

ON NONLOCAL PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS AND ON A NONLOCAL PARABOLIC TRANSMISSION PROBLEM

M. DENCHE

*Université de Constantine
Institut de Mathématiques
Constantine
Algeria*

(Received July, 1997; Revised August, 1998)

In the present paper we study nonlocal problems for ordinary differential equations with a discontinuous coefficient for the high order derivative. We establish sufficient conditions, known as regularity conditions, which guarantee the coerciveness for both the space variable and the spectral parameter, as well as guarantee the completeness of the system of root functions. The results obtained are then applied to the study of a nonlocal parabolic transmission problem.

Key words: Nonlocal Problems, Ordinary Differential Equations, Parabolic Differential Equation, Boundary and Transmission Conditions, Coerciveness Estimate, Completeness, Root Functions, Elementary Solutions.

AMS subject classifications: 34A30, 34B05, 34L40, 47E05, 35K20, 35K99.

1. Introduction

Many physical problems, the problem of heat and mass diffusion in anisotropic media, diffraction problems, and others lead to the study of equations with a discontinuous coefficient for the high order derivative [13]. In the present paper, we start with the study of boundary value problems for ordinary differential equations with a discontinuous coefficient for the high order derivative and with boundary conditions containing abstract functionals. We establish sufficient conditions of regularity type which guarantee the coerciveness for both the space variable and the spectral parameter, and which also guarantee the completeness of root functions. Regular problems for differential operators are studied in [3, 9, 10]. Completeness of the system of root functions for differential operators with functional boundary conditions is analyzed in [4, 6, 11, 14]. The coefficient of the high order derivative is assumed to be constant in [6, 11], whereas it is assumed to be continuous in [14]. The results obtained are used to show the existence and uniqueness of the solution, of a mixed

problem for a parabolic partial differential equation with a discontinuous coefficient for the high order derivative, and multipoint boundary values and transmission conditions containing abstract functionals. They are also used to show the completeness of elementary solutions of this given mixed problem. Thus, the study is reduced to a Cauchy problem for a parabolic abstract differential equation, where the analysis of the operator coefficient is given in detail in this paper.

2. Preliminaries

Let $W_q^m(a, b)$ denote a Sobolev space, defined so that

$$W_q^m(a, b) = \{u \in L_q(a, b); D^\alpha u \in L_q(a, b); \alpha \leq m\}, \quad q \in (1, \infty).$$

Lemma 1: For $u \in W_q^m(a, b)$, we have the following inequality [2]:

$$\max_{x \in [a, b]} |u^{(k)}(x)| \leq c \left(h^{1-\chi} \|u^{(m)}\|_{L_q(a, b)} + h^{-\chi} \|u\|_{L_q(a, b)} \right),$$

where $0 \leq k < m; 0 < h < h_0; \chi = \frac{1}{m} \left(k + \frac{1}{q} \right); q \in (1, \infty)$.

Lemma 2: For $y \in W_2^k(a, b)$ and $\lambda \in \mathbb{C}$, we have

$$|\lambda|^{k-s} \|y\|_{W_2^s(a, b)} \leq C_{ks} \left(\|y\|_{W_2^k(a, b)} + |\lambda|^k \|y\|_{L_2(a, b)} \right), \quad 0 \leq s \leq k.$$

Remark 1: Lemma 2 is a particular case of a result given in [1].

3. Nonlocal Problem for an Ordinary Differential Equation

In $[0, 1] \setminus b$, we consider the equation

$$L(\lambda)u = -a(x)u''(x) + (Bu)(x) - \lambda u(x) = f(x), \tag{1}$$

where λ is a complex parameter. To equation (1) we add the boundary conditions:

$$\begin{cases} L_1 u = \alpha_{11} u^{(K_1)}(0) + \beta_{11} u^{(K_1)}(1) + \sum_{p=1}^N \delta_{1p} u^{(K_1)}(x_{1p}) + T_1 u = f_1 \\ L_2 u = \alpha_{21} u^{(K_2)}(0) + \beta_{21} u^{(K_2)}(1) + \sum_{p=1}^M \delta_{2p} u^{(K_2)}(x_{2p}) + T_2 u = f_2, \end{cases} \tag{2}$$

and we include the transmission conditions:

$$\begin{cases} L_3 u = \alpha_{31} u(b-0) + \beta_{31} u(b+0) + T_3 u = f_3 \\ L_4 u = \alpha_{41} u'(b-0) + \beta_{41} u'(b+0) + T_4 u = f_4, \end{cases} \tag{3}$$

where $a(x) = \begin{cases} a_0 & \text{for } x \in (0, b] \\ a_1 & \text{for } x \in (b, 1] \end{cases}$, $a_0, a_1 \neq 0$, $x_{1p}, x_{2p} \in (0, b)$. B is a linear operator and each T_ν is a linear functional ($\nu = \overline{1, 4}$). $\alpha_{j1}, \beta_{j1}, \delta_{ip} \in \mathbb{C}$; $K_i = 0, 1$; $i = \overline{1, 2}$; $j = \overline{1, 4}$; $p = \overline{1, N}$. Let

$$\bar{\omega} := \max\{\arg a_0, \arg a_1\}, \underline{\omega} := \min\{\arg a_0, \arg a_1\}$$

and

$$S_\varepsilon := \{\lambda \in \mathbb{C} : \varepsilon + \bar{\omega} < \arg \lambda < 2\pi - \varepsilon + \underline{\omega}\}.$$

