THE ASYMPTOTIC BEHAVIOR OF ELEMENTARY SYMMETRIC FUNCTIONS ON A PROBABILITY DISTRIBUTION

V.S. KOZYAKIN
Institute for Information Transmission Problems
19 Bolshoi Karetny Lane
Moscow 101447, Russia

A.V. POKROVSKII[1]
National University of Ireland
Institute for Nonlinear Science, Department of Physics
University College, Cork, Ireland

The problem on asymptotic of the value

$$\pi(m,n) = m!\sigma_m(p(1,n), p(2,n), \ldots, p(n,n))$$

is considered, where $\sigma_m(x_1, x_2, \ldots, x_n)$ is the mth elementary symmetric function of n variables. The result is interpreted in the context of non-equiprobable random mappings theory.

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1. Statement of the Problem

Let

$$p(k,n) = \int_{\frac{k-1}{n}}^{\frac{k}{n}} q(t)dt, \quad k = 1, 2, \ldots, n, \tag{1}$$

with

$$\int_0^1 q(t)dt = 1 \tag{2}$$

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[1]Permanent address: Institute of Information Transmission Problems, Russian Academy of Science, 19 Bolshoi Karetny Lane, Moscow 101447, Russia.

where

$$q(t) = \alpha\beta(t)t^{\gamma-1} \geq 0, \quad \beta(0) = 1, \quad \alpha > 0, \quad 0 < \gamma \leq 1 \tag{3}$$

and the function $\beta(t)$ is continuous on the interval $[0,1]$.

The problem on asymptotic of the value

$$\pi(m,n) = m!\sigma_m(p(1,n), p(2,n), \ldots, p(n,n))$$

is considered, where $\sigma_m(x_1, x_2, \ldots, x_n)$ is the mth elementary symmetric function of n variables. This problem arises in particular in the analysis of the distribution of transient time and other combinatorial characteristics in random nonequiprobable graphs [7] whose distributions have proved essential to the analysis of discretizations of dynamic systems with quasi-chaotic behavior. See [3, 6] and references therein.

Theorem 1: *The following relations are valid for $n \to \infty$*

$$\pi(m,n) \to e^{-\frac{1}{2}\left(\int_0^1 q^2(t)dt\right)x^2} \quad as \quad \frac{m}{\sqrt{n}} \to x, \quad \frac{1}{2} < \gamma \leq 1; \tag{4}$$

$$\pi(m,n) \to e^{-\frac{1}{2}\alpha^2 x^2} \quad as \quad \frac{m}{\sqrt{n(\log n)^{-1}}} \to x, \quad \gamma = \frac{1}{2}; \tag{5}$$

$$\pi(m,n) \to \pounds(\gamma, \alpha x) \quad as \quad \frac{m}{n^\gamma} \to x, \quad 0 < \gamma < \frac{1}{2} \tag{6}$$

where $\pounds(\gamma, z)$ is the entire analytic function defined by the infinite product

$$\pounds(\gamma, z) = \prod_{k=1}^{\infty} \left\{ \left[1 + \frac{k^\gamma - (k-1)^\gamma}{\gamma}z\right] e^{-\frac{k^\gamma - (k-1)^\gamma}{\gamma}z} \right\}. \tag{7}$$

For $0 < \gamma < \frac{1}{2}$ and for the values $|x| < \frac{\gamma}{\alpha}$ the following representation

$$\pi(m,n) \to e^{-\sum_{r=2}^{\infty} \frac{(-1)^r}{r}\left\{\sum_{k=1}^{\infty}\left(\frac{k^\gamma - (k-1)^\gamma}{\gamma}\right)^r\right\}(\alpha x)^r} \quad as \quad \frac{m}{n^\gamma} \to x$$

is also valid.

Note a specification of the theorem above in the spirit of random mappings theory [1]. Denote $\Xi(n) = \{1, 2, \ldots, n\}$ for a positive integer n. Let also the set $\Xi(n)$ be endowed with a probability measure $\mu_{n,q}$ given by equalities $\mu_{n,q}(k) = p(k,n)$, where the function q and the numbers $p(k,n)$ are as in (1). Consider now the random mapping $F_{q,n}$ defined by

$$\mathbb{P}_\mu(F = f) = \prod_{k=1}^{n} p(f(k), n).$$

Generally speaking, this means that the "appeal" of a point n is proportional to its weight $p(k,n)$. If $q \equiv 1$, then $F_{q,n}$ is a completely random mapping. We emphasize here that the completely random mapping is essentially a purely combinatorial object. The theory of completely random mappings is quite well developed, using

specifically combinatorial methods. See [1] and the bibliography therein.

