Journal of Applied Mathematics and Stochastic Analysis, **13**:4 (2000), 347-364.

ON FILTERING OVER ÎTO-VOLTERRA OBSERVATIONS

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(Received September, 1998; Revised January, 2000)

In this paper, the Kalman-Bucy filter is designed for an Îto-Volterra process over Ito-Volterra observations that cannot be reduced to the case of a differential observation equation. The Kalman-Bucy filter is then designed for an Ito-Volterra process over discontinuous Ito-Volterra observations. Based on the obtained results, the filtering problem over discrete observations with delays is solved. Proofs of the theorems substantiating the filtering algorithms are given.

Key words: Kalman Filtering, Îto-Volterra Observations, Delayed Observations.

AMS subject classifications: 34K, 93D.

1. Introduction

Îto-Volterra processes and their applications to the optimal control theory have been studied from [1, 7]. The first optimal filter for an Îto-Volterra process over scalar observations given by a differential equation was designed in [8], and then the optimal filter was obtained [14] over vector observations.

Continuing the research initiated in [8, 14], this paper develops the optimal filter for an Îto-Volterra process over Îto-Volterra observations that cannot be reduced, unlike [8, 14], to the case of a differential observation equation. Based on the obtained filtering equations over continuous observations, the Kalman-Bucy filter is then designed for an Îto-Volterra process over discontinuous Îto-Volterra observations. In this sense, this paper follows the series of papers [2, 3, 5, 13] devoted to filtering over discontinuous observations. Some remarks concerning significance and applicability of these type of observations can be found there. As a consequence of the obtained filtering equations over discontinuous Îto-Volterra observations, the filtering problem over discrete observations with delays is also solved.

The paper is divided into two parts describing continuous and discontinuous observations, respectively. Proofs of theorems are given in the appendix.

2. Filtering Over Continuous Observations

2.1 Problem Statement

Let (Ω, F, P) be a complete probability space with an increasing right-continuous family of σ-algebras F_t, $t \geq 0$, and let $(W^1(t), F_t, t \geq 0)$ and $(W^2(t), F_t, t \geq 0)$ be independent Wiener processes. The partly observed F_t-measurable random process $(x(t), y(t))$ satisfies the Îto-Volterra equations

$$x(t) = \int_0^t (a_0(t,s) + a(t,s)x(s))ds + \int_0^t b(t,s)dW^2(s), \tag{1}$$

$$y(t) = \int_0^t (A_0(t,s) + A(t,s)x(s))ds + \int_0^t B(t,s)dW^1(s). \tag{2}$$

Here $x(t) \in R^n$ is a nonobserved component and $y(t) \in R^m$ is an observed one for the process $(x(t), y(t))$. Functions $a_0(t,s)$, $a(t,s)$, $b(t,s)$ are smooth in t uniformly in s and continuous in s, and functions $A_0(t,s)$, $A(t,s)$ and $B(t,s)$ are continuous in t,s. Let $A(t,s)$ be a nonzero matrix and $B(t,s)B^T(t,s)$ be a positive definite matrix. All coefficients in the equations (1) and (2) are considered deterministic functions.

The estimation problem is to find the best estimate for the Îto-Volterra process $x(t)$ at time t based on the observation process $Y(t) = \{y(s), 0 \leq s \leq t\}$, that is the conditional expectation $m(t) = E(x(t) \mid F_t^Y)$. Let $P(t) = E((x(t) - m(t)(x(t) - m(t))^T \mid F_t^Y)$ be the correlation function, where the symbol a^T means transposition of a vector (matrix) a.

The above statement generalizes the problem statement given in [8, 14] to the Îto-Volterra observation equation (2), which cannot be reduced to a differential equation. As in [8, 14], it is impossible to obtain a closed system of filtering equations for variables $m(t)$ and $P(t)$ due to the Volterra nature of the equations (1) and (2). Designing a closed filter requires introducing the additional function $f(t)$ characterizing a deviation of the best estimate $m(t)$ from the real state $x(t)$:

$$f(t,s) = E((x_s^t - m_s^t)(x(s) - m(s))^T \mid F_{t,s}^Y), \tag{3}$$

where

$$x_s^t = \int_0^s (a_0(t,r) + a(t,r)x(r))dr + \int_0^s b(t,r)dW^2(r), \tag{4}$$

$F_{t,s}^Y$ is the σ-algebra generated by the stochastic process y_s^t

$$y_s^t = \int_0^s (A_0(t,r) + A(t,r)x(r))dr + \int_0^s B(t,r)dW^1(r), \tag{5}$$

and

$$m_s^t = E(x_s^t \mid F_{t,s}^Y).$$

2.2 Optimal Filter

The optimal filter over continuous observations is given by the following theorem.

Theorem 1: *The best estimate* $m(t)$ *of the system state* (1) *over the observations* (2), *its correlation function* $P(t)$, *and the function* $f(t)$ *(see* (3)) *satisfy the following filtering equations:*

$$m(t) = \int_0^t (a_0(t,s) + a(t,s)m(s))ds \tag{6}$$

$$+ \int_0^t f(t,s)A^T(t,s)(B(t,s)B^T(t,s))^{-1}[dy(s) - (A_0(t,s) + A(t,s)m(s))ds],$$

$$P(t) = \int_0^t [a(t,s)f^T(t,s) + f(t,s)a^T(t,s) \tag{7}$$

$$+ b(t,s)b^T(t,s)]ds - \int_0^t f(t,s)A^T(t,s)(B(t,s)B^T(t,s))^{-1}A(t,s)f^T(t,s)ds,$$

$$f(t,s) = \int_0^s [a(s,r)f^T(t,r) + f(s,r)a^T(t,r) \tag{8}$$

$$+ (1/2)(b(t,r)b^T(s,r) + b(s,r)b^T(t,r))]dr$$

$$- \int_0^s [f(t,r)A^T(s,r)(B(s,r)B^T(s,r))^{-1}(s,r)f^T(s,r)$$

$$+ f(s,r)A^T(t,r)(B(t,r)B^T(t,r))^{-1}A(t,r)f^T(t,r)$$

$$- (1/2)f(t,r)A^T(t,r)(B(t,r)B^T(s,r))^{-1}A(s,r)f^T(s,r)$$

$$- (1/2)f(s,r)A^T(s,r)(B(s,r)B^T(t,r))^{-1}A(t,r)f^T(t,r)]dr.$$

Proofs of this and the following theorems are given in the appendix.

