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CONVERGENCE OF AN ITERATION LEADING TO A SOLUTION OF A RANDOM OPERATOR EQUATION

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In the present paper, we define a random iteration scheme and consider its convergence to a solution of a random operator equation. There is also a related fixed point result.

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1. Introduction

In recent years, the study of different types of random equations have attracted much attention, some of which may be noted in [1, 5, 6] and [7]. In this paper, we discuss a random operator equation involving two operators in the context of Hilbert spaces. We have also a random fixed point result as a corollary. We also demonstrate our result for the corresponding deterministic case by an example.

Throughout this paper, (Ω, Σ) denotes a measurable space and H stands for a separable Hilbert space.

A function $f: \Omega \to H$ is said to be *measurable* if $f^{-1}(B) \in \Sigma$ for every Borel subset B of H.

A function $F: \Omega \times H \to H$ is said to be *H-continuous*, if $F(t, \cdot): H \to H$ is continuous for all $t \in \Omega$.

A function $F: \Omega \times H \to H$ is said to be a *random operator*, if $F(\cdot, x): \Omega \to H$ is measurable for every $x \in H$.

A measurable function $g: \Omega \to H$ is said to be a *random fixed point* of the random operator $F: \Omega \times H \to H$, if $F(t, g(t)) = g(t)$ for all $t \in \Omega$.

A measurable function $g: \Omega \to H$ is said to be a *solution of the random operator equation* $S(t, x(t)) = T(t, x(t))$, where $S, T: \Omega \times H \to H$ are random operators, if $S(t, g(t)) = T(t, g(t))$ for all $t \in \Omega$.

The following result was established in [4]. We present this result as a lemma.

Lemma 1.1: *Let H be a Hilbert space. Then for any $x, y, z \in H$ and any real λ, the following equality holds:*

$$\| (1-\lambda)x + \lambda y - z \|^2$$
$$= (1-\lambda) \| x - z \|^2 + \lambda \| y - z \|^2 - \lambda(1-\lambda) \| x - y \|^2. \tag{1.1}$$

We define the random iteration scheme as follows:

Definition 1.2: *Random iteration scheme.* Let $S, T: \Omega \times H \rightarrow H$ be two random operators defined on a Hilbert space H. Let $g_0: \Omega \rightarrow H$ be any measurable function. Define the following sequence of functions $\{g_n\}$

$$g_{n+1}(t) = (1-\alpha_n)g_n(t) + \alpha_n h_n(t), \tag{1.2}$$

where

$$h_n(t) = (1-\beta_n)S(t, g_n(t)) + \beta_n T(t, g_n(t)), \tag{1.3}$$

$$0 < \alpha_n, \beta_n < 1 \text{ for all } n = 0, 1, 2, \ldots, \tag{1.4}$$

$$\overline{\lim_{n \rightarrow \infty}} \beta_n = M < 1, \tag{1.5}$$

and

$$\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty. \tag{1.6}$$

The construction of the iteration scheme is based on the same idea as that of Ishikawa's random iteration scheme [2]. But the present iteration is not a modification or generalization of that iteration.

A function $T: H \rightarrow H$ is said to satisfy *Tricomi's condition* if

$$Tp = p \text{ implies } \| Tx - p \| \leq \| x - p \|.$$

We define generalized Tricomi's condition for two operators in the following way.

Definition 1.3: *Generalized Tricomi's Condition.* Two functions $S, T: H \rightarrow H$ are said to satisfy *generalized Tricomi's condition* if

$$Sp = Tp \text{ implies } \| Sx - p \| \leq \| x - p \| \tag{1.7}$$

and

$$\| Tx - p \| \leq \| x - p \|. \tag{1.8}$$

2. Main Results

Theorem 2.1: *Let $S, T: \Omega \times H \rightarrow H$, where H is a separable Hilbert space, be two random operators such that*

(a) *S and T are H-continuous, and*
(b) *there exists* $f:\Omega \rightarrow H$ *(not necessarily measurable) such that*

$$(1-\lambda) \| S(t,x)-f(t) \|^2 + \lambda \| T(t,x)-f(t) \|^2 \leq \| x-f(t) \|^2 \tag{2.1}$$

for all $t \in \Omega, x \in H$ *and* $0 < \lambda < 1$.