For a mapping $f: \Xi(n) \to \Xi(n)$ and for an element $k \in \Xi(n)$ we denote by $Q(k, f)$ the first recurrence time, that is $Q(k, f) = \min\{i: f^i(k) f^j(k), \text{ for some } j < i\}$. Designate further by $\mathbb{Q}(x, f)$ the scaled distribution function:

$$\mathbb{Q}(x, f) = \frac{1}{n} \# \{k: Q(k, f) < xn^{\gamma}\}$$

where $\#S$ denotes the cardinality of the finite set S. The function $\mathbb{Q}$ is a random function if f is considered as realization of the random mapping F. Thus, we can consider the mathematical expectation $\mathbb{Q}_{\gamma, n}(x)$. The theorem above implies the following corollary immediately.

Corollary 1: *The following relations are valid for $n \to \infty$*

$$\mathbb{Q}_{\gamma, n}(x) \to \mathfrak{L}(\gamma, \alpha x) \text{ as } 0 < \gamma < \tfrac{1}{2}. \tag{8}$$

Random mapping with similar asymptotic of weights of elements arises naturally, for instance, in the analysis of discretizations of random mappings where the box counting dimension of the invariant measure differs from its correlation dimension [3].

To conclude this section we note that the equality (5) can be obtained also as a corollary of Theorem 1 [2].

2. Proof

The proof of the theorem is based on the following two lemmas:

Lemma 1: *Let $f(z)$ and $f_n(z)$, $n = 1, 2, \ldots$, be entire analytic functions of $z \in C$ such that*

$$f_n(z) \to f(z) \text{ as } n \to \infty$$

uniformly with respect to z from any bounded circle $|z| \le R$. Let $\mu(n)$ be an integer-valued function and $\nu(n)$ be a real-valued one such that

$$\nu(n) \to \infty, \quad \frac{\mu(n)}{\nu(n)} \to x \text{ as } n \to \infty.$$

Then

$$F(n) = \frac{1}{2\pi i} \int_{|z| = \rho} \frac{e^{\nu(n)z}}{z^{\mu(n)+1}} f_n(z) dz = \frac{(\nu(n))^{\mu(n)}}{\mu(n)!} f(x)(1 + o(1)) \text{ as } n \to \infty.$$

Lemma 2: *Let $f(n, z)$ be the polynomial*

$$f(n, z) = \prod_{k=1}^{n} (1 + p(k, n)z). \tag{9}$$

Then, uniformly with respect to z from any bounded circle $|z| \le R$ the following relations are valid

$$f(n, \sqrt{n}z)e^{-\sqrt{n}z} \to e^{-\frac{1}{2}(\int_0^1 q^2(t)dt)z^2} \text{ as } n \to \infty, \ \tfrac{1}{2} < \gamma \le 1;$$

$$f(n, \sqrt{n(\log n)^{-1}}z)e^{-\sqrt{n(\log n)^{-1}}z} \to e^{-\frac{1}{2}\alpha^2 z^2} \text{ as } n \to \infty, \ \gamma = \tfrac{1}{2};$$

$$f(n, n^\gamma z)e^{-n^\gamma z} \to \mathcal{L}(\gamma, \alpha z) \text{ as } n \to \infty, \ \ 0 < \gamma < \tfrac{1}{2}$$

where $\mathcal{L}(\gamma, z)$ is the entire analytic function defined by (7).

Proof of Theorem 1: Since the proof is identical for the case $0 < \gamma < \frac{1}{2}$, $\gamma = \frac{1}{2}$ and $\frac{1}{2}, \gamma \leq 1$ it will be presented only for the last case.

Remark that the value of $\frac{1}{m!}\pi(m, n)$ by the Viète theorem coincides with the coefficient at the term z^m in the Taylor expansion of the function $f(n, z)$ defined by (9). Then, by the theorem on residue of an analytic function

$$\pi(m, n) = m! \frac{1}{2\pi i} \int\limits_{|z| = \rho} \frac{f(n, z)}{z^{m+1}} dz$$

and, after substitution $z = \sqrt{n}w$,

$$\pi(m, n) = \frac{m!}{(\sqrt{n})^m} \frac{1}{2\pi i} \int\limits_{|w| = \widetilde{\rho}} \frac{e^{\sqrt{n}w}}{w^{m+1}} f_n(w) dw$$

where

$$f_n(w) = f(n, \sqrt{n}w)e^{-\sqrt{n}w}.$$

From this and from Lemmas 1 and 2 the statement of Theorem 1 for the case $\frac{1}{2} < \gamma \leq 1$ immediately follows.   □

Although Lemma 1 is in line with statements from the theory of integral's asymptotic proven with the help of the saddle point method (see, e.g., [4, 5]), we failed to find the appropriate reference to the exact formulation. Because of this and for the sake of completeness of presentation, the full proof of Lemma 1 is given below.