3.1 Coerciveness of a Principal Boundary Value Problem

Consider, in $[0, 1] \setminus b$ the simplified problem:

$$L_0(\lambda)u = -a(x)u''(x) - \lambda u(x) = f(x), \tag{4}$$

with

$$\begin{cases} L_{10}u = \alpha_{11}u^{(K_1)}(0) + \beta_{11}u^{(K_1)}(1) = f_1 \\ L_{20}u = \alpha_{21}u^{(K_2)}(0) + \beta_{21}u^{(K_2)}(1) = f_2 \end{cases} \tag{5}$$

and with

$$\begin{cases} L_{30}u = \alpha_{31}u(b-0) + \beta_{31}u(b+0) = f_3 \\ L_{40}u = \alpha_{41}u'(b-0) + \beta_{41}u'(b+0) = f_4. \end{cases} \tag{6}$$

Theorem 1: *Suppose that*

$$\frac{(-1)^{K_2} \alpha_{11} \beta_{21}}{(\sqrt{a_0})^{K_1} (\sqrt{a_1})^{K_2}} - \frac{(-1)^{K_1} \alpha_{21} \beta_{11}}{(\sqrt{a_0})^{K_2} (\sqrt{a_1})^{K_1}} \neq 0 \text{ and } \frac{\alpha_{31} \beta_{41}}{\sqrt{a_0}} - \frac{\beta_{31} \alpha_{41}}{\sqrt{a_1}} \neq 0.$$

Then for each $\varepsilon > 0$ there exists $R_\varepsilon > 0$, such that for any complex number $\lambda \in S_\varepsilon$ with $|\lambda| > R_\varepsilon$, the operator

$$\mathfrak{L}_0(\lambda): u \mapsto (L_0u, L_{10}u, L_{20}u, L_{30}u, L_{40}u),$$

from $W_q^l(0, b) \times W_q^l(b, 1)$ onto $W_q^{l-2}(0, b) \times W_q^{l-2}(b, 1) \times \mathbb{C}^4$, is an isomorphism. Furthermore, for any such λ and any solution of problem (4)-(6), the following estimate holds.

$$\begin{aligned} & \sum_{k=0}^l |\lambda|^{\frac{1}{2}(l-k)} \|u\|_{W_q^k(0, b) \times W_q^k(b, 1)} \\ & \leq C(\varepsilon) \left(\|f\|_{W_q^{l-2}(0, b) \times W_q^{l-2}(b, 1)} + |\lambda|^{\frac{1}{2}(l-2)} \|f\|_{L_q(0, b) \times L_q(b, 1)} \right. \\ & \quad \left. + \sum_{\nu=1}^4 |\lambda|^{\frac{1}{2}(l-k_\nu - \frac{1}{q})} |f_\nu| \right). \end{aligned} \tag{7}$$

Proof: Clearly, operator $\mathcal{L}_0(\lambda)$ acts linearly and continuously from $W_q^l(0, b) \times W_q^l(b, 1)$ into $W_q^{l-2}(0, b) \times W_q^{l-2}(b, 1) \times \mathbb{C}^4$. Let us show that, for every $f \in W_q^{l-2}(0, b) \times W_q^{l-2}(b, 1)$ and for every $f_\nu \in \mathbb{C}$, $\nu = \overline{1, 4}$, problem (4)-(6) has a unique solution belonging to $W_q^l(0, b) \times W_q^l(b, 1)$.

We seek the solution y of problem (4)-(6) in the form $y = y_1 + y_2$, with $y_1 = (u_1, u_2)$ and $y_2 = (u_3, u_4)$, so that y_1 will be a restriction on $[0, 1]$ of the solution \tilde{y}_1 to the equation

$$L_0(\lambda)\tilde{y}_1 = \tilde{f}(x), \quad x \in \mathbb{R}, \tag{8}$$

where

$$\tilde{f}(x) = \begin{cases} f(x), & x \in [0, 1] \\ 0, & x \notin [0, 1], \end{cases}$$

and y_2 will be the solution of the problem:

$$\begin{cases} L_0(\lambda)y_2 = 0 \\ L_{\nu 0}y_2 = -L_{\nu 0}y_1 + f_\nu, \nu = \overline{1, 4}. \end{cases} \tag{9}$$

$$\tag{10}$$

By applying a Fourier transform, denoted by \mathfrak{F} , to equation (8) and by making the change of variable $\lambda = -a_0\rho^2$ and $\lambda = -a_1\rho^2$, we obtain

$$\begin{cases} a_0[(i\sigma)^2 - \rho^2]\mathfrak{F}u_1 = -\mathfrak{F}\tilde{f}_1 \\ a_1[(i\sigma)^2 - \rho^2]\mathfrak{F}u_2 = -\mathfrak{F}\tilde{f}_2. \end{cases}$$

As $\frac{\pi}{2} + \varepsilon < \arg \rho < \frac{3\pi}{2} - \varepsilon$ and $\sigma \in \mathbb{R}$, using geometrical arguments, we have

$$|(i\sigma)^2 - \rho^2| \geq c(\varepsilon)(|\sigma|^2 + |\rho|^2). \tag{11}$$

Then $u_j^{(k)} = \mathfrak{F}^{-1}\mathfrak{F}u_j^{(k)} = \mathfrak{F}^{-1}(i\sigma)^k \mathfrak{F}u_j = -\mathfrak{F}^{-1}(i\sigma)^k a_{j-1}^{-1} [(i\sigma)^2 - \rho^2]^{-1} \mathfrak{F}\tilde{f}_j$. Now, let $T_k^j(\sigma, \lambda) = \rho^{2-k} (i\sigma)^k a_{j-1}^{-1} [(i\sigma)^2 - \rho^2]^{-1}$ $j = \overline{1, 2}$ and $k = \overline{0, 2}$. Clearly, each $T_k^j(\sigma, \lambda)$ is continuously differentiable with respect to σ in \mathbb{R} and, from inequality (11), we get

$$|T_k^j(\sigma, \lambda)| \leq c(\varepsilon), \quad \left| \frac{\partial T_k^j(\sigma, \lambda)}{\partial \sigma} \right| \leq \frac{c(\varepsilon)}{|\sigma|}.$$