Remark: Let us note that a filtering problem similar to the considered one was treated in [9] for scalar state and observation equations with all coefficients depending on the observation process y an coefficient B not depending on t. The filtering equations obtained in [9] compose a system of integral-differential equations with respect to five filtering variables: two first-order moments (expectations) and three second-order ones (cross-correlation functions). This paper presents solution to the filtering problem in a multidimensional case, considering all coefficients dependent on t but independent of y. The last assumption enables one (as shown by (6)-(8)) to obtain the optimal filter as a system of integral equations closed with respect to only two filtering variables: the expectation $m(t)$ and the cross-correlation function $f(t,s)$, as it was done in [8, 14] for a differential observation equation. In particular, this enables one to apply conventional numerical algorithms to solving the obtained equations (6)-(8) as a regular Kalman-Bucy filter.

3. Filtering Over Discontinuous Observations

3.1 Problem Statement

Consider a generalization of the filtering problem examined in Section 2 to the case of discontinuous observations. Let the partly observed F_t-measurable random process $(x(t), y(t))$ be given by the following Îto-Volterra equations:

$$x(t) = \int_0^t (a_0(t,s) + a(t,s)x(s))ds + \int_0^t b(t,s)dW^2(s), \tag{9}$$

$$y_i(t) = \int_0^t (A_{0i}(t,s) + (A_i(t,s), x(s)))du_i(s)$$

$$+ \int_0^t B_i(t,s)dW_i^1(u_i(s)), i = 1,\dots,m, \tag{10}$$

where $A(t,s) = (A_1(t,s),\dots,A_m(t,s)), A_i(t,s) \in R^n, i = 1,\dots,m$; $B(t,s) = (B_1(t,s),\dots, B_m(t,s))^T$, $B_i(t,s) \in R^k$ is the ith row of the matrix $B(t,s)$; (a,b) is the scalar product in R^n, and the rest of the notation is the same as previous.

The observation process is characterized by a vector bounded variation function $u(t) = (u_1(t),\dots,u_m(t)) \in R^m$, which is nondecreasing in the following sense: $u(t_2) \geq u(t_1)$ as $t_2 \geq t_1$ if $u_i(t_2) \geq u_i(t_1)$ for $i = 1,\dots,m$. This model of observations enables one to consider continuous and discrete observations in the common form: continuous observations correspond to the continuous component of a bounded variation function $u(t)$, and discrete observations correspond to its function of jumps.

The estimation problem is formulated as in Section 2. All the remarks of Section 2 concerning the possibility of obtaining a closed system of filtering equations remains valid in this case. We also retain the notation of Section 2 for functions $f(t,s)$, x_s^t, m_s^t, and y_s^t.

3.2 Optimal Filter

In [13], the filtering procedure is suggested to obtain filtering equations over discontinuous observations proceeding from the known filtering equations over continuous ones. To apply the filtering procedure to the examined problem is to complete the following actions:

- assuming a vector function $u(t) \in R^m$ in an observation equation (10) to be absolutely continuous, write the Îto-Volterra filtering equations over continuous observations obtained in Section 2 (see (6)-(8));
- in thus obtained equations, assume a vector bounded variation function $u(t) \in R^m$ to be an arbitrary nondecreasing one again, keeping in mind that a derivative $\dot{u}(t)$ can be a generalized function of zero singularity order (for example, δ-function).

As a result, the following system of Îto-Volterra equations over discontinuous observations (10) is obtained:

$$m(t) = \int_0^t (a_0(t,s) + a(t,s)m(s))ds \tag{11}$$

$$+ \int_0^t f(t,s)A^T(t,s)(B(t,s)B^T(t,s))^{-1}[dy(s) - (A_0(t,s) + A(t,s)m(s))du(s)],$$

$$P(t) = \int_0^t [a(t,s)f^T(t,s) + f(t,s)a^T(t,s) + b(t,s)b^T(t,s)]ds \tag{12}$$

$$- \int_0^t f(t,s)A^T(t,s)(B(t,s)B^T(t,s))^{-1}A(t,s)f^T(t,s)du(s),$$

$$f(t,s) = \int_0^s [a(s,r)f^T(t,r) + f(s,r)a^T(t,r) \tag{13}$$

$$+ (1/2)(b(t,r)b^T(s,r) + b(s,r))b^T(t,r))]dr$$

$$- \int_0^s [f(t,r)A^T(s,r)(B(s,r)B^T(s,r))^{-1}A(s,r)f^T(s,r)$$

$$+ f(s,r)A^T(t,r)(B(t,r)B^T(t,r))^{-1}A(t,r)f^T(t,r)$$

$$- (1/2)f(t,r)A^T(t,r)(B(t,r)B^T(s,r))^{-1}A(s,r)f^T(s,r)$$

$$- (1/2)f(s,r)A^T(s,r)(B(s,r)B^T(t,r))^{-1}A(t,r)f^T(t,r)]du(r).$$

Here, multiplication by an m-dimensional measure $du(t)$ should be regarded in the componentwise sense, as in the observation equation (10).