Proof of Lemma 1: By the theory on residue of an analytic function

$$\frac{1}{2\pi i} \int\limits_{|z| = \rho} \frac{e^{\nu(n)z}}{z^{\mu(n)+1}} dz = \frac{(\nu(n))^{\mu(n)}}{\mu(n)!}$$

for any value of $\rho > 0$. Hence, choosing $\rho = \frac{\mu(n)}{\nu(n)}$, the value of $F(n)$ may be represented as

$$F(n) = \frac{(\nu(n))^{\mu(n)}}{\mu(n)!} f(x) + \xi(n), \tag{10}$$

where

$$\xi(n) = \frac{1}{2\pi i} \int\limits_{|z| = \frac{\mu(n)}{\nu(n)}} \frac{e^{\nu(n)z}}{z^{\mu(n)+1}} (f_n(z) - f(x)) dz.$$

Estimate the value of $|\xi(n)|$. Let us make the substitution $z = \frac{\mu(n)}{\nu(n)}e^{i\varphi}$ in the above integral. Then

$$|\xi(n)| \leq \frac{1}{2\pi} \left| \int\limits_{-\pi}^{\pi} \frac{e^{\mu(n)(\cos\varphi + i\sin\varphi)}}{\left(\frac{\mu(n)}{\nu(n)}\right)^{\mu(n)+1} e^{i(\mu(n)+1)\varphi}} \left[f_n\left(\frac{\mu(n)}{\nu(n)}e^{i\varphi}\right) - f(x) \right] \frac{\mu(n)}{\nu(n)} i e^{i\varphi} d\varphi \right|$$

$$\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{\mu(n)\cos\varphi}}{\left(\frac{\mu(n)}{\nu(n)}\right)^{\mu(n)}} \left| f_n\left(\frac{\mu(n)}{\nu(n)}e^{i\varphi}\right) - f(x) \right| d\varphi$$

$$= \frac{1}{2\pi} \frac{e^{\mu(n)}(\nu(n))^{\mu(n)}}{(\mu(n))^{\mu(n)}} \int_{-\pi}^{\pi} e^{\mu(n)(\cos\varphi - 1)} \left| f_n\left(\frac{\mu(n)}{\nu(n)}e^{i\varphi}\right) - f(x) \right| d\varphi.$$

Now, using the Stirling's formula

$$\mu(n)! = \frac{1}{\sqrt{2\pi\mu(n)}} \left(\frac{\mu(n)}{e}\right)^{\mu(n)} \left(1 + O\left(\frac{1}{\mu(n)}\right)\right),$$

we may write that

$$|\xi(n)| \leq \frac{(\nu(n))^{\mu(n)}}{\mu(n)!} c_0 \eta(n) \tag{11}$$

with an appropriate constant c_0 where

$$\eta(n) = \sqrt{\frac{\mu(n)}{2\pi}} \int_{-\pi}^{\pi} e^{\mu(n)(\cos\varphi - 1)} \left| f_n\left(\frac{\mu(n)}{\nu(n)}e^{i\varphi}\right) - f(x) \right| d\varphi.$$

In view of (10) and (11), it remains to show that

$$\eta(n) \to 0 \quad \text{as } n \to \infty. \tag{12}$$

This proof introduces auxiliary constants

$$\chi = \min_{-\pi \leq \varphi \leq \pi} \frac{2(1 - \cos\varphi)}{\varphi^2}$$

and

$$c_1 = \sup_{n \geq 1} \sup_{|z| \leq 2x} |f_n(z) - f(x)|, \quad \varepsilon(n) = \max_{|\varphi| \leq x^{-1/2}(\mu(n))^{-1/4}} \left| f_n\left(\frac{\mu(n)}{\nu(n)}e^{i\varphi}\right) - f(x) \right|.$$

