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TRANSFORMATIONS OF INDEX SET FOR SKOROKHOD INTEGRAL WITH RESPECT TO GAUSSIAN PROCESSES

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We consider a Gaussian process $\{X_t, t \in T\}$ with an arbitrary index set T and study consequences of transformations of the index set on the Skorokhod integral and Skorokhod derivative with respect to X. The results applied to Skorokhod SDEs of diffusion type provide uniqueness of the solution for the time-reversed equation and, to Ogawa line integral, give an analogue of the fundamental theorem of calculus.

Key words: Skorokhod Integral, Anticipative Stochastic Calculus.
AMS subject classifications: 60H05, 60H10.

1. Introduction

The purpose of this article is to prove that, in a general case of Gaussian processes and under mild assumptions, transformations of a parameter set do not change the Skorokhod integral and Skorokhod derivative, and to indicate some applications of this fact.

Let T be any set, C a covariance on T and $H(C) = H$ the reproducing kernel Hilbert space (RKHS) on C (note that H may not be separable). With covariance C, we associate a Gaussian process $\{X_t,\ t \in T\}$ defined on $(\Omega, \mathfrak{F}, P)$, where $\mathfrak{F} = \sigma\{X_t, t \in T\}$. For the details of the constructions above, see [3]. Let $H^{\otimes p}$ be the p-fold tensor product of H. The p-Multiple Wiener Integral (MWI) $I_p: H^{\otimes p} \to L_2(\Omega, \mathfrak{F}, P)$ was defined in [6] (see also [5]) as a linear mapping satisfying the following properties. Here $\widetilde{f}$ is the symmetrization of f.

a) $EI_p(f) = 0,$

b) $EI_p(f)I_q(g) = \begin{cases} 0 & \text{if } p \neq q \\ p!(\widetilde{f}, \widetilde{g})_{H^{\otimes p}} & \text{if } p = q, \end{cases}$ for $f \in H^{\otimes p}$, $g \in H^{\otimes q}$.

c) $I_{p+1}(gh) = I_p(g)I_1(h) - \sum_{k=1}^{p} I_{p-1}(g \underset{k}{\otimes} h)$, for $g \in H^{\otimes p}$, $h \in H$.

Above, $(g \underset{k}{\otimes} h)\ (t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_p) = (g(t_1, \ldots, t_{k-1}, \cdot, t_{k+1}, \ldots, t_p), h(\cdot))_H$.

We note that $I_p(f) = I_p(\widetilde{f})$ and hence $I_p(H^{\otimes p}) = I_p(H^{\odot p})$ where $H^{\odot p}$ is the p-fold symmetric tensor product.

Let $u_. : \Omega \to H$ be a Bochner measurable function with $\| u \|_H \in L_2(\Omega, \mathfrak{F}, P)$. Using Wiener chaos decomposition, $L_2(\Omega, \mathfrak{F}, P) = \sum_{p=0}^{\infty} \oplus I_p(H^{\odot p})$, we have a unique representation $u_t(\omega) = \sum_{p=0}^{\infty} I_p(f_p(\cdot, t))$, with $f_p(\cdot, *) \in H^{\otimes p+1}$ and $f_p(\cdot, t) \in H^{\odot p}$. The Skorokhod derivative and integral of u. with respect to Gaussian processes are defined in [6] (for Skorokhod's original definition, see [12]). The Skorokhod derivative $\{D_s u_t, s \in T\}$ of u_t, for a fixed t is an $L_2(\Omega, H)$-valued random variable,

$$D_s u_t = \sum_{p=1}^{\infty} p I_{p-1}(f_p(t_1, \ldots, t_{p-1}, s, t)).$$

The Skorokhod derivative exists iff $E \| D.u_t \|_H^2 = \sum_{p=1}^{\infty} p p! \| f_p(\cdot, t) \|_{H^{\otimes p}}^2 < \infty$ and $\{D_s u_t \in L_2(\Omega, H^{\otimes 2}),\ s, t \in T\}$, with $H^{\otimes 2}$ identified with the space of Hilbert-Schmidt operators on H, iff $E \| D.u_* \|_{H^{\otimes 2}}^2 = \sum_{p=1}^{\infty} p p! \| f_p \|_{H^{\otimes (p+1)}}^2 < \infty$.

