda_0 \notin \sigma(S),$$

which contradicts that $\lambda_0 \in \sigma(S)$. Therefore, σ is an usco mapping.

Definition 3.1: For each $T \in B(X)$,

(*i*) $\lambda \in (T)$ is an *essential spectrum value relative to* $B(X)$ if, for each open neighborhood $N(\lambda)$ of λ in C, there exists an open neighborhood $O(T)$ of T in $B(X)$ such that $\sigma(T') \cap N(\lambda) \neq \emptyset$ for each $T' \in O(T)$;

(*ii*) T is *essential relative to* $B(X)$ if, every $\lambda \in \sigma(T)$ is an essential spectrum value relative to $B(X)$.

Theorem 3.2: (1) *σ is lower semicontinuous at $T \in B(X)$ if and only if T is essential relative to $B(X)$;*

(2) *σ is continuous at $T \in B(X)$ if and only if T is essential relative to $B(X)$.*

Proof: (1) σ is lower semicontinuous at $T \in B(X)$ if and only if each $\lambda \in \sigma(T)$ is an essential spectrum value relative to $B(X)$ and T is essential relative to $B(X)$.

(2) The proof follows from (1) and Theorem 3.1.

Theorem 3.3: *If $T \in B(X)$ such that $\sigma(T)$ is a singleton set, then T is essential relative to $B(X)$.*

Proof: Suppose $\sigma(T) = \{\lambda\}$, and let G be any open set in C such that $\sigma(T) \cap G \neq \emptyset$. Then $\lambda \in G$, so that $\sigma(T) \subset G$. Since σ is upper semicontinuous at T, by Theorem 3.1, there exists an open neighborhood $O(T)$ of T in $B(X)$ such that $\sigma(T') \subset G$ for each $T' \in O(T)$. In particular, $G \cap \sigma(T') \neq \emptyset$ for each $T' \in O(T)$. Thus σ is lower semicontinuous at T, and by Theorem 3.2 (1), T is essential relative to $B(X)$.

Theorem 3.4: *Let C be complex plane, and $X \neq \{0\}$ be a complex Banach space. Then there exists a dense residual subset Q of $B(X)$ such that T is essential relative to $B(X)$ for each $T \in Q$.*

Proof: By Theorem 3.1 and Theorem 2.1, σ is lower semicontinuous on some dense residual subset Q of $B(X)$. Consequently, by Theorem 3.2 (1), T is essential relative to $B(X)$ for each $T \in Q$.

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