Clearly, $\chi > 0$, $c_1 < \infty$ and $\varepsilon(n) \to 0$ as $n \to \infty$ in view of uniform convergence of the sequence $\{f_n(z)\}$ to $f(z)$ on any bounded circle. Hence

$$\eta(n) \leq \sqrt{\frac{\mu(n)}{2\pi}} \int_{-\pi}^{\pi} e^{-\frac{\mu(n)\chi}{2}\varphi^2} \left| f_n\left(\frac{\mu(n)}{\nu(n)}e^{i\varphi}\right) - f(x) \right| d\varphi$$

and, by substitution $\sqrt{\mu(n)\chi}\varphi = \psi$, we obtain that

$$\eta(n) \leq \frac{1}{\sqrt{2\pi\chi}} \int_{-\sqrt{\mu(n)\chi}\pi}^{\sqrt{\mu(n)\chi}\pi} e^{-\frac{\psi^2}{2}} \left| f_n\left(\frac{\mu(n)}{\nu(n)}e^{i\frac{\psi}{\sqrt{\mu(n)\chi}}}\right) - f(x) \right| d\psi = I_1(n) + I_2(n) + I_3(n)$$

where

$$I_1(n) = \frac{1}{\sqrt{2\pi\chi}} \int\limits_{-(\mu(n))^{1/4}}^{(\mu(n))^{1/4}} e^{-\frac{\psi^2}{2}} \left| f_n\left(\frac{\mu(n)}{\nu(n)} e^{i\frac{\psi}{\sqrt{\mu(n)\chi}}} \right) - f(x) \right| d\psi,$$

$$I_2(n) = \frac{1}{\sqrt{2\pi\chi}} \int\limits_{-\sqrt{\mu(n)\chi}\pi}^{-(\mu(n))^{1/4}} e^{-\frac{\psi^2}{2}} \left| f_n\left(\frac{\mu(n)}{\nu(n)} e^{i\frac{\psi}{\sqrt{\mu(n)\chi}}} \right) - f(x) \right| d\psi,$$

$$I_3(n) = \frac{1}{\sqrt{2\pi\chi}} \int\limits_{(\mu(n))^{1/4}}^{\sqrt{\mu(n)\chi}\pi} e^{-\frac{\psi^2}{2}} \left| f_n\left(\frac{\mu(n)}{\nu(n)} e^{i\frac{\psi}{\sqrt{\mu(n)\chi}}} \right) - f(x) \right| d\psi.$$

By definition of the value $\varepsilon(n)$, we have

$$\left| f_n\left(\frac{\mu(n)}{\nu(n)} e^{i\frac{\psi}{\sqrt{\mu(n)\chi}}} \right) - f(x) \right| \le \varepsilon(n) \text{ for } |\psi| \le (\mu(n))^{1/4}$$

and thus

$$I_1(n) \le \left(\frac{1}{\sqrt{2\pi\chi}} \int\limits_{-(\mu(n))^{1/4}}^{(\mu(n))^{1/4}} e^{-\frac{\psi^2}{2}} d\psi \right) \varepsilon(n) \le \sqrt{\frac{2\pi}{\chi}} \varepsilon(n). \tag{13}$$

At the same time, by definition of the constant c_1, we have

$$I_2(n), I_3(n) \le \left(\frac{1}{\sqrt{2\pi\chi}} \int\limits_{(\mu(n))^{1/4}}^{\sqrt{\mu(n)\chi}\pi} e^{-\frac{\psi^2}{2}} d\psi \right) c_1 = \zeta(n) \to 0 \text{ as } n \to \infty. \tag{14}$$

From (13), (14) it follows (12) which completes the proof of the lemma. □

In the proof of Lemma 2, properties of the numbers $p(k,n)$ play an important role. Therefore, we establish these properties prior to proceeding to the proof of Lemma 2. From the theorem on mean value of an integral it follows that

$$p(k,n) = \alpha\beta(\tau_{k,n})\frac{k^\gamma - (k-1)^\gamma}{\gamma n^\gamma}, \ \frac{k-1}{n} \le \tau_{k,n} \le \frac{k}{n}, \ k = 1,2,\ldots,n. \tag{15}$$

Therefore,

$$p(k,n) = \alpha\left(\beta\left(\frac{k}{n} \right) + \widetilde{\beta}(k,n) \right) \frac{k^\gamma - (k-1)^\gamma}{\gamma n^\gamma}, k = 1,2,\ldots,n \tag{16}$$

where

$$\widetilde{\beta}(k,n) \le \max_{\frac{k-1}{n} \le t \le \frac{k}{n}} \left| \beta\left(\frac{k}{n} \right) - \beta(t) \right|$$

and thus, in view of continuity of the function $\beta(t)$,

$$\widehat{\beta}(n) = \max_{1 \leq k \leq n} \widetilde{\beta}(k,n) \to 0 \text{ as } n \to \infty. \tag{17}$$