Then, by virtue of the Mikhlin theorem [3], the function $T_k^j(\sigma, \lambda)$ is a Fourier multiplier of type (q, q) . Hence, if $\tilde{f} \in L_q(\mathbb{R}) \times L_q(\mathbb{R})$, then a function \tilde{y}_1 is a solution to equation (8) belonging to $W_q^2(\mathbb{R}) \times W_q^2(\mathbb{R})$, and so we have

$$\sum_{k=0}^2 |\rho|^{2-k} \|\tilde{y}_1^{(k)}\|_{L_q(\mathbb{R}) \times L_q(\mathbb{R})} \leq c(\varepsilon) \|\tilde{f}\|_{L_q(\mathbb{R}) \times L_q(\mathbb{R})}. \tag{12}$$

Using (12) and Lemma 2 with a recursive argument, we can easily show that if $\tilde{f} \in W_q^{l-2}(\mathbb{R}) \times W_q^{l-2}(\mathbb{R})$, then $\tilde{y}_1 \in W_q^l(\mathbb{R}) \times W_q^l(\mathbb{R})$ and the following estimate is valid

$$\sum_{k=0}^l |\lambda|^{\frac{1}{2}(l-k)} \|\tilde{y}_1\|_{W_q^k(\mathbb{R}) \times W_q^k(\mathbb{R})} \tag{13}$$

$$\leq c(\varepsilon) \left(\|\tilde{f}\|_{W_q^{l-2}(\mathbb{R}) \times W_q^{l-2}(\mathbb{R})} + |\lambda|^{\frac{1}{2}(l-2)} \|\tilde{f}\|_{L_q(\mathbb{R}) \times L_q(\mathbb{R})} \right).$$

Thus, equation (8) possesses a unique solution $\tilde{y}_1 \in W_q^2(\mathbb{R}) \times W_q^2(\mathbb{R})$ and for y_1 we get

$$\sum_{k=0}^l |\lambda|^{\frac{1}{2}(l-k)} \|y_1\|_{W_q^k(0,b) \times W_q^k(b,1)}$$

$$\leq c(\varepsilon) \left(\|f\|_{W_q^{l-2}(0,b) \times W_q^{l-2}(b,1)} + |\lambda|^{\frac{1}{2}(l-2)} \|f\|_{L_q(0,b) \times L_q(b,1)} \right).$$

Now, let us establish that for any complex number $f_\nu \in \mathbb{C}$, $\nu = \overline{1,4}$, problem (9)-(10) has a unique solution y_2 belonging to $W_q^l(0,b) \times W_q^l(b,1)$; and for an estimate of the solution, we have $L_0(\lambda)y_2 = 0$, which is equivalent to

$$\begin{cases} -a_0 u_3'' - \lambda u_3 = 0 \text{ in } [0, b] \\ -a_1 u_4'' - \lambda u_4 = 0 \text{ in } (b, 1]. \end{cases} \tag{14}$$

Setting $\rho_0 = \sqrt{\frac{-\lambda}{a_0}}$ and $\rho_1 = \sqrt{\frac{-\lambda}{a_1}}$, we get that the general solutions of the equations of (14) are, respectively, $u_3 = c_1 \exp \rho_0 x + c_2 \exp[-\rho_0(x-b)]$ and $u_4 = c_3 \exp \rho_1(x-b) + c_4 \exp[-\rho_1(x-1)]$. Substituting these expressions into the conditions of (10), we obtain a system for c_k , $k = \overline{1,4}$. By a straightforward computation, it is easy to see that the determinant of this system is of the form:

$$\Delta = \theta + R(\rho_0, \rho_1),$$

where $\theta = [a_{11}\beta_{21}\rho_0^{k_1}(-\rho_1)^{k_2} - \alpha_{21}\beta_{11}\rho_0^{k_2}(-\rho_1)^{k_1}] (\alpha_{31}\beta_{41}\rho_1 + \beta_{31}\alpha_{41}\rho_0)$ and $R(\rho_0, \rho_1) \rightarrow 0$ for $|\rho_0|, |\rho_1| \rightarrow \infty$, in S_ε . According to the assumption, we have $\theta \neq 0$.

Again, by straightforward computation, we find that each c_i is of the form: $c_i = \frac{\theta_i + R_i(\rho_0, \rho_1)}{\theta + R(\rho_0, \rho_1)}$, where θ_i is obtained from θ by replacing the column i with the column formed by right-hand sides of the boundary conditions of (10) such that $R_i(\rho_0, \rho_1) \rightarrow 0$, $i = \overline{1,2}$ for $|\rho_0|, |\rho_1| \rightarrow \infty$, in S_ε . Substituting these values in the expressions of u_3 and u_4 , we find that problem (9)-(10) has a unique solution, given by

$$u_3 = \frac{\theta_1 + R_1(\rho_0, \rho_1)}{\theta + R(\rho_0, \rho_1)} \exp \rho_0 x + \frac{\theta_2 + R_2(\rho_0, \rho_1)}{\theta + R(\rho_0, \rho_1)} \exp [-\rho_0(x-b)]$$

$$u_4 = \frac{\theta_3 + R_3(\rho_0, \rho_1)}{\theta + R(\rho_0, \rho_1)} \exp \rho_1(x-b) + \frac{\theta_4 + R_4(\rho_0, \rho_1)}{\theta + R(\rho_0, \rho_1)} \exp [-\rho_1(x-1)].$$

In the sector S_ε , for $|\rho_0|, |\rho_1| \rightarrow \infty$ we have $\|\exp \rho_0 x\|_{L_q(0,b)} \leq c(\varepsilon) |\rho_0|^{-\frac{1}{q}}$,