The obtained equations (11)-(13) are integral equations with integration w.r.t. a vector discontinuous measure generated by a nondecreasing bounded variation function $u(t)$. Further investigation will follow the standard scheme suggested in [13]. It will be specified how to understand the solution of these equations and how to compute jumps of the solution at the discontinuity points of $u(t)$. Actually, these jumps reflect reaction of the filtering variables (the estimate $m(t)$ and its characteristics $P(t)$ and $f(t,s)$) to appearance of discrete measurements in the environment of a continuous signal. The final step is to prove that the introduced solution of the system (11)-(13) really yields the optimal estimate and its correlation characteristics.

3.3 Solution and Jumps: Theoretical Background

To avoid unnecessary complication of formulas, let us study an integral equation with integration w.r.t. a vector discontinuous measure generated by a nondecreasing bounded variation function in the general form and develop theoretical constructions for it. Namely, consider an integral equation in the form

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x,u,t,s)ds + b(x,u,t,s)du(s), \tag{14}$$

where functions $f(x,u,t,s)$ and $b(x,u,t,s)$ are continuous in x, u, t, s; $u(t) = (u_1(t), \ldots, u_m(t)) \in R^m$ is a nondecreasing bounded variation function. The set of discontinuity points of $u(t)$ is considered a countable set of isolated points.

The solution of the equation (14) is introduced as follows (cf. [13]).

Definition: The left-continuous function $x(t)$ is said to be a vibrosolution of the equation (14), if the $*$-weak convergence (see [10]) of an arbitrary sequence of absolutely continuous nondecreasing functions $u^k(t) \in R^m$ to a nondecreasing function $u(t) \in R^m$ in the bounded variation functions space

$$* - \lim u^k(t) = u(t)$$

implies the analogous convergence

$$* - \lim x^k(t) = x(t)$$

of corresponding solutions $x^k(t)$ of the equation

$$x^k(t) = x(t_0) + \int_{t_0}^{t} f(x^k,u^k,t,s)ds + b(x^k,u^k,t,s)du^k(s),$$

and the unique limit $x(t)$ is regardless of a choice of an approximating sequence $\{u^k(t)\}$, $k = 1,2,\ldots$.

The existence and uniqueness conditions for the vibrosolution of the equation (14) are given in the next theorem. Let us note that a vibrosolution is expected to be a function discontinuous at discontinuity points of $u(t)$.

Theorem 2: *Let*

(1) *functions* $f(x,u,t,s)$, $b(x,u,t,s)$, $\partial b(x,u,t,s)/\partial x$, $\partial b(x,u,t,s)/\partial t$, $\partial b(x,u,t,s)/\partial s$ *be continuous in* x,u,t,s;

(2) *functions* $f(x,u,t,s)$, $b(x,u,t,s)$ *satisfy the Lipschitz condition in* x;

(3) *the* $n \times m$*-dimensional system of differential equations in differentials*

$$\frac{d\xi}{du} = b(\xi,u,t,t), \quad \xi(\omega) = z, \tag{15}$$

is solvable on the cone of positive directions $K = \{u \in R^m : u_i \geq \omega_i, i = 1,\ldots,m\}$ *with arbitrary initial values* $\omega \in R^m$, $\omega \geq u(t_0)$, *and* $z \in R^n$.

Then there exists the unique vibrosolution of the equation (14).

Jumps of the vibrosolution of the equation (14) at the discontinuity points of the function $u(t)$ can be computed using the following equivalent equation with a measure.

Theorem 3: *Let the conditions of Theorem 2 hold. Then the integral equation* (14) *and the equivalent equation with a measure*

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x,u,t,s)ds + b(x,u,t,s)du^c(s)$$

$$+ \sum_{t_i} G(x(t_i-), u(t_i-), \Delta u(t_i), t_i) d\chi(t-t_i),$$

have the same unique solution regarded for the equation (14) *as a vibrosolution. Here* $G(x,\omega,u,t) = \xi(z,\omega,\omega+u,t) - z$ *and* $\xi(z,\omega,u,t)$ *is the solution of the system in differentials* (15)*;* $x(t_i-)$ *and* $x(t_i+)$ *are values of the function* $x(t)$ *at a discontinuity point* t_i *from the left and right, respectively;* $u^c(t)$ *is the continuous component of a nondecreasing function* $u(t)$*,* $\Delta u(t_i) = u(t_i+) - u(t_i-)$ *is the jump of a function* $u(t)$ *at* t_i*,* t_i *are the discontinuity points of a function* $u(t)$*,* $\xi(t-t_i)$ *is a Heaviside function.*

3.4 Equivalent Form of Filtering Equations

Thus, Theorem 2 yields existence and uniqueness of the vibrosolution $\{m(t), P(t), f(t,s)\}$ to the system of filtering equations (11)-(13), and Theorem 3 brings out the method for computing jumps of the vibrosolution $\{m(t), P(t), f(t,s)\}$ at the discontinuity points of the function $u(t)$ (i.e., at the discontinuity points of observations). Indeed, in view of Theorem 3, the equivalent equations with a measure for the filtering equations (11)-(13) take the form