From the continuity of the function $\beta(t)$ it follows that the product of the first two multipliers in (15) is uniformly (with respect to n and $k \in [1,n]$) bounded. At the same time, for a given $\gamma \in (0,1]$ the numbers $k^{1-\gamma}(k^\gamma - (k-1)^\gamma)$ are also uniformly bounded. Then, by (16) such a constant $p_* = p(\alpha, \gamma)$ can be chosen that

$$\alpha(k^\gamma - (k-1)^\gamma) \leq \gamma p_* k^{\gamma - 1} \tag{18}$$

and simultaneously

$$0 \leq p(k,n) \leq p_* \frac{k^{\gamma - 1}}{n^\gamma} \leq p_* n^{-\gamma}. \tag{19}$$

Proof of Lemma 2: Case $\frac{1}{2} < \gamma \leq 1$. Set $g(n,z) = f(n, \sqrt{n}z)e^{-\sqrt{nz}}$, then

$$g(n,z) = e^{-\sqrt{nz}} \prod_{k=1}^n (1 + p(k,n)\sqrt{n}z) = \prod_{k=1}^n \{(1 + p(k,n)\sqrt{n}z)e^{-p(k,n)\sqrt{nz}}\}. \tag{20}$$

(Here, the second equality follows from (1) and (2).) By (19), $0 \leq p(k,n)\sqrt{n} \leq p_* n^{1/2 - \gamma}$ and so $p(k,n)\sqrt{n} \to 0$ as $n \to \infty$ uniformly with respect to k. Hence, for any $R < \infty$ such a value $n(R)$ can be chosen that $|p(k,n)\sqrt{n}z| \leq \frac{1}{2}$ for $n \geq n(R)$, $|z| \leq R$ uniformly with respect to k. Then, for such n and z, the representation

$$1 + p(k,n)\sqrt{n}z = e^{\ln(1 + p(k,n)\sqrt{n}z)}$$

$$= e^{p(k,n)\sqrt{n}z - \frac{1}{2}(p(k,n)\sqrt{n}z)^2 + \theta(k,n,z)(p(k,n)\sqrt{n}z)^3} \tag{21}$$

is valid where

$$|\theta(k,n,z)| \leq \theta_0 < \infty, \ n \geq n(R), \ 1 \leq k \leq n, \ |z| \leq R. \tag{22}$$

By substituting (21) in (20) we obtain

$$g(n,z) = e^{-\frac{n}{2}(\sum_{k=1}^n p^2(k,n))z^2 + \widetilde{\theta}(n,z)} \tag{23}$$

where

$$\widetilde{\theta}(n,z) = n^{\frac{3}{2}} \left(\sum_{k=1}^n \theta(,n,z)p^3(k,n) \right) z^3. \tag{24}$$

From (1) it is seen that $p(k,n) = \frac{1}{n}q(t_{k,n})$ with an appropriate $t_{k,n} \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$. Thus, given arbitrary $\varepsilon > 0$, we can write

$$\frac{n}{2} \sum_{k=1}^n p^2(k,n) = S_1(\varepsilon, n) + S_2(\varepsilon, n)$$

where

$$S_1(\varepsilon,n) = \frac{n}{2} \sum_{1 \le k < \varepsilon n} p^2(k,n), \quad S_2(\varepsilon,n) = \frac{1}{2} \sum_{\varepsilon n \le k \le n} \frac{1}{n} q^2(t_{k,n}).$$

From (19) it follows that

$$S_1(\varepsilon,n) \le \frac{1}{2} p_*^2 n^{1-2\gamma} \sum_{1 \le k < \varepsilon n} k^{2(\gamma-1)} \le \frac{1}{2} c_0 p_*^2 n^{1-2\gamma} (\varepsilon n)^{2(\gamma-1)+1}$$

with an appropriate constant c_0, and thus

$$S_1(\varepsilon,n) \le \frac{1}{2} c_0 p_*^2 \varepsilon^{2\gamma-1} \to 0 \text{ as } \varepsilon \to 0. \tag{25}$$

On the other hand, in view of the summability of the function $q^2(t)$ on the interval $[\varepsilon, 1]$,

$$S_2(\varepsilon,n) \to \frac{1}{2} \int_{\varepsilon}^{1} q^2(t) dt \text{ as } n \to \infty.$$

From this and from (25) it follows that

$$\frac{n}{2} \sum_{k=1}^{n} p^2(k,n) \to \frac{1}{2} \int_{0}^{1} q^2(t) dt \text{ as } n \to \infty. \tag{26}$$