The Skorokhod integral of u. is an $L_2(\Omega)$-valued random variable,

$$I^s(u.) = \sum_{p=0}^{\infty} I_{p+1}(\widetilde{f}_p(\cdot, *)).$$

We note that u. is integrable iff $EI^s(u.)^2 = \sum_{p=0}^{\infty} (p+1)! \| \widetilde{f}_p(\cdot, *) \|_{H^{\odot p+1}}^2 < \infty$.

Example 1: Skorokhod derivative and integral for Brownian motion. In the case of standard Brownian motion, the MWI I_p and consequently, the Skorokhod derivate and integral defined above, coincide with the MWI I_p^i, the Malliavin derivative D^i and the Skorokhod integral I^i defined in [7]. With $V: L_2([0,1)] \to H$ defined by: $Vf = \int_0^{\cdot} f(s)ds$,

$$I_p^i(f_p) = I_p(V^{\otimes p} f),\ I^s(V(u)) = I^i(u) \text{ and } D_s(V(u)(t)) = D_s^i u_t$$

for $f_p \in L_2([0,1]^p)$ and $u \in L_2(\Omega, L_2([0,1]))$. The first two equalities hold in $L_2(\Omega)$ and the third holds in $L_2(\Omega, H)$ for a fixed t.

If u is adapted to the natural (resp. future) filtration of Brownian motion, $\mathfrak{F}_t = \sigma\{B_s, s \leq t\}$ $(\mathfrak{F}^t = \sigma\{B_1 - B_s, t \leq s \leq 1\})$, then the Skorokhod and Itô (backward Itô) integrals coincide (see [7]).

2. Skorokhod Integral Under Transformation of a Parameter Set

For a Gaussian process $\{X_t, t \in T\}$, let $H(X) = cl(span\{X_t, t \in T\})$, the closure being taken in $L_2(\Omega, \mathfrak{F}, P)$. With a transformation $R: S \to T$ we associate a Gaussian process $X^R = \{X_{R(s)},\ s \in S\}$ and we call R *nondegenerate* if it is onto and if $H(X^R) = H(X)$. Our main result on transformations of the Skorokhod derivative and integral is the following:

Theorem 1: *Let $\{X_t\}_{t \in T}$ be a Gaussian process and $R: S \to T$ be a nondegenerate transformation. Denote by I_X^s and $I_{X^R}^s$ the Skorokhod integrals with respect to X and X^R, respectively. Then:*

1) $f_p \mapsto f_p^R = f(R(s_1), \ldots, R(s_p))$ *is an isometry from* $H(C_X)^{\otimes p}$ *onto* $H(C_{X^R})^{\otimes p}$.

2) *If* $u \in \mathfrak{D}(I_X^s)$ *then* $u^R = \{u_{R(s)}, s \in S\} \in \mathfrak{D}(I_{X^R}^s)$ *and* $I_X^s(u) = I_{X^R}^s(u^R)$.

Moreover, denote by D^X *and* D^{X^R} *the Skorokhod derivatives with respect to* X *and* X^R*, respectively.*

3) *If for* $t \in T$ $u_t \in \mathfrak{D}(D^X)$*, then* $u_s^R \in \mathfrak{D}(D^{X^R})$ *for* $s \in R^{-1}\{t\}$ *and* $D_{s'}^{X^R} u_s^R = D^X_{R(s')} u_{R(s)}$ *P-a.e., for* $s, s' \in S$*. The equality is in* $H(C_{X^R})$*, with* $s' \in S$ *as the variable.*

Also, $D_{t'} u_t \in H(C_X)^{\otimes 2}$*,* $(t, t' \in T)$ *implies* $D_{s'}^{X^R} u_s^R \in H(C_{X^R})^{\otimes 2}$*,* $(s, s' \in S)$*, and equality of norms* $\| D_{t'} u_t \|_{L_2(\Omega, H(C_X)^{\otimes 2})} = \| D_{s'}^{X^R} u_s^R \|_{L_2(\Omega, H(C_{X^R})^{\otimes 2})}$.