$$m(t) = \int_0^t (a_0(t,s) + a(t,s)m(s))ds \tag{16}$$

$$+ \int_0^t f(t,s-)[I + A^T(t,s)(B(t,s)B^T(t,s))^{-1}$$

$$\times A(t,s)f(t,s-)\Delta u(s)]^{-1} A^T(t,s)(B(t,s)B^T(t,s))^{-1}$$

$$\times [dy(s) - (A_0(t,s) + A(t,s)m(s-))du(s)],$$

$$P(t) = \int_0^t [a(t,s)f^T(t,s) + f(t,s)a^T(t,s) \tag{17}$$

$$+ b(t,s)b^T(t,s)]ds - \int_0^t f(t,s-)[I + A^T(t,s)$$

$$\times (B(t,s)B^T(t,s))^{-1} A(t,s)f(t,s-)\Delta u(s)]^{-1}$$

$$\times A^T(t,s)(B(t,s)B^T(t,s))^{-1}A(t,s)f^T(t,s-)du(s),$$

$$f(t,s) = \int_0^s [a(s,r)f^T(t,r) + f(s,r)a^T(t,r) \tag{18}$$

$$+ (1/2)(b(t,r)b^T(s,r) + b(s,r)b^T(t,r)]dr$$

$$- \int_0^s [f(t,r-)[I + (A^T(s,r)(B(s,r)B^T(s,r))^{-1}A(s,r)f(s,r-)$$

$$+ A^T(t,r)(B(t,r)B^T(t,r))^{-1}A(t,r)f(t,r-)$$

$$- (1/2)A^T(s,r)(B(s,r)B^T(t,r))^{-1}A(t,r)f(t,r-)$$

$$- (1/2)A^T(t,r)(B(t,r)B^T(s,r))^{-1}A(s,r)f(s,r-))\Delta u(r)]^{-1}$$

$$\times A^T(s,r)(B(s,r)B^T(s,r))^{-1}A(s,r)f^T(s,r-)$$

$$+ f(s,r-)[I + (A^T(s,r)(B(s,r)B^T(s,r))^{-1}A(s,r)f(s,r-)$$

$$+ A^T(t,r)(B(t,r)B^T(t,r))^{-1}A(t,r)f(t,r-)$$

$$- (1/2)A^T(s,r)(B(s,r)B^T(t,r))^{-1}A(t,r)f(t,r-)$$

$$- (1/2)A^T(t,r)(B(t,r)B^T(s,r))^{-1}A(s,r)f(s,r-))\Delta u(r)]^{-1}$$

$$\times A^T(t,r)(B(t,r)B^T(t,r))^{-1}A(t,r)f^T(t,r-)$$

$$- (1/2)f(s,r-)[I + (A^T(s,r)(B(s,r)B^T(s,r))^{-1}A(s,r)f(s,r-)$$

$$+ A^T(t,r)(B(t,r)B^T(t,r))^{-1}A(t,r)f(t,r-)$$

$$- (1/2)A^T(s,r)(B(s,r)B^T(t,r))^{-1}A(t,r)f(t,r-)$$

$$- (1/2)A^T(t,r)(B(t,r)B^T(s,r))^{-1}A(s,r)f(s,r-))\Delta u(r)]^{-1}$$

$$\times A^T(s,r)(B(s,r)B^T(t,r))^{-1}A(t,r)f^T(t,r-)$$

$$- (1/2)f(t,r-)[I + (A^T(s,r)(B(s,r)B^T(s,r))^{-1}A(s,r)f(s,r-)$$

$$+ A^T(t,r)(B(t,r)B^T(t,r))^{-1}A(t,r)f(t,r-)$$

$$- (1/2)A^T(s,r)(B(s,r)B^T(t,r))^{-1}A(t,r)f(t,r-)$$

$$- (1/2)A^T(t,r)(B(t,r)B^T(s,r))^{-1}A(s,r)f(s,r-))\Delta u(r)]^{-1}$$

$$\times A^T(t,r)(B(t,r)B^T(s,r))^{-1}A(s,r)f^T(s,r-)]du(r),$$

where I is the $n \times n$-dimensional identity matrix. The function $f(t,s)$ is continuous in t.

It is readily verified that jumps of the filtering variables $m(t)$, $P(t)$, and $f(t,s)$ at a discontinuity point t_i of $u(t)$ are equal to the expressions under the integral signs in the right-hand sides of (16)-(18), upon substituting the jumps $\Delta u(t_i)$ and $\Delta y(t_i)$ for differentials $du(t)$ and $dy(t)$, respectively.

The equations with a measure (16)-(18) completely determine the behavior of the filtering variables $m(t)$, $P(t)$ and $f(t,s)$, i.e., the complete reaction of the filtering variables to a composition of continuous and discrete measurements. The next optimality theorem is the final step in solution of the filtering problem for an Îto-Volterra process over Îto-Volterra discontinuous observations.

Theorem 4: *The solutions $m(t)$, $P(t)$ and $f(t,s)$ of the equations* (16)-(18) *are the optimal estimate in the filtering problem* (9), (10), *its correlation function, and its correlation characteristic* (3), *respectively.*

3.5 Filtering Over Discrete Observations with Delays

Finally, consider the filtering problem for an Îto-Volterra process over discrete observations with delays, whose general solution has not been published previously. Let the state equation be the same as (9) and the observation equation be as follows:

$$y(t_j) = A_0(t_j,t_i) + A(t_j,t_i)x(t_i) + B(t_j,t_i)\psi(t_i),$$

where $y(t_j) \in R^m$ are discrete observations at time moments t_j, $j = 0,1,\ldots, x(t_i)$ are values of the system state as moments t_i available at the observation moments t_j, $A(t_j,t_i) \in R^{m \times n}$ are transition matrices, and $\psi(t_i)$ are independent Gaussian noises acting at the moments t_i. We consider the model of discrete observations with one time delay $t_j - t_i$, although the observation equation (10) allows a set of delays up to the power of continuum.

