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A k-OUT-OF-n RELIABILITY SYSTEM WITH AN UNRELIABLE SERVER AND PHASE TYPE REPAIRS AND SERVICES: THE (N,T) POLICY

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In this paper we study a k-out-of-n reliability system in which a single unreliable server maintains n identical components. The reliability system is studied under the (N,T) policy. An idle server takes a vacation for a random amount of time T and then attends to any failed component waiting in line upon completion of the vacation. The vacationing server is recalled instantaneously upon the failure of the N^{th} component. The failure times of the components are assumed to follow an exponential distribution. The server is subject to failure with failure times exponentially distributed. Repair times of the component, fixing times of the server, and vacationing times of the server are assumed to be of phase type. Using matrix-analytic methods we perform steady state analysis of this model. Time spent by a failed component in service, total time in the repair facility, vacation time of the server, non-vacation time of the server, and time until failure of the system are all shown to be of phase type. Several performance measures are evaluated. Illustrative numerical examples are presented.

Key words: Reliability System, k-out-of-n System, (N,T)-Policy, Phase Type Distribution, Matrix-Analytic Methods, Algorithmic Probability, and Optimization.

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1. Introduction

When dealing with critical systems such as aircrafts, space shuttles, nuclear plants, satellites, electric generators, and computer systems, it is imperative that redundancy be in the system. The redundancy is in the form of multiple (often identical) components connected in such a way that when on component fails the others not only keep the system functioning but share the increased load due to fewer operating components. Also, redundancy is highly cost effective in achieving a certain reliability level of the system. A common form of redundancy is a k-out-of-n system in which at least k components of n must be functioning for the system to be operational. k-out-of-n reliability systems have been studied extensively in the context of computing the reliability, optimization of the system, common cause failures, and repair facility availability for fixing failed components [1-4, 7-10, 13-18, 20, 21]. Though there are many papers on such systems, very few papers address load-sharing k-out-of-n systems [12, 19]. Load-sharing systems are natural and very useful in dealing with systems such as nuclear plants, aircraft engines, electric generators, computer systems, suspension bridge cables and other workload balancing systems.

When analyzing the reliability of systems with a repair facility to fix failed components (or machines), a review of past work on machine repairman problems shows two distinct areas of research:

(1) systems with reliable servers and
(2) systems with unreliable servers

Most of the research addressing machine repairman problems has focused on reliable servers Research work dealing with unreliable servers is limited in the literature [6, 11, 22-29]. Recently a comparative analysis of steady state performance measures under a variety of distributions for repair times and service times for a 1-out-of-n reliability system with a single unreliable repairman was given in Chakravarthy and Agarwal [6].

In this paper, we will extend the work of [6] for a k-out-of-n system in which

(a) the server is unreliable,
(b) the failed server is fixed by a repair facility,
(c) the server goes on vacation for a random amount of time whenever the system is functioning with all n components,
(d) upon completing a vacation, the server goes for another vacation of random duration when there are no failed components waiting to be fixed, and
(e) the vacationing server is recalled instantaneously when the number of failed components reach a predetermined threshold N assumed to be less than or equal to $n-k$.

We consider a k-out-of-n system that can be *cold, warm,* or *hot.*

This paper is organized as follows. In Section 2, we describe the model under study and give a brief review of PH-distributions. The Markov chain description of the three reliability systems is presented in Section 3. The steady state analysis of the systems is presented in Section 4. The performance measures for studying the qualitative behavior of the reliability systems are developed in Section 5, and a few representative numerical examples are discussed in Section 6.

2. Model Description

We consider a k-out-of-n load-sharing reliability system with n identical components which can fail independently of each other, and a single unreliable server that attends to failed components. The system will be functional if and only if there are k or more working components. Failure times of the components are assumed to be exponential with a parameter that depends on the number of working components. That is, if there are i working components, then the failure times are exponential with parameter λ_i.

Upon failure, a component enters into repair facility and the repair times follow a PH-distribution with representation (β_1, S_1) of of order m_1. When the server is busy or when the server is unavailable, the failed component joins the queue. Once repaired, the component is returned back to the system as new.