Then the random iteration scheme (Definition 1.2), if convergent, converges to a solution of the random operator equation

$$S(t,x(t)) = T(t,x(t)). \tag{2.2}$$

Proof: For any $t \in \Omega$,

$$\| g_{n+1}(t)-f(t) \|^2 = \| (1-\alpha_n)g_n(t)+\alpha_n h_n(t)-f(t) \|^2 \quad \text{(by (1.2))}$$

$$= (1-\alpha_n) \| g_n(t)-f(t) \|^2 + \alpha_n \| h_n(t)-f(t) \|^2$$

$$-\alpha_n(1-\alpha_n) \| g_n(t)-h_n(t) \|^2 \quad \text{(by (1.1))}$$

$$\leq (1-\alpha_n) \| g_n(t)-f(t) \|^2$$

$$+\alpha_n \| (1-\beta_n)S(t,g_n(t))+\beta_n T(t,g_n(t))-f(t) \|^2 \quad \text{(by (1.4) and (1.3))}$$

$$= (1-\alpha_n) \| g_n(t)-f(t) \|^2 + \alpha_n\{(1-\beta_n) \| S(t,g_n(t)-f(t) \|^2$$

$$+\beta_n \| T(t,g_n(t))-f(t) \|^2 - \beta_n(1-\beta_n) \| S(t,g_n(t))-T(t,g_n(t)) \|^2\} \quad \text{(by (1.1)}$$

or,

$$\alpha_n\beta_n(1-\beta_n) \| S(t,g_n(t))-T(t,g_n(t)) \|^2$$

$$\leq (1-\alpha_n) \| g_n(t)-f(t) \|^2 - \| g_{n+1}(t))-f(t) \|^2$$

$$+\alpha_n\{(1-\beta_n) \| S(t,g_n(t))-f(t) \|^2 + \beta_n \| T(t,g_n(t))-f(t) \|^2\}$$

$$\text{for all } t \in \Omega \tag{2.3}$$

or,

$$\alpha_n\beta_n(1-\beta_n) \| S(t,g_n(t))-T(t,g_n(t)) \|^2 \leq (1-\alpha_n) \| g_n(t)-f(t) \|^2$$

$$- \| g_{n+1}(t)-f(t) \|^2 + \alpha_n \| g_n(t)-f(t) \|^2 \text{ for all } t \in \Omega \quad \text{(by (2.1))}$$

or,

$$\alpha_n\beta_n(1-\beta_n) \| S(t,g_n(t))-T(t,g_n(t)) \|^2 \leq \| g_n(t)-f(t) \|^2$$

$$- \| g_{n+1}(t)-f(t) \|^2 \text{ for all } t \in \Omega. \tag{2.4}$$

Summing up the inequalities in (2.4) over n, we obtain for all $t \in \Omega$,

$$\sum_{n=0}^{\infty} \alpha_n \beta_n (1-\beta_n) \| S(t, g_n(t)) - T(g, g_n(t)) \|^2 \leq \| g_0(t) - f(t) \|^2 < \infty. \quad (2.5)$$

Let $M < M' < 1$. Then, by (1.5), there exists a positive integer m_0 such that $\beta_m < M'$, that is $1 - \beta_m > 1 - M'$ for all $m > m_0$.

This shows that

$$\sum_{m=m_0}^{\infty} \alpha_m \beta_m (1-\beta_m) \geq (1-M') \sum_{m=m_0}^{\infty} \alpha_m \beta_m = \infty. \quad (2.6)$$

(2.5) and (2.6) imply that, for all $t \in \Omega$,

$$\underline{\lim}_{n \to \infty} \| S(t, g_n(t)) - T(t, g_n(t)) \|^2 = 0. \quad (2.7)$$

Let

$$g_n(t) \to g(t) \text{ as } n \to \infty. \quad (2.8)$$

Since g_0 is measurable and H is separable, according to Himmelberg [3], g_n's are measurable and, therefore, $g: \Omega \to H$ is measurable.