$\| \exp[-\rho_0(x-b)] \|_{L_q(0,b)} \leq c(\varepsilon) |\rho_0|^{-\frac{1}{q}}$, $\| \exp \rho_1(x-b) \|_{L_q(b,1)} \leq c(\varepsilon) |\rho_1|^{-\frac{1}{q}}$
 and $\| \exp[-\rho_1(x-1)] \|_{L_q(b,1)} \leq c(\varepsilon) |\rho_1|^{-\frac{1}{q}}$. Hence, for $k \geq 0$ and $\lambda \in S_\varepsilon$, we get

$$\| y_2^{(k)} \|_{L_1(0,b) \times L_q(b,1)} \leq c(\varepsilon) \sum_{\nu=0}^4 |\lambda|^{\frac{1}{2}(k-\frac{1}{q}-k_\nu)} (\| L_{\nu 0} y_1 \| + \| f_\nu \|). \tag{15}$$

To evaluate $\| L_{\nu 0} y_1 \|$, we apply Lemma 1 for $\chi = j + \frac{1}{q}$ and $h = |\lambda|^{-1}$, since

$$\begin{aligned} & \| u_1 \|_{c^{k_\nu}[0,b]} \\ & \leq c \sum_{j=0}^{k_\nu} \left(|\rho_0|^{-2[1-\frac{1}{2}(j+\frac{1}{q})]} \| u_1'' \|_{L_q(0,b)} + |\rho_0|^{2[\frac{1}{2}(j+\frac{1}{q})]} \| u_1 \|_{L_q(0,b)} \right). \end{aligned}$$

This, from (13), gives

$$\begin{aligned} & \| u_1 \|_{c^{k_\nu}[0,b]} \\ & \leq c(\varepsilon) \sum_{j=0}^{k_\nu} |\rho_0|^{j+\frac{1}{q}-l} \left(\| f_1 \|_{W_q^{l-2}(0,b)} + |\rho_0|^{l-2} \| f_1 \|_{L_q(0,b)} \right). \end{aligned}$$

By an analogous argument for u_2 , we obtain

$$\begin{aligned} & \| u_2 \|_{c^{k_\nu}(b,1)} \\ & \leq c(\varepsilon) \sum_{j=0}^{k_\nu} |\rho_1|^{j+\frac{1}{q}-l} \left(\| f_2 \|_{W_q^{l-2}(0,b) \times W_q^{l-2}(b,1)} + |\rho_1|^{l-2} \| f_2 \|_{L_q(b,1)} \right). \end{aligned}$$

As $\| L_{\nu 0} y \| \leq c \| y_1 \|_{c^{k_\nu}[0,b] \times c^{k_\nu}(b,1)}$, so

$$\begin{aligned} \| L_{\nu 0} y_1 \| & \leq c(\varepsilon) \sum_{j=0}^{k_\nu} |\lambda|^{\frac{1}{2}(-l+j+\frac{1}{q})} (\| f \|_{W_q^{l-2}(0,b) \times W_q^{l-2}(b,1)} \\ & \quad + |\lambda|^{\frac{1}{2}(l-2)} \| f \|_{L_q(0,b) \times L_q(b,1)}) \\ & \leq c(\varepsilon) |\lambda|^{\frac{1}{2}(-l+k_\nu+\frac{1}{q})} (\| f \|_{W_q^{l-2}(b,1)} + |\lambda|^{\frac{1}{2}(l-2)} \| f \|_{L_q(0,b) \times L_q(b,1)}), \end{aligned}$$

$$\nu = \overline{1,4}. \tag{16}$$

Applying (16), we get that inequality (15) gives estimate (7). The uniqueness of the solution of problem (4)-(6) follows from estimate (7).

3.2 Coerciveness of the General Problem

First, consider the following definition.

Definition 1: Boundary value and transmission conditions (2)-(3) are said to be regular if the following hold.

1. $\frac{(-1)^{K_2} \alpha_{11} \beta_{21}}{(\sqrt{a_0})^{K_1} (\sqrt{a_1})^{K_2}} - \frac{(-1)^{K_1} \alpha_{21} \beta_{11}}{(\sqrt{a_0})^{K_2} (\sqrt{a_1})^{K_1}} \neq 0$ and $\frac{\alpha_{31} \beta_{41}}{\sqrt{a_0}} - \frac{\beta_{31} \alpha_{41}}{\sqrt{a_1}} \neq 0$.
2. For some $q \in (1, \infty)$, each functional T_ν is continuous in $W_q^{k_\nu}(0, b) \times W_q^{k_\nu}(b, 1)$, where $\nu = \overline{1, 4}$, $0 \leq k_\nu \leq 1$ for $\nu = \overline{1, 2}$ and $k_3 = 0$ and $k_4 = 1$.

Remark 2: The above definition coincides with the Yakubov definition ([14], p. 86) in the case where $\alpha_{31} = \alpha_{41} = \beta_{31} = \beta_{41} = 0$.

Theorem 2: Suppose the following hold.

1. Boundary and transmission conditions (2)-(3) are regular.
2. Operator B , from $W_q^l(0, b) \times W_q^l(b, 1)$ into $W_q^{l-2}(0, b) \times W_q^{l-2}(b, 1)$ is compact, where $l \geq 2$ and $q \in (1, \infty)$.

Then, for all $\lambda \in S_\varepsilon$, the operator

$$\mathcal{L}(\lambda): u \mapsto (L_0 u, L_{10} u, L_{20} u, L_{30} u, L_{40} u),$$

from $W_q^l(0, b) \times W_q^l(b, 1)$ into $W_q^{l-2}(0, b) \times W_q^{l-2}(b, 1) \times \mathbb{C}^4$, is a Fredholm operator.