It is assumed that the server is subject to failure and the failure times of the server are assumed to follow an exponential distribution with parameter μ_{SF}. The fixing times of the server follow a PH-distribution with representation (β_2, S_2) of order m_2. Once fixed, the server resumes the work of failed component at the stage where it left servicing the failed component.

Whenever there is no component waiting for repair, the server takes a vacation for a random time T having a Ph-distribution with representation (β_3, S_3) of order m_3.

We consider a k-out-of-n reliability system that is *cold*, *warm*, or *hot*. By a *cold* system, we mean a reliability system in which the individual components cease to fail upon system failure. Note that the system failure occurs as soon as the number of functioning components drops below k. By a *warm* system, we mean a system in which the individual components continue to fail, but with a smaller failure rate, even after a system failure. In a *hot* system, individual components continue to fail at the same rate after a system failure.

In the sequel, we will assume that the failure rate of components is inversely proportional to the number of functioning components. That is, if there are i functioning components, then $\lambda_i = \lambda/i$. Note that for a *cold* system, $\lambda_i = \lambda/i$, for $1 \leq i \leq n-k$, $\lambda_i = 0$, for $n-k+1 \leq i \leq n$. For a *warm* system, $\lambda_i = \lambda/i$, for $1 \leq i \leq n-k$, $\lambda_i = \theta/i$, for $n-k+1 \leq i \leq n$, where $\theta < \lambda$. In the case of *hot* system, $\lambda_i = \lambda/i$, for $1 \leq i \leq n$. For the sake of simplicity, we will make this assumption, though the model is valid for the general case of the failure rate. In the following, e and I will represent a column vector of 1's, and the identity matrix of appropriate dimensions, respectively. The notation $\otimes$ will stand for the Kronecker product of two matrices. Thus, if A is an $m \times n$ matrix and if B is a $p \times q$ matrix, then $A \otimes B$ will denote a $mp \times nq$ matrix whose $(i,j)^{th}$ block matrix is given by $a_{ij}B$. For more details on Kronecker products, we refer the reader to [5].

Before we describe the Markov chain of the repairman model, we present a review of PH-distributions.

Phase type distributions: Poisson arrival processes and exponential service time distributions have mathematical properties that make queueing models very attractive and tractable. However, in applications these assumptions are highly restrictive. To remove some of the restrictions, Neuts [13] developed the theory of PH-distributions and related point processes. In stochastic modeling, PH-distributions lend themselves naturally to algorithmic implementation and have nice closure properties with a related matrix formalism that make them attractive for practical use. In this section we review continuous-time PH-distributions.

Continuous-time PH-distributions: Consider a finite Markov chain (MC) with m transient states and one absorbing state with the infinitesimal generator Q partitioned as

$$Q = \begin{bmatrix} S & \boldsymbol{S^0} \\ \mathbf{0} & 0 \end{bmatrix},$$

where S is a matrix of order m and $\boldsymbol{S^0}$ is a column vector such that $Se + \boldsymbol{S^0} = 0$. The vector e is a column of 1's. It is necessary and sufficient that S be nonsingular for eventual absorption into the absorbing state, starting from any initial state. Suppose that the initial state of the MC is chosen according to the probability vector (β, β_{m+1}) and let X denote the time until absorption. Then X is a continuous random variable on nonnegative values with a probability distribution function $F(x)$ given by $F(x) = 1 - \beta e^{Sx} e$ for $x \geq 0$.

Note that

(*a*) the i^{th} component of the vector $e^{Sx}e$ is the probability that starting in state i, absorption has not occurred by time x;

(*b*) the $(i,j)^{th}$ component of the matrix $(-S)^{-1}$ is the *conditional expected total time* spent in state j before absorption given that the initial state is i.

The function $F(\cdot)$ has a jump β_{m+1} at the origin, but in most cases, $F(0) = 0 = \beta_{m+1}$. Such a probability function constructed from a finite MC with a single absorbing state is a continuous PH-distribution. The pair (β, S) of order m is said to be a *representation* of the PH-distribution.