4) *If* $v \in L_2(\Omega, H(C_{X^R}))$ *then* $v = u^R$ *for some* $u \in L_2(\Omega, H(C_X))$ *and* $\| v \|_{L_2} = \| u \|_{L_2}$.

Moreover, $v \in \mathfrak{D}(I^s_{X^R})$ *implies* $u \in \mathfrak{D}(I^s_X)$ *and* $v_s \in \mathfrak{D}(D^{X^R})$ *implies* $u_{R(s)} \in \mathfrak{D}(D^X)$ *with* $D_{s'}^{X^R} v_s = D^X_{R(s')} u_{R(s)}$ *for* $s, s' \in S$.

If $D_{s'}^{X^R} v_s \in H(C_{X^R})^{\otimes 2}$*,* $(s, s' \in S)$*, then* $D_{t'} u_t \in H(C_X)^{\otimes 2}$*,* $(t, t' \in T)$*, and the H-S norms of those derivatives are equal.*

Proof: 1) Let us denote $f^R(s_1, \ldots, s_n) = f(R(s_1), \ldots, R(s_n))$ for $(s_1, \ldots, s_n) \in S^p$, (thus $f_p^R(s_1, \ldots, s_p, s) = f_p(R(s_1), \ldots, R(s_p), R(s))$, $(s_1, \ldots, s_p, s) \in S^{p+1}$). Let $f(t) \in H(C_X)$, then $f(t) = E(X_t I_1^X(f))$, with $I_1^X(f) \in H(X)$ and, for any $s \in S$,

$$f^R(s) = f(R(s)) = E(X_{R(s)} I_1^X(f)) = E(X_s^R I_1^X(f))$$

(I_p^X or $I_p^{X^R}$ denotes the p^{th} order Wiener integral with respect to either X or X^R). By definition and uniqueness of representation, $f^R \in H(C_{X^R})$ and $I_1^{X^R}(f^R) = I_1^X(f)$. Also, if $g \in H(C_{X^R})$ then, for $s \in S$, $g(s) = E(X_{R(s)} I_1^{X^R}(g))$. But, $I_1^{X^R}(g) \in H(X)$, thus $f(t) = E(X_t I_1^{X^R}(g))$ defines an element of $H(C_X)$, with $g(s) = f(R(s))$, $s \in S$ and $\| g \|_{H(C_{X^R})} = \| I_1^{X^R} g \|_{L_2(\Omega, \mathfrak{F}, P)} = \| f \|_{H(C_X)}$, proving (1).

2) - 3) Let us first show that $I_X^p(f_p) = I^p_{X^R}(f_p^R)$, $p = 0, 1, \ldots$.