Proof: The operator $\mathcal{L}(\lambda)$ can be represented in the form $\mathcal{L}(\lambda) = \mathcal{L}_0(\lambda) + \mathcal{L}_1(\lambda)$, where

$$\mathcal{L}_0(\lambda) = (-a(x)u''(x) - \lambda u(x), u(0), u'(1), u(b-0), u'(b+0))$$

and

$$\mathcal{L}_1(\lambda) = (Bu, L_{10}u - u(0), L_{20}u - u'(1), L_{30}u - u(b-0), L_{40}u - u'(b+0)).$$

From Theorem 1, it follows that $\mathcal{L}_0(\lambda)$ is an isomorphism for $\lambda \in S_\varepsilon$. Hence, it is a Fredholm operator. By virtue of hypotheses 1 and 2, $\mathcal{L}_1(\lambda)$, from $W_q^l(0, b) \times W_q^l(b, 1)$ into $W_q^{l-2}(0, b) \times W_q^{l-2}(b, 1) \times \mathbb{C}^4$, is compact. Then, by a theorem on perturbation of Fredholm operators (see [7]), $\mathcal{L}(\lambda) = \mathcal{L}_0(\lambda) + \mathcal{L}_1(\lambda)$ is a Fredholm operator.

Theorem 3: Suppose that conditions 1 and 2 of Theorem 2 are satisfied. Then, for any $\varepsilon > 0$, there is $R_\varepsilon > 0$ such that for any complex number λ where $\lambda \in S_\varepsilon$ and $|\lambda| > R_\varepsilon$, the operator

$$\mathcal{L}(\lambda): u \mapsto (L(\lambda)u, L_1(\lambda)u, L_2(\lambda)u, L_3(\lambda)u, L_4(\lambda)u),$$

from $W_q^l(0, b) \times W_q^l(b, 1)$ into $W_q^{l-2}(0, b) \times W_q^{l-2}(b, 1) \times \mathbb{C}^4$, is an isomorphism. Furthermore, for such a λ , estimate (7) holds for the solution of problem (1)-(3).

Proof: By displacing the perturbed terms of problem (1)-(3) to the right-hand side members, and by applying Theorem 1 to the obtained problem, we find that

$$\sum_{k=0}^l |\lambda|^{-\frac{1}{2}(l-k)} \|u\|_{W_q^k(0, b) \times W_q^k(b, 1)}$$

$$\begin{aligned} &\leq C(\varepsilon) \left(\|f\|_{W_q^{l-2}(0,b) \times W_q^{l-2}(b,1)} + |\lambda|^{\frac{1}{2}(l-2)} \|f\|_{L_q(0,b) \times L_q(b,1)} \right. \\ &\quad \left. + \sum_{\nu=1}^4 |\lambda|^{\frac{1}{2}(l-k_\nu-\frac{1}{q})} |f_\nu| + \|Bu\|_{W_q^{l-2}(0,b) \times W_q^{l-2}(b,1)} \right. \\ &\quad \left. + |\lambda|^{\frac{1}{2}(l-2)} \|Bu\|_{L_q(0,b) \times L_q(b,1)} + \sum_{\nu=1}^4 |\lambda|^{\frac{1}{2}(l-k_\nu-\frac{1}{q})} |T_\nu u| \right). \end{aligned}$$

The above inequality, Lemma 2.7 from [14] and the continuity of T_ν give

$$\begin{aligned} &\sum_{k=0}^l |\lambda|^{\frac{1}{2}(l-k)} \|u\|_{W_q^k(0,b) \times W_q^k(b,1)} \\ &\leq C(\varepsilon) \left(\|f\|_{W_q^{l-2}(0,b) \times W_q^{l-2}(b,1)} + |\lambda|^{\frac{1}{2}(l-2)} \|f\|_{L_q(0,b) \times L_q(b,1)} \right. \\ &\quad \left. + \sum_{\nu=1}^4 |\lambda|^{\frac{1}{2}(l-k_\nu-\frac{1}{q})} |f_\nu| \right. \\ &\quad \left. + c(\varepsilon)(\delta + c(\delta) |\lambda|^{-\frac{1}{2q}} \sum_{k=0}^l |\lambda|^{\frac{1}{2}(l-k)} \|u\|_{W_q^k(0,b) \times W_q^k(b,1)}) \right). \end{aligned}$$

Choosing δ such that $c(\varepsilon)(\delta + c(\delta) |\lambda|^{-\frac{1}{2q}}) < 1$, we get inequality (7), which implies that $\mathfrak{L}(\lambda)$ is injective. Since operator B , from $W_q^l(0,b) \times W_q^l(b,1)$ into $W_q^{l-2}(0,b) \times W_q^{l-2}(b,1)$, is compact, and since according to Theorem 2 $\mathfrak{L}(\lambda): W_q^l(0,b) \times W_q^l(b,1) \rightarrow W_q^{l-2}(0,b) \times W_q^{l-2}(b,1) \times \mathbb{C}^4$ is a Fredholm operator, then by a Fredholm alternative $\mathfrak{L}(\lambda)$ is surjective. Therefore, it is an isomorphism.

4. Completeness of Root Functions

In the space $L_2(0,b) \times L_2(b,1)$, consider the operator \mathfrak{L} defined as follows.

$$\mathfrak{L}u = -a(x)u''(x) + (Bu)(x)$$

$$D(\mathfrak{L}) = (W_2^2(0,b) \times W_2^2(b,1), L_\nu u = 0, \nu = \overline{1,4}).$$

The root functions of operator \mathfrak{L} are root functions of the following problem:

$$\begin{cases} L(\lambda)u = 0 \\ L_\nu(\lambda)u = 0, \nu = \overline{1,4}. \end{cases} \tag{17}$$

To establish the completeness of the root functions of \mathfrak{L} , we shall use a theorem given in [14] (Theorem 3.6, with $n = 1$). This theorem is actually a variation of the well-

known theorem of N. Danford ad J.T. Schwartz [3]. Consider the following.