In view of the equations (16)-(18), the optimal estimate $m(t)$, correlation function $P(t)$, and correlation characteristic (3) $f(t,s)$ satisfy the following equations between the observation moments t_j

$$m(t) = m(t_j+) + \int_{t_j+}^{t} (a_0(t,s) + a(t,s)m(s))ds,$$

$$P(t) = P(t_j+) + \int_{t_j+}^{t} [a(t,s)f^T(t,s) + f(t,s)a^T(t,s) + b(t,s)b^T(t,s)]ds,$$

$$f(t,s) = f(t,t_j+) + \int_{t_j+}^{s} [a(s,r)f^T(t,r) + f(s,r)a^T(t,r)$$

$$+ (1/2)(b(t,r)b^T(s,r) + b(s,r)b^T(t,r))]dr,$$

and their jumps at the moments t_j of discrete observations are equal to

$$\Delta m(t_j) = f(t_j, t_j -)[I + A^T(t_j, t_j)(B(t_j, t_j)B^T(t_j, t_j))^{-1}$$
$$\times A(t_j, t_j)f(t_j, t_j -)]^{-1}A^T(t_j, t_j)(B(t_j, t_j)B^t(t_j, t_j))^{-1}y(t_j)$$
$$- f(t_j, t_i -)[I + A^T(t_j, t_i)(B(t_j, t_i)B^T(t_j, t_i))^{-1}$$
$$\times A(t_j, t_i)f(t_j, t_i -)]^{-1}A^T(t_j, t_i)(B(t_j, t_i)B^T(t_j, t_i))^{-1}$$
$$\times (A_0(t_j, t_i) + A(t_j, t_i)m(t_i -)),$$
$$\Delta P(t_j) = f(t_j, t_i -)[I + A^T(t_j, t_i)$$
$$\times (B(t_j, t_i)B^T(t_j, t_i))^{-1}A(t_j, t_i)f(t_j, t_i -)]^{-1}$$
$$\times A^T(t_j, t_i)(B(t_j, t_i)B^T(t_j, t_i))^{-1}A(t_j, t_i)f^T(t_j, t_i -),$$
$$\Delta f(t, t_j) = [f(t, t_i -)[I + (A^T(t_j, t_i)(B(t_j, t_i)B^T(t_j, t_i))^{-1}A(t_j, t_i)f(t_j, t_i -)$$
$$+ A^T(t, t_i)(B(t, t_i)B^T(t, t_i))^{-1}A(t, t_i)f(t, t_i -)$$
$$- (1/2)A^T(t_j, t_i)(B(t_j, t_i)B^T(t, t_i))^{-1}A(t, t_i)f(t, t_i -)$$
$$- (1/2)A^T(t, t_i)(B(t, t_i)B^T(t_j, t_i))^{-1}A(t_j, t_i)f(t_j, t_i -))]^{-1}$$
$$\times A^T(t_j, t_i)(B(t_j, t_i)B^T(t_j, t_i))^{-1}A(t_j, t_i)f^T(t_j, t_i -)$$
$$+ f(t_j, t_i -)[I + (A^T(t_j, t_i)(B(t_j, t_i)B^T(t_j, t_i))^{-1}A(t_j, t_i)f(t_j, t_i -)$$
$$+ A^T(t, t_i)(B(t, t_i)B^T(t, t_i))^{-1}A(t, t_i)f(t, t_i -)$$
$$- (1/2)A^T(t_j, t_i)(B(t_j, t_i)B^T(t, t_i))^{-1}A(t, t_i)f(t, t_i -)$$
$$- (1/2)A^T(t, t_i)(B(t, t_i)B^T(t_j, t_i))^{-1}A(t_j, t_i)f(t_j, t_i -))]^{-1}$$
$$\times A^T(t, t_i)(B(t, t_i)B^t(t, t_i))^{-1}A(t, t_i)f^T(t, t_i -)$$
$$- (1/2)f(t_j, t_i -)[I + (A^T(t_j, t_i)(B(t_j, t_i)B^T(t_j, t_i))^{-1}A(t_j, t_i)f(t_j, t_i -)$$
$$+ A^T(t, t_i)(B(t, t_i)B^T(t, t_i))^{-1}A(t, t_i)f(t, t_i -)$$
$$- (1/2)A^T(t_j, t_i)(B(t_j, t_i)B^T(t, t_i))^{-1}A(t, t_i)f(t, t_i -)$$
$$- (1/2)A^T(t, t_i)(B(t, t_i)B^T(t_j, t_i))^{-1}A(t_j, t_i)f(t_j, t_i -))]^{-1}$$
$$\times A^T(t_j, t_i)(B(t_j, t_i)B^T(t, t_i))^{-1}A(t, t_i)f^T(t, t_i -)$$
$$- (1/2)f(t, t_i -)[I + (A^T(t_j, t_i)(B(t_j, t_i)B^T(t_j, t_i))^{-1}A(t_j, t_i)f(t_j, t_i -)$$
$$+ A^T(t, t_i)(B(t, t_i)B^T(t, t_i))^{-1}A(t, t_i)f(t, t_i -)$$

$$- (1/2)A^T(t_j,t_i)(B(t_j,t_i)B^T(t,t_i))^{-1}A(t,t_i)f(t,t_i-)$$

$$- (1/2)A^T(t,t_i)(B(t,t_i)B^T(t_j,t_i))^{-1}A(t_j,t_i)f(t_j,t_i-))]^{-1}$$

$$\times A^T(t,t_i)(B(t,t_i)B^T(t_j,t_i))^{-1}A(t_j,t_i)f^T(t_j,t_i-)].$$

Thus, the solution of the filtering problem for an Îto-Volterra process over discrete observations with delays readily follows from the solution of the filtering problem over discontinuous observations. This gives us one more point for significance of the model of discontinuous observations (10).

4. Conclusion

This paper presents an addition to the Kalman-Bucy filtering theory, which is related to filtering over observations given by Îto-Volterra equations. The filtering equations have been obtained first over continuous observations, then over discontinuous ones, for the model of discontinuous observations enables one to consider continuous and discrete observations in the common form. Solution of the filtering problem over discontinuous observations has allowed us to solve the filtering problem over discrete observations with delays, whose general solution has not been previously published.