The above is clear for $p = 0$ and $p = 1$. Let $f_p \in H(C_X)^{\otimes p}$, $f(t_1, t_2, \ldots, t_p) = \sum_{\alpha_1, \alpha_2, \ldots, \alpha_p} a_{\alpha_1, \alpha_2, \ldots, \alpha_p} e_{\alpha_1}(t_1) e_{\alpha_2}(t_2) \ldots e_{\alpha_p}(t_p)$, with $\sum_{\alpha_1, \alpha_2, \ldots, \alpha_p} a^2_{\alpha_1, \alpha_2, \ldots, \alpha_p} < \infty$ and $\{e_\alpha, \alpha = 1, 2, \ldots\}$ an ONB in $H(C_X)$. For $f_p = e_{\alpha_1}(t_1) e_{\alpha_2}(t_2) \ldots\ e_{\alpha_p}(t_p)$ we have $[(f_p \underset{k}{\otimes} g_1)^X]^R(s_1, \ldots, s_{k-1}, s_{k+1}, \ldots, s_p) = (f_p^R \underset{k}{\otimes} g_1^R)^{X^R}(s_1, \ldots,\ s_{k-1}, s_{k+1}, \ldots, s_p)$, where the superscripts X and X^R indicate that the operation "$\underset{k}{\otimes}$" is taken either with respect to the process X or X^R. Thus, $I_p^X((f_p \underset{k}{\otimes} g_1)^X) = I_p^{X^R}([(f_p \underset{k}{\otimes} g_1)^X]^R) = I_p^{X^R}((f_p^R \underset{k}{\otimes} g_1^R)^{X^R})$, which allows us to use the inductive relation (c) for MWI to complete the proof. For $f_p \in H(C_X)$ arbitrary, we have

$$I_p^X(f_p) = \lim_{n_1,\dots,n_p\to\infty} I_p^X\left(\left(\sum_{\alpha_1=1}^{n_1}\dots\sum_{\alpha_p=1}^{n_p} a_{\alpha_1,\dots,\alpha_p} e_{\alpha_1}\dots e_{\alpha_p}\right)\right)$$

$$= \lim_{n_1,\dots,n_p\to\infty} I_p^{X^R}\left(\left(\sum_{\alpha_1=1}^{n_1}\dots\sum_{\alpha_p=1}^{n_p} a_{\alpha_1,\dots,\alpha_p} e^R_{\alpha_1}\dots e^R_{\alpha_p}\right)\right)$$

$$= I_p^{X^R}\left(\lim_{n_1,\dots,n_p\to\infty}\left(\sum_{\alpha_1=1}^{n_1}\dots\sum_{\alpha_p=1}^{n_p} a_{\alpha_1,\dots,\alpha_p} e^R_{\alpha_1}\dots e^R_{\alpha_p}\right)\right) = I_p^{X^R}(f_p^R).$$

Now if $u \in \mathfrak{D}(I_X^s)$ and $u_t = \sum_{p=0}^{\infty} I_p(f_p(t_1,\dots,t_p,t))$ then, for $s \in S$,

$$u_{R(s)} = \sum_{p=0}^{\infty} I_p^X(f_p(\cdot, R(s))) = \sum_{p=0}^{\infty} I_p^{X^R}(f_p^R(\cdot, s))$$

and 2) and 3) follow.

4) Let $v \in L_2(\Omega, H(C_{X^R}))$; then for $s \in S$, using 1),

$$v_s = \sum_{p=0}^{\infty} I_p^{X^R}(g_p(\cdot, s)) = \sum_{p=0}^{\infty} I_p^{X^R}(f_p^R(\cdot, s)),$$

because for any $g \in H(C_{X^R})^{\otimes(p+1)}$ there exists $f \in H(C_X)^{\otimes(p+1)}$ with $g = f^R$. Hence, for $s \in S$, $v_s = \sum_{p=0}^{\infty} I_p^{X^R}(f_p^R(\cdot, s)) = \sum_{p=0}^{\infty} I_p^X(f_p(\cdot, R(s)))$.

According to 1), $u_t = \sum_{p=0}^{\infty} I_p^X(f_p(\cdot, t)) \in L_2(\Omega, H(C_X))$ and equality of norms claimed in 4) is satisfied. The last part of assertion 4) follows from 1), 2) and 3) since failure to satisfy any stated condition by u implies violation of this condition by v. □

Example 2: Transformations of parameter set and Skorokhod integral.