Theorem 4: *Suppose that the conditions given below are satisfied.*

1. *There exist two Hilbert spaces, H and H_1 , with the compact embedding $H \subset H_1$, and $\overline{H_1} \big|_H = H$.*
2. *The embedding operator \mathfrak{J} belongs to $\sigma_p(H_1, H)$ for some $p > 0$.*
3. *The linear operator A from H_1 and H is bounded.*
4. *There exists a set of rays l_k , in the complex plane such that angles between the neighboring rays are less than $\frac{\pi}{p}$; and there exists a number $m \in \mathbb{N}$ such that $\|R(\lambda, A)\|_{B(H, H_1)} \leq c |\lambda|^m$, with $\lambda \in l_k$ and with $|\lambda| \rightarrow \infty$.*

Then the spectrum of operator A is discrete and the system of root vectors of operator A is complete in the space H_1 .

Applying the method used in proving Theorem 2.1 in [14] and Theorem 3, we can prove the following lemma.

Lemma 3:

$$\overline{(W_2^2(0, b) \times W_2^2(b, 1), L_\nu u = 0, \nu = \overline{1, 4})} \big|_{L_2(0, b) \times L_2(b, 1)} = L_2(0, b) \times L_2(b, 1).$$

Theorem 5: *Suppose that the conditions below hold.*

1. *Boundary and transmission conditions of problem (17) are regular.*
2. *Operator B , from $W_2^2(0, b) \times W_2^2(b, 1)$ into $L_2(0, b) \times L_2(b, 1)$ is compact.*

Then the spectrum of problem (17) is discrete, and the system of root functions of problem (17) is complete in $(W_2^2(0, b) \times W_2^2(b, 1), L_\nu u = 0, \nu = \overline{1, 4})$ and, therefore, in $L_2(0, b) \times L_2(b, 1)$.

Proof: Set $H = L_2(0, b) \times L_2(b, 1)$ and $H_1 = (W_2^2(0, b) \times W_2^2(b, 1), L_\nu u = 0, \nu = \overline{1, 4})$. Since embeddings $W_2^2(0, b) \subset L_2(0, b)$ and $W_2^2(b, 1) \subset L_2(b, 1)$ are compact [12], then embeddings $W_2^2(0, b) \times W_2^2(b, 1) \subset L_2(0, b) \times L_2(b, 1)$ is compact. Using hypothesis 1 and Lemma 3, we find $\overline{H_1} \big|_H = H$. take $p = \frac{1}{2} + \delta$, where δ is an arbitrary positive number. From [12], we get $S_j(\mathfrak{J}, W_2^2(0, b), L_2(0, b)) \sim j^{-2}$ and $S_j(\mathfrak{J}, W_2^2(b, 1), L_2(b, 1)) \sim j^{-2}$, and so, $\mathfrak{J} \in \sigma_{\frac{1}{2} + \delta}(W_2^2(0, b), L_2(0, b))$ and $\mathfrak{J} \in \sigma_{\frac{1}{2} + \delta}(W_2^2(b, 1), L_2(b, 1))$. It is easy to see that $\mathfrak{J} \in \sigma_{\frac{1}{2} + \delta}(W_2^2(0, b) \times W_2^2(b, 1), L_2(0, b) \times L_2(b, 1))$. Since H_1 is a closed subspace in $W_2^2(0, b) \times W_2^2(b, 1)$, applying [5], we have $\mathfrak{J} \in \sigma_{\frac{1}{2} + \delta}(H_1, H)$. It is obvious that operator \mathcal{L} is bounded from H_1 into H . From Theorem 3, we see that in the sector S_ε we have $\|R(\lambda, \mathcal{L})\|_{B(H, H_1)} \leq c(\varepsilon)$, for $|\lambda| \rightarrow \infty$. From this sector S_ε , take two rays l_1 and l_2 centered at the origin and choose a number $\delta > 0$ such that the angle between the two rays is less than $\frac{\pi}{\frac{1}{2} + \delta}$. Since all conditions of Theorem 4 are satisfied we get the desired result.

5. A Nonlocal Parabolic Transmission Problem

5.1 Correct Solvability

In $[0, T] \times ([0, 1] \setminus b)$, consider the equation

$$\frac{\partial u(t, x)}{\partial t} + a(x) \frac{\partial^2 u(t, x)}{\partial x^2} + (Bu(t, \cdot))(x) = 0 \tag{18}$$

with the functional boundary conditions:

$$\begin{cases} L_{10}u = \alpha_{11}u^{(K_1)}(t, 0) + \beta_{11}u^{(K_1)}(t, 1) + \sum_{p=1}^N \delta_{1p}u^{(K_1)}(t, x_{1p}) + T_1u(t, \cdot) = 0 \\ L_{20}u = \alpha_{21}u^{(K_2)}(t, 0) + \beta_{21}u^{(K_2)}(t, 1) + \sum_{p=1}^M \delta_{2p}u^{(K_2)}(t, x_{2p}) + T_2u(t, \cdot) = 0, \end{cases} \tag{19}$$

with the functional transmission conditions:

$$\begin{cases} L_{30}u = \alpha_{31}u(t, b - 0) + \beta_{31}u(t, b + 0) + T_3u(t, \cdot) = 0 \\ L_{40}u = \alpha_{41}u'(t, b - 0) + \beta_{41}u'(t, b + 0) + T_4u(t, \cdot) = 0, \end{cases} \tag{20}$$

and with the initial condition:

$$u(0, x) = \varphi_0(x), \tag{21}$$

where $a(x) = \begin{cases} a_0 & \text{for } x \in [0, b) \\ a_1 & \text{for } x \in (b, 1]. \end{cases}$ $x_{1p}, x_{2p} \in (0, b)$; B is a linear operator; each T_ν is a linear functional ($\nu = \overline{1, 4}$); $\alpha_{j1}, \beta_{j1}, \delta_{ip} \in \mathbb{C}$; $K_i = 0, 1$; $i = \overline{1, 2}$; $j = \overline{1, 4}$; $p = \overline{1, N}$.