At the same time from (19), (22) and (24) it follows that

$$|\tilde{\theta}(n,z)| \le n^{\frac{3}{2}} \left(\sum_{k=1}^{n} |\theta(k,n,z)| p^3(k,n) \right) z^3$$

$$\le \theta_0 p_*^3 n^{3(\frac{1}{2}-\gamma)} \left(\sum_{k=1}^{n} k^{3(\gamma-1)} \right) z^3 \le \theta_0 c_1 p_*^3 n^{3(\frac{1}{2}-\gamma)} n^{3(\gamma-1)+1} z^3$$

with an appropriate constant c_1. Thus,

$$|\tilde{\theta}(n,z)| \le \theta_0 c_1 p_*^3 n^{-\frac{1}{2}}. \tag{27}$$

From (23), (26), (27) and from the definition of the function $g(n,z)$ the statement of the lemma for the case $\frac{1}{2} < \gamma \le 1$ follows:

Case $\gamma = \frac{1}{2}$. Set $g(n,z) = f(n, \sqrt{n(\log n)^{-1}} z) e^{-\sqrt{n(\log n)^{-1}} z}$, then by (1) and (2)

$$g(n,z) = \prod_{k=1}^{n} \left\{ \left(1 + p(k,n)\sqrt{n(\log n)^{-1}} z \right) e^{-p(k,n)\sqrt{n(\log n)^{-1}} z} \right\}. \tag{28}$$

By (19), $0 \le p(k,n)\sqrt{n(\log n)^{-1}} \le p_*(\log n)^{-1/2}$ and so $p(k,n)\sqrt{n(\log n)^{-1}} \to 0$ as $n \to \infty$ uniformly with respect to k. Hence, for any $R < \infty$ such a value $n(R)$ can be chosen that $|p(k,n)\sqrt{n(\log n)^{-1}} z| \le \frac{1}{2}$ for $n \ge n(R)$, $|z| \le R$ uniformly with respect to k. Then, for such n and z the representation

$$1 + p(k,n)\sqrt{n(\log n)^{-1}} z = e^{\ln(1 + p(k,n)\sqrt{n(\log n)^{-1}} z)}$$

$$= e^{p(k,n)\sqrt{n(\log n)^{-1}}z - \frac{1}{2}(p(k,n)\sqrt{n(\log n)^{-1}}z)^2 + \theta(k,n,z)(p(k,n)\sqrt{n(\log n)^{-1}}z)^3} \quad (29)$$

is valid where

$$|\theta(k,n,z)| \le \theta_0 < \infty, \ n \ge n(R), \ 1 \le k \le n, \ |z| \le R. \quad (30)$$

By substituting (29) in (28) we obtain that

$$g(n,z) = e^{-\frac{n}{2\log n}(\sum_{k=1}^{n} p^2(k,n))z^2 + \tilde{\theta}(n,z)} \quad (31)$$

where

$$\tilde{\theta}(n,z) = \left(\frac{n}{\log n}\right)^{\frac{3}{2}} \left(\sum_{k=1}^{n} \theta(k,n,z)p^3(k,n)\right) z^3.$$

Estimation of $\tilde{\theta}(n,z)$ is done analogously to the previous case. From (19) and (30) it follows that

$$|\tilde{\theta}(n,z)| \le \left(\frac{n}{\log n}\right)^{\frac{3}{2}} \left(\sum_{k=1}^{n} |\theta(k,n,z)| \, p^3(k,n)\right) z^3$$

$$\le \theta_0 p_*^3 (\log n)^{-\frac{3}{2}} \left(\sum_{k=1}^{n} k^{-\frac{3}{2}}\right) z^3$$

$$\le \theta_0 c_1 p_*^3 n^{-\frac{1}{2}} (\log n)^{-\frac{3}{2}} z^3 \to 0 \text{ as } n \to \infty. \quad (32)$$

Before estimating the first summand in the power of the number e in (31), we recall Euler's formula:

$$\lim_{n \to \infty} \left(\sum_{m=1}^{n} \frac{1}{m} - \log n\right) = C,$$

from which it immediately follows

$$\left|\sum_{m=k+1}^{n} \frac{1}{m} - \log\frac{n}{k}\right| \le C_*, \ 1 \le k \le n, \quad (33)$$

with an appropriate constant C_*.