1) Brownian motion and time reversal. Let $\{u_t, t \in [0,1]\}$ be an $L_2(\Omega, L_2[0,1])$-valued process adapted to the natural filtration $(\mathfrak{F}_t)_{t\in[0,1]}$ of Brownian motion. Note that $\{\widetilde{B}_t = B_1 - B_{1-t}, t \in [0,1]\}$ is also a Brownian motion and $\{\overline{u}_t = u_{1-t}, t \in [0,1]\}$ is adapted to filtration $\widetilde{\mathfrak{F}}^t = \sigma\{\widetilde{B}_1 - \widetilde{B}_s, t \le s \le 1\}$. Denote $\overline{B}_t = B_{1-t}$. We have

$$\int_0^1 u_t dB_t = I_B^s\left(\int_0^{\cdot} u_r dr\right) = I_{\overline{B}}^s\left(\int_0^{1-\cdot} ur_r dr\right). \tag{1}$$

By the same method as in the proof of Theorem 1 we can show that $I_{\widetilde{B}}^s((\int_0^{\cdot} u_r dr)^{\sim}) = I_B^s(\int_0^{\cdot} u_r dr)$ with $(\int_0^{\cdot} u_r dr)^{\sim} = \int_0^1 u_r dr - \int_0^{1-\cdot} u_r dr$. Hence we get

$$\int_0^1 u_t dB_t = I_{\widetilde{B}}^s\left(\left(\int_0^{\cdot} u_r dr\right)^{\sim}\right) = I_{\widetilde{B}}^i(\overline{u}) = \int_0^1 \overline{u}_t * d\widetilde{B}_t$$

where "*" denotes the backward Itô integral. We have just obtained the relation

$I_B^i(u) = I_{\widetilde{B}}^i(\bar{u})$ given in [8]. Note also that $\bar{B}_t$ is not a Brownian motion and equation (1) is reversed pathwise in H. In the case of Brownian motion, we also have

$$I_{\bar{B}}^s\left(\int_0^{1-\cdot} us_s ds\right) = I_{\widetilde{B}}^s\left(\left(\int_0^{\cdot} u_s ds\right)^{\sim}\right).$$

Indeed, $I_{\bar{B}}^s(\int_0^{1-\cdot} u_s ds) = I_B^s(\int_0^{\cdot} u_s ds) = I_B^i(u) = I_{\widetilde{B}}^i(\bar{u}) = I_{\widetilde{B}}^s(\int_0^{\cdot} u_{1-s} ds) = I_{\widetilde{B}}^s(\int_0^1 u_s ds - \int_0^{1-\cdot} u_s ds)$.

2. Ogawa Line Integral. We recall the definition of the Ogawa integral ([4, 9]) with respect to a Gaussian process $\{X_t, t \in [0,1]\}$ with the RKHS H. Let $u.: \Omega \to H$ be an H-valued Bochner measurable function. Then, on a set of P-measure one, $u.(\omega)$ takes values in a separable subspace of H. Let $\{e_n, n \in N\}$ be an ONB of this subspace. The (universal) Ogawa integral of u is defined as follows:

$$\delta(u) = \sum_{n=1}^{\infty} (u, e_n)_H I_1(e_n) \text{ (limit in probability)}$$

if it exists with respect to all ONBs and is independent of the choice of basis.

The relation between Skorokhod and Ogawa integrals is explained in [4].

Let $\gamma: S \to T$ be a bijective parametrization. Let $Y_s = X_{\gamma(s)}$. Then

(*i*) $C_X(\gamma(s_1), \gamma(s_2)) = C_Y(s_1, s_2)$;

(*ii*) $H(C_X)$ and $H(C_Y)$ are isometric under the mapping $f \mapsto f \circ \gamma$;

(*iii*) $I_1^X(f) = I_1^Y(f \circ \gamma)$ for $f \in H(C_X)$.

Thus, $\delta_X(u) = \delta_Y(v)$ for $v_s = u_{\gamma(s)}$, provided either of the integrals exists.