5. Appendix

Proof of Theorem 1: Let us consider the filtering problem for the state x_s^t over the observation process y_s^t. The equation (4) for x_s^t and the equation (5) for y_s^t are actually differential equations with respect to s, where t is the parameter. Therefore, the principal filtering theorem (the correlation theorem for conditionally Gaussian processes, see Theorem 8.6 in [11]) is applicable, and we obtain

$$m_s^t = \int_0^s (a_0(t,r) + a(t,r)m(r))dr$$

$$+ \int_0^s E(x_r^t x^T(r) - m_r^t m^T(r) \mid F_{t,r}^Y)A^T(t,r)(B(t,r)B^T(t,r))^{-1}$$

$$\times [dy_r^t - (A_0(t,r) + A(t,r)m(r))dr].$$

In view of nonstochastic coefficients in the observation equation, the innovations process $dy_s^t - A_0(t,s)ds - A(t,s)m(s)ds$ generates the same σ-algebra for any t, and, therefore, can be replaced by the innovations process $dy(s) - A_0(t,s)ds - A(t,s)m(s)ds$. Thus, equating $s = t$ yields, in view of (3), the equations (6) for $m(t)$.

Let us now prove the equation (8) for $f(t,s)$. The Îto formula yields

$$d(x_r^t - m_r^t) = [a(t,r) - f(t,r)A^T(t,r)(B(t,r)B^T(t,r))^{-1}A(t,r)]$$

$$\times (x(r) - m(r))dr + b(t,r)dW^2(r)$$

$$-f(t,r)A^T(t,r)(B(t,r)B^T(t,r))^{-1}B(t,r)dW^1(r),$$

and

$$d((x_r^t - m_r^t)(x_r^u - m_r^u)^T) = (x_r^t - m_r^t)(x(r) - m(r))^T$$
$$\times [a(u,r) - f(u,r)A^T(u,r)(B(u,r)B^T(u,r))^{-1}A(u,r)]^T dr$$
$$+ [a(t,r) - f(t,r)A^T(t,r)(B(t,r)B^T(t,r))^{-1}$$
$$\times A(t,r)](x(r) - m(r))(x_r^u - m_r^u)^T dr$$
$$+ (1/2)(b(t,r)b^T(u,r) + b(u,r)b^T(t,r))dr$$
$$+ (1/2)f(t,r)A^T(t,r)(B(t,r)B^T(u,r))^{-1}A(u,r)f^T(u,r)dr$$
$$+ (1/2)f(u,r)A^T(u,r)(B(u,r)B^t(t,r))^{-1}A(t,r)f^T(t,r)]dr$$
$$-(x_r^t - m_r^t)[f(u,r)A^T(u,r)(B(u,r)B^T(u,r))^{-1}B(u,r)dW^1(r) - b(u,r)dW^2(r)]^T$$
$$-[f(t,r)A^T(t,r)(B(t,r)B^T(t,r))^{-1}B(t,r)dW^1(r) - b(t,r)dW^2(r)](x_r^u - m_r^u)^T.$$

Integrating with r from 0 to s, equating $u = s$, and using Theorem 8.6 in [11], we obtain

$$E((x_s^t - m_s^t)(x(s) - m(s))^T \mid F_{t,s}^Y)$$
$$= \int_0^s [E((x_r^t - m_r^t)(x(r) - m(r))^T \mid F_{t,r}^Y)a^T(s,r)$$
$$+ a(t,r)E((x(r) - m(r))(x_r^s - m_r^s)^T \mid F_{s,r}^Y)$$
$$+ (1/2)(b(t,r)b^T(s,r) + b(s,r)b^T(t,r))$$
$$+ (1/2)f(t,r)A^T(t,r)(B(t,r)B^T(s,r))^{-1}A(s,r)f^T(s,r)$$
$$+ (1/2)f(s,r)A^T(s,r)(B(s,r)B^T(t,r))^{-1}A(t,r)f^T(t,r)$$
$$- E((x_r^t - m_r^t)(x(r) - m(r))^T \mid F_{t,r}^Y)A^T(s,r)(B(s,r)B^T(s,r))^{-1}A(s,r)f(s,r)$$
$$- f(t,r)A^T(t,r)(B(t,r)B^T(t,r))^{-1}A(t,r)E((x(r) - m(r))(x_r^s - m_r^s)^T \mid F_{s,r}^Y)]dr$$
$$+ \int_0^s [E((x_r^t - m_r^t)(x_r^s - m_r^s)^T(x(r) - m(r)) \mid F_{t,r}^Y)]$$
$$\times A^T(t,r)(B(t,r)B^T(t,r))^{-1}[dy_r^t - (A_0(t,r) + A(t,r)m(r))dr].$$

Let us note that the latter term is equal to zero because all the processes under the expectation sign are zero mean conditionally Gaussian. Thus, in view of (3), the last equation yields the equation (8). Finally, the equation (7) follows from (8), because $P(t) = f(t,t)$.

Proof of Theorem 2: By virtue of the theorem conditions, the equation (14) has the unique absolutely continuous solution on the continuity intervals of the function $u(t)$ (see [7]). Moreover, this absolutely continuous solution is a vibrosolution, in view of the Lebesgue bounded convergence theorem (see [10]) which yields existence of the limit required in the definition of a vibrosolution on the continuity intervals of $u(t)$. Thus, it remains to prove existence of the vibrosolution of (14) only in neighborhoods of the isolated discontinuity points t_i, $i = 1, 2, \ldots$, of the function $u(t)$.

In accordance with the theorem conditions, the system (15) has the solution $\xi(z, \omega, u, t_i)$ on the cone K, where t_i is an isolated discontinuity point of $u(t)$. Let us seek the solution of (14) corresponding to a nondecreasing function $u(t)$ in the form

$$x(t) = \xi(z(t), u_i, u(t), t_i), \tag{19}$$

where $u_i = u(t_i)$.

In accordance with the definition of a solution of the system in differentials (15), the expression (19) implies the representation

$$x(t) = z(t) + \int_{u_i}^{u(t)} b(\xi(v), v, t_i, t_i)dv,$$

or

$$x(t) = z(t) + \int_0^T b(z(t) + y(r), u_i + w(r), t_i, t_i)\dot{w}(r)dr, \tag{20}$$

where T is the time, for which the trajectory of (15) reaches the point $x(t)$, and $y(r)$ is the solution of (15) corresponding to the nondecreasing function $w(r) = u(r) - u_i$.