Now, fix an $\varepsilon > 0$ and choose such a real number $\delta > 0$ that

$$|1 - \beta^2(t)| = |\beta^2(0) - \beta^2(t)| \le \varepsilon, \ 0 \le t \le \delta. \quad (34)$$

(This can be done by continuity of the function $\beta(t)$.) Then, we can write

$$\frac{n}{2\log n} \sum_{k=1}^{n} p^2(k,n) = S_1(\delta,n) + S_2(\delta,n) \quad (35)$$

where

$$S_1(\delta,n) = \frac{n}{2\log n} \sum_{1 \le k \le \delta n} p^2(k,n), \quad S_2(\delta,n) = \frac{n}{2\log n} \sum_{\delta n < k \le n} p^2(k,n).$$

From (19) and (33) it follows that

$$S_2(\delta,n) \le \frac{1}{2\log n} p_*^2 \sum_{\delta n < k \le n} \frac{1}{k} \le \frac{1}{2\log n} p_*^2 \left(C_* + \log\frac{n}{\delta n}\right)$$

$$\le \frac{1}{2\log n} p_*^2 (C_* + |\log\delta|) \to 0 \text{ as } n \to \infty. \quad (36)$$

At the same time by (15)

$$S_1(\delta, n) = \frac{\alpha^2}{2\log n} \sum_{1 \le k \le \delta n} 4\beta^2(\tau_{k,n})(\sqrt{k} - \sqrt{k-1})^2 \text{ with } \tau_{k,n} \le \delta.$$

Thus

$$S_1(\delta, n) = \frac{\alpha^2}{2} + S_{11}(\delta, n) + S_{12}(\delta, n) + S_{13}(\delta, n) \tag{37}$$

where

$$S_{11}(\delta, n) = \frac{\alpha^2}{2\log n} \left(\sum_{1 \le k \le \delta n} \frac{1}{k} - \log n \right),$$

$$S_{12}(\delta, n) = \frac{\alpha^2}{2\log n} \sum_{1 \le k \le \delta n} 4\Big(\beta^2(\tau_{k,n}) - 1 \Big)\Big(\sqrt{k} - \sqrt{k-1} \Big)^2,$$

$$S_{13}(\delta, n) = \frac{\alpha^2}{2\log n} \sum_{1 \le k \le \delta n} \left\{ 4(\sqrt{k} - \sqrt{k-1})^2 - \frac{1}{k} \right\}.$$

From (33) it follows that

$$|S_{11}(\delta, n)| \le \frac{\alpha^2(c_* + |\log \delta|)}{2\log n}. \tag{38}$$

From (34) it follows that

$$|S_{12}(\delta, n)| \le \frac{\alpha^2 \varepsilon}{2\log n} \sum_{1 \le k \le \delta n} 4(\sqrt{k} - \sqrt{k-1})^2 \le \frac{\alpha^2 \varepsilon}{2\log n} \sum_{1 \le k \le \delta n} \frac{4}{k}$$

$$\le 2\alpha^2 \varepsilon \frac{C_* + |\log \delta| + \log n}{\log n}. \tag{39}$$

Finally, since

$$\frac{1}{k} \le 4(\sqrt{k} - \sqrt{k-1})^2 = \left(\frac{2}{\sqrt{k} + \sqrt{k-1}} \right)^2 \le \frac{1}{k-1},$$

then

$$|S_{13}(\delta, n)| \le \frac{\alpha^2}{2\log n} \sum_{1 \le k \le \delta n} \left| 4(\sqrt{k} - \sqrt{k-1})^2 - \frac{1}{k} \right|$$

$$\le \frac{\alpha^2}{2\log n} \left(3 + \sum_{k \ge 2} \left(\frac{1}{k-1} - \frac{1}{k} \right) \right) = \frac{2\alpha^2}{\log n}. \tag{40}$$

From (37), (38), (39) and (40) it follows that

$$\limsup_{n \to \infty} \left| S_1(\delta, n) - \frac{\alpha^2}{2} \right| \le 2\alpha^2 \varepsilon,$$

and thus in view of (35) and (36)

$$\limsup_{n \to \infty} \left| \frac{n}{2\log n} \sum_{k=1}^{n} p^2(k, n) - \frac{\alpha^2}{2} \right| \le 2\alpha^2 \varepsilon.$$

Since ε is arbitrary, then

$$\lim_{n \to \infty} \frac{n}{2\log n} \sum_{k=1}^{n} p^2(k, n) = \frac{\alpha^2}{2}.$$

From this and from (31), (32) the statement of the lemma in the case $\gamma = \frac{1}{2}$ follows.