Again, S and T are H-continuous, which shows that $\lim_{n \to \infty} S(t, g_n(t)) = S(t, g(t))$ and

$$\lim_{n \to \infty} T(t, g_n(t)) = T(t, g(t)) \text{ for all } t \in \Omega.$$

By (2.7) and (2.8), we have for all $t \in \Omega$,

$$S(t, g(t)) = T(t, g(t)), \text{ where } g: \Omega \to H \text{ is a measurable function.} \quad (2.9)$$

This shows that the random iteration scheme if convergent, converges to a solution of (2.2). □

Corollary 2.2: *Let H be a separable Hilbert space and $S, T: \Omega \times H \to H$ be two random operators such that*

(a) *S, T are H-continuous, and*

(b) *there exists $f: \Omega \to H$ (not necessarily measurable) such that*

$$\| S(t,x) - f(t) \| \leq \| x - f(t) \| \quad (2.10)$$

and

$$\| T(t,x) - f(t) \| \leq \| x - f(t) \|. \quad (2.11)$$

Then the random iteration scheme if convergent, converges to a solution of $S(t, x(t)) = T(t, x(t))$.

Proof: It is easily seen that (2.10) and (2.11) imply (2.1). The corollary then follows by Theorem 2.1. □

Setting S as the identify random operator, that is, $S(t,x) = x$ for all $t \in \Omega$ and

$x \in H$, we obtain the following fixed point result as a corollary.

Corollary 2.3: *Let H be a separable Hilbert space and T be a random operator which is H-continuous. Assume that there exists $f: \Omega \to H$ (not necessarily measurable) such that for all $t \in \Omega$*

$$\| T(t,x) - f(t) \| \leq \| x - f(t) \| . \tag{2.12}$$

Then the sequence of functions $\{g_n\}$, where $g_0: \Omega \to H$, is measurable and

$$g_{n+1}(t) = (1-\alpha_n)g_n(t) + \alpha_n((1-\beta_n)g_n(t) + \beta_n T(t, g_n(t))), \; n = 0,1,2,\ldots \tag{2.13}$$

for all $t \in \Omega$, where $0 < \alpha_n$, $\beta_n < 1$, $\overline{\lim}_{n\to\infty} \beta_n < 1$, and $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$ if convergent, converges to a random fixed point of T.

Corollary 2.4: *Let $S, T: H \to H$ be two operators such that the following holds: there exists a $z \in H$ such that*

$$(1-\lambda) \| Sx - z \|^2 + \lambda \| Tx - z \|^2 \leq \| x - z \|^2, \tag{2.14}$$

for all $x \in H$ and $0 < \lambda < 1$.

Then the sequence $\{x_n\}$, obtained by the iteration

$$x_0 \in H, \tag{2.15}$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n((1-\beta_n)Sx_n + \beta_n Tx_n), \tag{2.16}$$

where

$$0 < \alpha_n, \beta_n < 1 \tag{2.17}$$

$$\overline{\lim_{n\to\infty}} \beta_n < 1 \tag{2.18}$$

and

$$\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty, \tag{2.19}$$

if convergent, converges to a solution of $Sx = Tx$.

The proof trivially follows from Theorem 2.1. It may be noted that the separability of H was required to ensure that g_n's are measurable. In the statement of the corollary, H need not be separable.

Theorem 2.5: *Let C be a convex and compact subset of a separable Hilbert space H and $S, T: \Omega \times H \to C$ be two random operators such that the following conditions are satisfied:*

(*a*) *S, T are H-continuous,*

(*b*) *there exists $f: \Omega \to H$ (not necessarily measurable) such that*

$$(1-\lambda) \| S(t,x) - f(t) \|^2 + \lambda \| T(t,x) - f(t) \|^2 \leq \| x - f(t) \|^2 \tag{2.20}$$

for all $t \in \Omega$, $x \in H$ and $0 < \lambda < 1$, and *(c) $S(t,\cdot), T(t,\cdot): C \rightarrow C$ satisfy Generalized Tricomi's condition for all $t \in \Omega$. Then for any measurable function $g_0: \Omega \rightarrow C$, the sequence of functions $\{g_n\}$ constructed by the random iteration scheme ((1.2)-(1.6)) actually converges to a solution of the random operator equation $S(t, x(t)) = T(t, x(t))$.*

Proof: By the construction of $\{g_n\}$ it is seen that g_n's are measurable functions from Ω to C for all $n = 0, 1, 2, \ldots$. Proceeding exactly in the same way as in Theorem 2.1, we have as in (2.7) that

$$\underline{\lim}_{n \rightarrow \infty} \| S(t, g_n(t)) - T(t, g_n(t)) \|^2 = 0.$$

Therefore, for a fixed $t \in \Omega$, there exists a subsequence

$$\{g_{n_i}(t)\} \subset \{g_n(t)\} \text{ such that } \lim_{i \rightarrow \infty} \| S(t, g_{n_i}(t)) - T(t, g_{n_i}(t)) \| = 0. \tag{2.21}$$

Again, C is compact, therefore, there exists $\{g_{n_{i_k}}(t)\} \subset \{g_{n_i}(t)\}$ such that $\{g_{n_{i_k}}(t)\}$ is convergent.