Theorem 6: *Let the following conditions be satisfied.*

1. $a_i \neq 0$ and $|\arg a_i| > \frac{\pi}{2}$.
2. $\frac{(-1)^{K_2} \alpha_{11} \beta_{21}}{(\sqrt{a_0})^{K_1} (\sqrt{a_1})^{K_2}} - \frac{(-1)^{K_1} \alpha_{21} \beta_{11}}{(\sqrt{a_0})^{K_2} (\sqrt{a_1})^{K_1}} \neq 0$ and $\frac{\alpha_{31} \beta_{41}}{\sqrt{a_0}} - \frac{\beta_{31} \alpha_{41}}{\sqrt{a_1}} \neq 0$.
3. The operator B , from $W_2^2(0, b) \times W_2^2(b, 1)$ into $L_2(0, b) \times L_2(b, 1)$ is compact.
4. Each functional T_ν , $\nu = \overline{1, 4}$, is continuous in $W_2^{k\nu}(0, b) \times W_2^{k\nu}(b, 1)$.
5. $\varphi_0 \in (W_2^2(0, b) \times W_2^2(b, 1))$, $L_\nu u = 0$, $\nu = \overline{1, 4}$.

Then problem (18)-(21) has a unique solution u in

$$C([0, T], L_2(0, b) \times L_2(b, 1)) \cap C^1([0, T], W_2^2(0, b) \times W_2^2(b, 1), L_2(0, b) \times L_2(b, 1));$$

and we have the following inequalities:

$$\|u(t)\|_{L_2(0, b) \times L_2(b, 1)} \leq c \|\varphi_0\|_{W_2^2(0, b) \times W_2^2(b, 1)}, \quad t \in (0, T]$$

and

$$\begin{aligned} & \|u'(t)\|_{L_2(0, b) \times L_2(b, 1)} + \|u(t)\|_{W_2^2(0, b) \times W_2^2(b, 1)} \\ & \leq c \cdot t^{-1} \|\varphi_0\|_{W_2^2(0, b) \times W_2^2(b, 1)}, \quad t \in (0, T]. \end{aligned}$$

Proof: Let A denote the operator defined on $L_2(0, b) \times L_2(b, 1)$ by $Au(x) = -a(x)u''(x)$ with $D(A) = (W_2^2(0, b) \times W_2^2(b, 1), L_\nu u = 0, \nu = \overline{1, 4})$. Then problem (18)-(21), in $L_3(0, b) \times L_2(b, 1)$, can be rewritten as follows.

$$\begin{cases} u'(t) = Au(t) - Bu(\bar{t}) & (22) \\ u(0) = \varphi_0, & (23) \end{cases}$$

where $u(t) = u(t, \cdot)$ and $f(t) = f(t, \cdot)$ are functions with values in $L_2(0, b) \times L_2(b, 1)$, $\varphi_0 = \varphi_0(\cdot) \in L_2(0, b) \times L_2(b, 1)$. Using Theorem 3 in sector S_ε , we get $\|R(\lambda, A)\| \leq c|\lambda|^{-1}$, $|\lambda| \rightarrow \infty$. From hypothesis 1, the number $\varepsilon > 0$ can be chosen sufficiently small so that for some $\alpha > 0$ we have

$$\|R(\lambda, A)\| \leq c|\lambda|^{-1}, \quad |\arg \lambda| \leq \frac{\pi}{2} + \alpha, \quad |\lambda| \rightarrow \infty.$$

We know that operator B is compact, from $W_2^2(0, b) \times W_2^2(b, 1)$, into $L_2(0, b) \times L_2(b, 1)$, and operator $R(\lambda, A)$ is bounded, from $L_2(0, b) \times L_2(b, 1)$ into $W_2^2(0, b) \times W_2^2(b, 1)$ (by Theorem 3). Consequently, operator $T = BR(\lambda, A)$ is compact, from $L_2(0, b) \times L_2(b, 1)$ into $L_2(0, b) \times L_2(b, 1)$. Now, by Lemma 3, $D(A)$ is dense in $L_2(0, b) \times L_2(b, 1)$; and since the space $L_2(0, b) \times L_2(b, 1)$ has a basis and is reflexive, then, by Lemma 2.7 from [14] we have that for arbitrary $\varepsilon > 0$ and for arbitrary $u \in D(A)$

$$\begin{aligned} & \|Bu\|_{L_2(0, b) \times L_2(b, 1)} \\ & \leq \varepsilon \|(A - \lambda I)u\|_{L_2(0, b) \times L_2(b, 1)} + c(\varepsilon) \|u\|_{L_2(0, b) \times L_2(b, 1)}. \end{aligned}$$

Since $\varphi_0 \in D(A)$, then Lemma 2.7 from [14] can be applied to problem (22)-(23), which gives the desired result.

5.2 Completeness of Elementary Solutions

It is not difficult to show (see Lemma 0.1 from [14]) that a function U_j , given by the formula

$$U_j(t) = e^{\lambda_0 t} \left(\frac{t^k}{k!} u_0 + \frac{t^{k-1}}{(k-1)!} u_1 + \dots + \frac{t}{1!} u_{k-1} + u_k \right), \tag{24}$$

where $j = \overline{0, k}$, becomes a solution to equation (22) if and only if u_0, u_1, \dots, u_k is a chain of root functions of the operator $A + B$ corresponding to eigenvalue λ_0 . A solution of form (24) is called an elementary solution to equation (22).