Case $0 < \gamma < \frac{1}{2}$. Set $g(n, z) = f(n, n^\gamma z)e^{-n^\gamma z}$, then

$$g(n, z) = e^{-n^\gamma z} \prod_{k=1}^{n} (1 + p(k,n)n^\gamma z)$$

$$= \prod_{k=1}^{n} \{(1 + p(k,n)n^\gamma z)e^{-p(k,n)n^\gamma z}\}. \tag{41}$$

(Here the latter equality follows from (1) and (2).) Set also

$$g(m, n, z) = \prod_{k=1}^{m} \{(1 + p(k,n)n^\gamma z)e^{-p(k,n)n^\gamma z}\}, \tag{42}$$

$$\mathcal{L}(m, \gamma, z) = \prod_{k=1}^{m} \left\{ \left[1 + \frac{k^\gamma - (k-1)^\gamma}{\gamma} z \right] e^{-\frac{k^\gamma - (k-1)^\gamma}{\gamma} z} \right\}. \tag{43}$$

Now, using the evident inequality

$$|(1+z)e^{-z}| \leq e^{4|z|^2}$$

and inequalities (18), (19) we can estimate the values $|g(m,n,z)|$ and $|\mathcal{L}(m,\gamma,\alpha z)|$ as follows

$$|g(m,n,z)|, |\mathcal{L}(m,\gamma,\alpha z)| \leq e^{4p_*^2(\sum_{k=1}^{m} k^{2(\gamma-1)})|z|^2}$$

$$\leq e^{4p_*^2(\sum_{k=1}^{\infty} k^{2(\gamma-1)})|z|^2}.$$

Since the power series $\sum_{k=1}^{\infty} k^{2(\gamma-1)}$ is summable for $0 < \gamma < \frac{1}{2}$, then

$$|g(m,n,z)|, |\mathcal{L}(m,\gamma,\alpha z)| \leq e^{\omega|z|^2} \tag{44}$$

with an appropriate constant ω uniformly with respect to all possible combinations of n and m. Analogously, for the functions

$$\widehat{g}(m, n, z) = \prod_{k=m+1}^{n} \left\{(1 + p(k,n)n^\gamma z)e^{-p(k,n)n^\gamma z}\right\}, \tag{45}$$

$$\widehat{\mathcal{L}}(m, \gamma, \alpha z) = \prod_{k=m+1}^{\infty} \left\{ \left[1 + \frac{k^\gamma - (k-1)^\gamma}{\gamma} z \right] e^{-\frac{k^\gamma - (k-1)^\gamma}{\gamma} z} \right\} \tag{46}$$

we can get the following estimates

$$|\widehat{g}(m,n,z) - 1|, |\widehat{\mathcal{L}}(m,\gamma,\alpha z)| \leq e^{\omega(m)|z|^2} - 1 \tag{47}$$

where

$$\omega(m) = 4p_*^2 \left(\sum_{k=m+1}^{\infty} k^{2(\gamma-1)} \right) \to 0 \text{ as } m \to \infty.$$

Represent the difference $g(n,z) - \mathcal{L}(\gamma, \alpha z)$ in the form

$$g(n,z) - \mathcal{L}(\gamma, \alpha z) = g(m,n,z)\widehat{g}(m,n,z) - \mathcal{L}(m,\gamma,\alpha z)\widehat{\mathcal{L}}(m,\gamma,\alpha z)$$

$$= (g(m,n,z) - \mathcal{L}(m,\gamma,\alpha z))$$

$$+ g(m,n,z)(\widehat{g}(m,n,z) - 1) + \mathcal{L}(m,\gamma,\alpha z)(\widehat{\mathcal{L}}(m,\gamma,\alpha z) - 1)$$

where the integer m will be chosen later. Then, by (44) and (47)

$$| g(n,z) - \mathcal{L}(\gamma, \alpha z) | \leq | g(m,n,z) - \mathcal{L}(m, \gamma, \alpha z) | + 2e^{\omega |z|^2} | e^{\omega(m)|z|^2} - 1 |. \tag{48}$$

Now, fix a number $R > 0$, $\varepsilon > 0$ and choose such a big integer $m = m(R, \varepsilon)$ that

$$2e^{\omega R^2} | e^{\omega(m)R^2} - 1 | < \varepsilon.$$

Then by (48)

$$| g(n,z) - \mathcal{L}(\gamma, \alpha z) | \leq | g(m,n,z) - \ell(m, \gamma, \alpha z) | + \varepsilon$$

$$| z | \leq R, n > m(R, \varepsilon). \tag{49}$$

But, in view of (16), (17)

$$g(m,n,z) \to \mathcal{L}(m, \gamma, \alpha z) \text{ as } n \to \infty$$

for any fixed m uniformly with respect to z from any bounded circle. From here and from (49) we get that

$$\limsup_{n \to \infty} | g(n,z) - \mathcal{L}(\gamma, \alpha z) | \leq \varepsilon, \quad | z | \leq R$$

and thus, since ε is arbitrary,

$$g(n,z) \to \mathcal{L}(\gamma, \alpha z) \text{ as } n \to \infty$$

uniformly with respect to $| z | \leq R$.

The lemma is completely proved and so is the theorem.   □

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