The solvability of the system in differentials (15) on the cone K implies (see [6] for further details) that the integral form

$$\int_0^{T'} b(z(t) + y(r), u_i + w(r), t_i, t_i)\dot{w}(r)dr \tag{21}$$

is equal to zero for any nonnegative function $w(r)$ in R^m: $w(r) \in R^m$, $w_j(r) \geq 0$, $j = 1, \ldots, m$, which is piecewise smooth on the interval $[0, T']$ and equal to zero at its terminal points 0 and T'. In other words, the integral form (21) is equal to zero for any piecewise smooth loop $w(r) \in R^m$ inside the nonnegative orthant of R^m, which starts and ends at zero. Here, T' is the time of passing the loop.

Let $w(r)$ be such a piecewise smooth loop in R^m that the corresponding solution of (15) with the initial value $z(t)$ reaches the point $x(t)$ for the time T, where $w(r) \geq 0$. Then, the equality (21) takes the form

$$\int_0^T b(z(t) + y(r), u_i + w(r), t_i, t_i)\dot{w}(r)dr$$

$$+\int_{T}^{T'} b(z(t)+y(r),u_i+w(r),t_i,t_i)\dot{w}(r)dr$$

$$=\int_{0}^{T} b(z(t)+y(r),u_i+w(r),t_i,t_i)\dot{w}(r)dr$$

$$+\int_{0}^{T'-T} b(x(t)+y(r),u(t)+w(r),t_i,t_i)\dot{w}(r)dr=0, \tag{22}$$

where $u(t)=u_i+w(T)$.

Upon substituting the representation (20) into (22), we obtain

$$x(t)-z(t)+\int_{0}^{T'-T} b(x(t)+y(r),u(t)+w(r),t_i,t_i)\dot{w}(r)dr=0,$$

i.e.,

$$z(t)=x(t)+\int_{0}^{T'-T} b(x(t)+y(r),u(t)+w(r),t_i,t_i)\dot{w}(r)dr,$$

or

$$z(t)=x(t)+\int_{u(t)}^{u_i} b(\xi(v),v,t_i,t_i)dv. \tag{23}$$

The representation (23) implies that the inversion formula

$$z(t)=\xi(x(t),u(t),u_i,t_i) \tag{24}$$

is valid for $u(t)\geq u_i$. In particular, $z(t_i)=x(t_i)$.

Based on the existence of the derivatives $\partial\xi/\partial z$, $\partial\xi/\partial t$, $\partial\xi/\partial v$, and solvability of the system in differentials (15) (the first and third conditions of this theorem), we obtain, using the transforming technique from [12], that $z(t)$ satisfies the equation with a discontinuous right-hand side

$$\dot{z}(t)=\varphi(z(t),u_i,u(t),t_i),\quad z(t_i-)=x(t_i-) \tag{25}$$

where

$$\varphi(z,u_i,u,t)=\frac{\partial\xi(\xi(z,u_i,u,t),u,u_i,t)}{\partial z}$$

$$\times f(\xi(z,u_i,u,t),u,t)+\frac{\partial\xi(\xi(z,u_i,u,t),u,u_i,t)}{\partial t}.$$

The function $\varphi(z,u_i,u,t)$ is continuous as a combination of the continuous functions $\partial\xi/\partial z$, f, and $\partial\xi/\partial t$. Thus, a solution of the equation (25) exists.

Let $u^k(t)$, $k=1,2,\ldots$, where $*-\lim u^k(t)=u(t)$, $k\to\infty$, $t\geq t_i$, be a sequence of absolutely continuous nondecreasing functions converging to $u(t)$ in the $*$-weak topology of the bounded variation functions space. The equation (14) with functions $u^k(t)$ in the right-hand side becomes an ordinary integral equation without integration w.r.t. a discontinuous measure

$$x^k(t) = x(t_i -) + \int_{t_i -}^{t} f(x^k, u_k, t, s)ds + b(x^k, u^k, t, s)du^k(s), \tag{26}$$

In view of the first and second conditions of this theorem, there exists the unique solution of the equation (26). The inverse formula (24) implies that $\xi(x^k(t), u^k(t), u_i, t_i) = z^k(t)$ is the unique solution of the equation

$$\dot{z}^k(t) = \varphi(z^k(t), u_i, u_k(t), t_i), \quad z^k(t_i -) = x(t_i -). \tag{27}$$

Let $* - \lim u^k(t) = u(t)$, $k \to \infty$, $t \geq t_i$. Based on the continuous dependence of a solution of a differential equation on its right-hand side, we obtain that $* - \lim z^k(t) = z^*(t)$, $k \to \infty$, $t \geq t_i$, where $z^*(t)$ is a solution of the equation (25). This solution is unique due to uniqueness of the solutions $z^k(t)$ of (27) for pre-limiting functions $u^k(t)$. Thus, $z^*(t)$ is the vibrosolution of (25). Based on the continuity and one-to-one correspondence of the relation (19), we conclude that $x^*(t) = \xi(z^*(t), u_i, u(t), t_i)$ is the desired vibrosolution of the equation (14). Moreover, $\sup\limits_k \mathrm{Var}[t_i, t] x^k(t) < \infty$ for $t \geq t_i$, in view of uniform boundedness of the variations of the functions $z^k(t)$ and $u^k(t)$, $k = 1, 2, \ldots$. Uniform boundedness of the variations of the functions $z^k(t)$ and $u^k(t)$ follows from the convergence

$$* - \lim u^k(t) = u(t), \quad * - \lim z^k(t) = z^*(t), \; k \to \infty, \; t \geq t_i,$$

in the $*$-weak topology of the bounded variation functions space.