Theorem 7: *Suppose that all conditions of Theorem 6 are satisfied. Then problem (18)-(21) has a unique solution:*

$$u \in C([0, T], L_2(0, b) \times L_2(b, 1)) \cap C^1((0, T], W_2^2(0, b) \times W_2^2(b, 1), L_2(0, b) \times L_2(b, 1));$$

and there exists a set of numbers, c_{jn} , such that

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \left\| u(t, \cdot) - \sum_{j=1}^n c_{jn} u_j(t, \cdot) \right\|_{L_2(0, b) \times L_2(b, 1)} = 0$$

and

$$\lim_{n \rightarrow \infty} \sup_{t \in (0, T]} t \left(\left\| u'_t(t, \cdot) - \sum_{j=1}^n c_{jn} u'_{jt}(t, \cdot) \right\|_{L_2(0, b) \times L_2(b, 1)} + \left\| u(t, \cdot) - \sum_{j=1}^n c_{jn} u_j(t, \cdot) \right\|_{W_2^2(0, b) \times W_2^2(b, 1)} \right) = 0,$$

where u is a solution of problem (18)-(21) and each u_j is an elementary solution of problem (18)-(20).

Proof: By Theorem 6, we get the existence and uniqueness of the solution to problem (18)-(21); and by Theorem 5, the completeness of root functions of problem (17) is guaranteed. Therefore, if we denote by $\lambda_j, j = \overline{1, \infty}$, the eigenvalues of problem (17), taking into consideration their order of algebraic multiplicity, there exists a set of numbers c_{jn} , such that

$$\lim_{n \rightarrow \infty} \left\| \varphi_0 - \sum_{j=1}^n c_{jn} u_{jk_j} \right\|_{W_2^2(0, b) \times W_2^2(b, 1)} = 0,$$

where $u_{j_0}, u_{j_1}, \dots, u_{j_k_j}$ form some chain of root functions of problem (17) corresponding to the eigenvalues λ_j . On the other hand, using Theorem 6, we find that problem (18)-(20), with the initial condition $u(0, x) = \varphi_0(x) - \sum_{j=1}^n c_{jn} u_{jk_j}(x)$, has a unique solution

$$(t, x) \rightarrow v(t, x) = u(t, x) - \sum_{j=1}^n c_{jn} u_j(t, x)$$

in the space

$$C([0, T], L_2(0, b) \times L_2(b, 1)) \cap C^1((0, T], W_2^2(0, b) \times W_2^2(b, 1), L_2(0, b) \times L_2(b, 1)).$$

We also have the following inequalities:

$$\left\| u(t, \cdot) - \sum_{j=1}^n c_{jn} u_j(t, \cdot) \right\|_{L_2(0, b) \times L_2(b, 1)} \leq c \left\| \varphi_0 - \sum_{j=1}^n c_{jn} u_{jk_j} \right\|_{W_2^2(0, b) \times W_2^2(b, 1)}$$

and

$$\left\| u'_t(t, \cdot) - \sum_{j=1}^n c_{jn} u'_{jt}(t, \cdot) \right\|_{L_2(0, b) \times L_2(b, 1)} + \left\| u(t, \cdot) - \sum_{j=1}^n c_{jn} u_j(t, \cdot) \right\|_{W_2^2(0, b) \times W_2^2(b, 1)}$$

$$\leq c \cdot t^{-1} \left\| \varphi_0 - \sum_{j=1}^n c_{jn} u_{jk_j} \right\|_{W_2^2(0,b) \times W_2^2(b,1)}.$$

Therefore, the proof of the theorem is complete.

References

- [1] Agranovich, M.S. and Vishik, M.I., Elliptic problems with a parameter and parabolic problems of general type, *Russian Math. Survey* **19** (1964), 53-159.
- [2] Besov, O.V., Il'in, V.P. and Nikolskii, S.M., *Integral Representations and Embedding Theorems*, Halsted Press, New York **I** 1978.
- [3] Danford, N. and Schwartz, J.T., *Linear Operators. Part II: Spectral Theory*, Interscience 1963.
- [4] Dezin, A.A., *General Questions of the Boundary Value Problem Theory*, Nauka, Moscow 1980 (in Russian).
- [5] Gohberg, I.C. and Krein, M.G., *Introduction to the Theory of Linear Non-Self-adjoint Operators*, Amer. Math. Soc., Providence, RI 1969.
- [6] Goureev, G.M. and Kovalenko, A.I., Completeness criterion for root subspaces of derivation operator with abstract boundary conditions, *Mat. Zametki* **4:30** (1981), 543-552 (in Russian).
- [7] Kato, T., *Perturbation Theory for Linear Operators*, Springer-Verlag 1966.
- [8] Krein, S.G., *Linear Differential Equations in Banach Space*, Providence 1971.
- [9] Naimark, M.A., *Linear Differential Operators*, Ungar, New York 1967.
- [10] Shkalikov, A.A., Boundary value problems for ordinary differential equations with a parameter in boundary conditions, *J. Soviet. Math.* **33** (1986), 1311-1342.
- [11] Shkalikov, A.A., Basis properties of root vectors for ordinary differential operators with integral boundary conditions, *Vestnik Mosk. Univ. Serie: Math-Mech. Series* **6** (1982), 12-21 (in Russian).
- [12] Triebel, H., *Interpolation Theory. Function Spaces. Differential Operators*, North-Holland, Amsterdam 1978.
- [13] Voitovich, N.N., Katsenelenbaum, B.Z. and Sivov, A.N., *Generalized Method of Eigen-Oscillations in the Diffraction Theory*, Nauka, Moscow 1977 (in Russian).
- [14] Yakubov, S.Ya., *Completeness of Root Functions of Regular Differential Operators*, Longman 1994.