Consider Brownian sheet $\{W_{(x,t)}, (x,t) \in [0,1]^2\}$. Assume that $\Gamma \subset [0,1]^2$ is a curve parametrized by a function $\gamma: [a,b] \to \Gamma$, $0 \le a \le b \le 1$. We define the *Ogawa line integral*, $\Gamma - \delta$, over Γ with respect to $\{W_{(x,t)}, (x,t) \in \Gamma\}$ using Γ as the parameter set. In addition, let $\gamma(s) = (\gamma_1(s), \gamma_2(s))$ with both coordinates nondecreasing and such that the map $\widetilde{\gamma}^{-1}(\gamma_1(r), \gamma_2(r)) = \gamma_1(r)\gamma_2(r)$ is bijective from Γ to $S = [\gamma_1(a)\gamma_2(a), \gamma_1(b)\gamma_2(b)]$. Then $\widetilde{\gamma}: S \to \Gamma$ is a bijective parametrization and the process $B_s = W_{\widetilde{\gamma}(s)}$ is a Brownian motion. Hence,

$$\Gamma - \delta_W(u) = \delta_B(v) = \int_S (V^{-1}v)(s) \circ dB_s,$$

where $v_s = u_{\widetilde{\gamma}(s)}$, V is the isometry from Example 1, and the last integral is in the sense of Fisk and Stratonovich and is assumed to exist. In particular, if $u_{(x,t)} = f(W_{(x,t)})$ and $f \in C^2$, then

$$\Gamma - \delta_W(V^{\otimes 2}(f'(W))) = \int_S f'(B_s) \circ dB_s = f(W(\gamma_1(b), \gamma_2(b))) - f(W(\gamma_1(a), \gamma_2(a))).$$

Thus, in this case, the Ogawa line integral satisfies the fundamental theorem of calculus. We conjecture that a counterpart of Green's formula for the Ogawa integral holds (see [2] for initial exposition and [11] for some recent results).

Example 3: Skorokhod-type stochastic differential equations. The following class of Skorokhod SDEs was considered by Buckdahn in [1], where, under smoothness assumptions, the author proved existence and uniqueness results

$$Z_t = \eta + \int_0^t b(Z(s))ds + I^i(\sigma(Z(s))1_{[0,t]}(s)),\ 0 \leq t \leq 1. \tag{2}$$

The initial condition η needs to be bounded. However, this restriction vanishes if equation (2) is reversed.

Lemma 1: *Let $\{u_s\}_{s \in [0,1]}$ be such that $u_s 1_{[0,t]}(s) \in \mathfrak{D}(I^i_B)$ $\forall t \in [0,1]$. Then for the time reversed process $\bar{u}_s = u_{1-s}$, we have $\bar{u}_s 1_{[0,t]}(s) \in \mathfrak{D}(I^i_{\tilde{B}})$ $\forall t \in [0,1]$ and if we denote $X_t = I^i_B(1_{[0,t]}(s)u_s)$, then*

$$X_{1-t} - X_1 = -I^i_{\tilde{B}}(1_{[0,t]}(s)\bar{u}_s).$$

Using time reversal and Lemma 1, Buckdahn's result can be extended to time reversed SDEs with the initial condition being a terminal value of the solution of the original equation.

Theorem 2: *Assume that coefficients b and σ of a Skorokhod SDE (2) satisfy assumptions for existence and uniqueness of the solution. If $\{Z_t\}_{t \in [0,1]}$ is the solution of Equation (2), then the time reversed process $\bar{Z}_t = Z_{1-t}$ is the unique solution in $L_1([0,1] \times \Omega)$ of the time reversed equation*

$$X_t = \bar{Z}_0 + \int_0^t -\bar{b}(X_s)ds + I^i_{\tilde{B}}(-1_{[0,t](s)}\bar{\sigma}(X(s))),$$

where $\bar{b}(X_t) = b(X_{1-t}), \bar{\sigma}(X_t) = \sigma(X_{1-t})$, and $\tilde{B}_t = B_1 - B_{1-t}$.

The above theorem gives a partial answer to a question in [8], Proposition 5.2.

The technique of time reversal has been used in [10] to solve a problem regarding anticipative stochastic models in finance.

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