Proof of Theorem 3: A proposition similar to Theorem 3 is proved in [12] for a vibrosolution of an ordinary differential equation in distributions. This proof can be repeated here using the existence and uniqueness theorem for a solution of an integral Volterra equation (see [7]), instead of that for a solution of an ordinary differential one. The rest of the proof can be carried over from [12].

Proof of Theorem 4: The last step is to prove optimality of the solutions of the equations (16)-(18) as filtering variables.

Define the functions $\alpha(t)$, $\beta(s)$, $\gamma(s)$ by:

$$\alpha(t) = t + \| u(t) - u(0) \|, \quad \beta(r) = \inf\{t : \alpha(t) > r\},$$

$$\gamma(r) = u(t-) + [\Delta u(t) / \| \Delta u(t) \|][r - \alpha(t-)], \text{ if } \beta(r) = t \notin D,$$

$$\gamma(r) = u(\beta(r)), \text{ if } \beta(r) \in D,$$

where D is set of continuity points of the function $u(t)$, $\Delta u(t)$ is the jump of the nondecreasing bounded variation function $u(t)$ at its discontinuity point t, and the symbol $\| \cdot \|$ denotes the Euclidean norm in R^m. Note that the functions $\gamma(r)$, $\beta(r)$ are absolutely continuous in r. Introduce a state vector $x''(r)$ and an observation process $y''(r)$ as solutions of the equations

$$x''(r) = \int_0^r (a_0(r,\beta(q)) + a(r,\beta(q))x''(q))\dot{\beta}(q)dq$$
$$+ \int_0^r b(r,\beta(q))dW^2(\beta(q)), \tag{28}$$

$$y_i''(r) = \int_0^r (A_{0i}(r,\beta(q)) + (A_i(r,\beta(q)), x''(q)))\dot{\gamma}_i(q)dq$$

$$+ \int_0^r B_i(r,\beta(q))dW_i^1(\gamma_i(q)). \tag{29}$$

The functions $x''(\alpha(t)), y''(\alpha(t))$ obviously satisfy the equations (9), (10), and therefore the following equalities hold:

$$x(t) = x''(\alpha(t)),\ y(t) = y''(\alpha(t)).$$

Hence, the filtering variables $m''(\alpha(t)), P''(\alpha(t))$, and $f''(t,\alpha(s))$ should be the optimal filtering variables in the initial filtering problem (9), (10).

Since functions $\beta(r)$, $\gamma(r)$ are absolutely continuous in r, the right-hand sides of the equations (28), (29) contain only continuous functions. Thus, the filtering equations for the state $x''(r)$ over the continuous observations $y''(r)$, which were obtained in Section 2 (see (6), (8)) take the form

$$m''(r) = \int_0^r (a_0(r,\beta(q)) + a(r,\beta(q))m''(q))dq \tag{30}$$

$$+ \int_0^r f''(r,q)^T(r,\beta(q))(B(r,\beta(q))B^T(r,\beta(q)))^{-1}$$

$$\times [dy''(q) - (A_0(r,\beta(q)) + A(r,\beta(q))m''(q))\dot{\gamma}(q)dq],$$

$$f''(r,p) = \int_0^p [a(p,\beta(q))f''^T(r,q) + f''(p,q)a^T(r,\beta(q)) \tag{31}$$

$$+ (1/2)(b(r,\beta(q))b^T(p,\beta(q)) + b(p,\beta(q))b^T(r,\beta(q)))]dr$$

$$- \int_0^p [f''(r,q)A^T(p,\beta(q))(B(p,\beta(q))B^T(p,\beta(q)))^{-1}$$

$$\times A(p,\beta(q))f''^T(p,q) + f''(p,q)A^T(r,\beta(q))(B(r,\beta(q))$$

$$\times B^T(r,\beta(q)))^{-1}A(r,\beta(q))f''^T(r,q) - (1/2)f''(r,q)A^T(r,\beta(q))$$

$$\times (B(r,\beta(q))B^T(p,\beta(q)))^{-1}A(p,\beta(q))f''^T(p,q)$$

$$-(1/2)f''(p,q)A^T(p,\beta(q))(B(p,\beta(q))B^T(r,\beta(q)))^{-1}$$

$$\times A(r,\beta(q))f''^T(r,q)]\dot{\gamma}(q)dq,$$

where $m''(s)$ is the optimal estimate (conditional expectation) of the state $x''(r)$ over the continuous observations $y''(r)$ and $f''(r,p)$ is its correlation characteristic (3). (The equation for $P''(r)$ is not used here.)

In view of the equations (30), (31), the variables $m''(\alpha(t))$ and $f''(t,\alpha(s))$ satisfy the equations (16), (18) in the continuity intervals of $u(t)$. Moreover, if $r \in [\alpha(t_i-),\alpha(t_i+)]$ and $p \in [\alpha(s_i-),\alpha(s_i+)]$, where t_i and s_i are discontinuity points of the function $u(t)$, then, in view of Theorem 3, jumps $m''(\alpha(t_i+))-m''(\alpha(t_i-))$ and $f''(t_i,\alpha(s_i+))-f''(t_i,\alpha(s_i-))$ of the variables $m''(\alpha(t))$ and $f''(t,\alpha(s))$ respectively coincide with jumps $m(t_i+)-m(t_i-)$ and $f(t_i,s_i+)-f(t_i,s_i-)$ of the solutions $m(t)$ and $f(t,s)$ of (16), (18). Thus, the optimal filtering variables $m''(\alpha(t))$ and $f''(t,\alpha(s))$ are solutions of (16), (18) everywhere. Finally, the equation (17) follows from the equation (18) because $P(t)=f(t,t)$.

Acknowledgement

The author would like to thank the reviewers of this manuscript and the associate and principal editors for their notes and comments, as well as their patience, which enabled the author to significantly improve the paper and finally make it publishable.

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