Let

$$\lim_{k \rightarrow \infty} g_{n_{i_k}}(t) = g(t) \text{ for } t \in \Omega. \tag{2.22}$$

Since S and T are H-continuous random operators, from (2.21), we have for any $t \in \Omega$,

$$S(t, g(t)) = T(t, g(t)). \tag{2.23}$$

For any $t \in \Omega$,

$$\| g_{n+1}(t) - g(t) \|^2 = \| (1-\alpha_n) g_n(t) + \alpha_n h_n(t) - g(t) \|^2$$

$$= (1-\alpha_n) \| g_n(t) - g(t) \|^2 + \alpha_n \| h_n(t) - g(t) \|^2$$

$$- (1-\alpha_n)\alpha_n \| g_n(t) - h_n(t) \|^2$$

$$\leq (1-\alpha_n) \| g_n(t) - g(t) \|^2 + \alpha_n \| (1-\beta_n) S(t, g_n(t)) + \beta_n T(t, g_n(t)) - g(t) \|^2$$

(by (1.3) and (1.4))

$$= (1-\alpha_n) \| g_n(t) - g(t) \|^2 + \alpha_n \{ (1-\beta_n) \| S(t, g_n(t)) - g(t) \|^2$$

$$+ \beta_n \| T(t, g_n(t)) - g(t) \|^2 \} - \alpha_n \beta_n (1-\beta_n) \| S(t, g_n(t)) - T(t, g_n(t)) \|^2$$

(by (1.1))

$$\leq (1-\alpha_n) \| g_n(t) - g(t) \|^2 + \alpha_n \| g_n(t) - g(t) \|^2$$

(by (2.23) and Generalized Tricomi's condition)

or

$$\| g_{n+1}(t) - g(t) \| \leq \| g_n(t) - g(t) \| . \tag{2.24}$$

(2.22) and (2.24) together imply that

$$g_n \rightarrow g \text{ as } n \rightarrow \infty. \tag{2.25}$$

Since H is separable, g_n's are measurable [3], and, hence g is also measurable. From (2.23), g is a random solution of $S(t, x(t)) = T(t, x(t))$. This completes the proof. □

We have the following obvious corollary.

Corollary 2.6: *Let $S, T: H \rightarrow C$, where C is a compact and convex subset of a Hilbert space H are such that the following are satisfied:*

(*a*) *S, T are continuous,*

(*b*) *there exists $z \in H$ such that*

$$(1-\lambda) \| Sx - z \|^2 + \lambda \| Tx - z \|^2 \leq \| x - z \|^2 \tag{2.26}$$

for all $x \in H$ and $0 < \lambda < 1$, and

(*c*) *S, T satisfy Generalized Tricomi's condition.*

Then the sequence, defined as $x_0 \in C$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n((1-\beta_n)Sx_n + \beta_n Tx_n),\ n = 0, 1, 2, \ldots, \tag{2.27}$$

where

$$0 < \alpha_n, \beta_n < 1, \tag{2.28}$$

$$\overline{\lim_{n \rightarrow \infty}} \beta_n < 1, \tag{2.29}$$

and

$$\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty, \tag{2.30}$$

converges to a solution of the equation $Sx = Tx$.

Example: Let $C = [0,1], S, T: R \rightarrow [0,1]$ be defined as

$$Sx = x^2/2 \text{ if } x \in [0,1]$$

$$= 1/2 \text{ if } x > 1$$

$$= 0 \text{ if } x < 0$$

and

$$Tx = x^2/4 \text{ if } x \in [0,1]$$

$$= 1/4 \text{ if } x > 1$$

$$= 0 \text{ if } x < 1.$$

With the choice of $z = 0$, the conditions of Corollary 2.6 are seen to be satisfied. Thus Corollary 2.6 applies to this example.

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