The transition matrix $P(x) = e^{Qx}$ is of the form

$$e^{Qx} = \begin{bmatrix} e^{Sx} & e - e^{Sx}e \\ 0 & 1 \end{bmatrix},$$

suggesting that the density $f(x)$ of $F(x)$ is either identically 0 or strictly positive for all $x > 0$ (see [5]). In the latter case, $f(x)$ is given by $f(x) = \beta e^{Sx} S^0$, for $x \geq 0$. The k^{th} noncentral moment of $F(x)$ is given by

$$\mu'_k = k!\alpha(-S)^{-k}\boldsymbol{e}, \text{ for } k \geq 0.$$

When $m = 1$, $S = -\lambda$, the underlying PH-distribution becomes exponential. A generalized Erlang distribution of order m is a PH-distribution with representation (β, S)

$$\beta = (1, 0, \ldots, 0) \text{ and } S = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 & \ldots & 0 \\ 0 & -\lambda_2 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_{m-1} \\ 0 & 0 & 0 & \ldots & -\lambda_m \end{bmatrix}.$$

A hyperexponential distribution of order m is a PH-distribution with representation (β, S)

$$\beta = (\beta_1, \beta_2, \ldots, \beta_m) \text{ and } S = \begin{bmatrix} -\lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & -\lambda_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & -\lambda_m \end{bmatrix}.$$

For any PH-distribution, there exist infinitely many different representations. For example, if $\beta = u$ where u is the left eigenvector of S (with u normalized to be a probability vector) corresponding to the eigenvalue $-\eta$ of maximum real part, then the representation (u, S) of order m is an exponential distribution with parameter η.

3. The Markov Process

The model outlined in Section 1 can be described by a Markov process. The state space of the Markov process depends on whether the system is *cold*, *warm* or *hot*. In the case of a *cold* reliability system, the state space is $\Omega_c = \{\{\{(i, j_3): 1 \le j_3 \le m_3\}, 0 \le i \le N-1\} \cup \{(i, j_1, j_2): 1 \le j_1 \le m_1,\ 1 \le j_2 \le m_2\}, 1 \le i \le n-k+1\} \cup \{\{\{(i, j_1): 1 \le j_1 \le m_1\},\ 1 \le i \le n-k+1\}\}$. For the other two cases, the state space is $\Omega_{wh} = \{\{\{(i, j_3): 1 \le j_3 \le m_3\}, 0 \le i \le N-1\} \cup \{(i, j_1, j_2): 1 \le j_1 \le m_1,\ 1 \le j_2 \le m_2\},\ 1 \le i \le n\} \cup \{\{\{(i, j_1): 1 \le j_1 \le m_1\},\ 1 \le i \le n\}\}$. The sets of states $\boldsymbol{i}^* = \{(i, j_3):\ 1 \le j_3 \le m_3\}$ corresponds to the case where i components are under repair and the server is on vacation in phase j_3. The set of states $\boldsymbol{i}' = \{(i, j_1, j_2):\ 1 \le j_1 \le m_1,\ 1 \le j_2 \le m_2\}$ corresponds to the case where i components and the server are under repair, the server repair in phase j_2, and the server failed while attending a failed component in phase j_1. The set of states $\boldsymbol{i} = \{(i, j_1): 1 \le j_1 \le m_1\}$ corresponds to the case where i components are under repair and the current service is in phase j_1.

The Markov process for a *cold* k-out-of-n reliability system has the infinitesimal generator Q_c given by

$$Q_c = \begin{bmatrix} B_{11}^c & B_{12}^c & 0 \\ B_{21}^c & B_{22}^c & \mu_{SF}(I \otimes \beta_2) \\ 0 & I \otimes S_2^0 & B_{33}^c \end{bmatrix}, \tag{1}$$

where

$$B_{11}^c = \begin{bmatrix} S_3 + S_3^0\beta_3 - \lambda I & \lambda I & 0 & \ldots & 0 & 0 \\ 0 & S_3 - \lambda I & \lambda I & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & S_3 - \lambda I & \lambda I \\ 0 & 0 & 0 & \ldots & 0 & S_3 - \lambda I \end{bmatrix} \tag{2}$$

$$B_{12}^c = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ S_3^0\beta_1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & S_3^0\beta_1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & S_3^0\beta_1 & \lambda e\beta_1 & \dots & 0 & 0 \end{bmatrix}, \tag{3}$$

$$B_{21}^c = \begin{bmatrix} S_1^0\beta_3 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \tag{4}$$

$$B_{22}^c = \begin{bmatrix} A_1 & \lambda I & 0 & \dots & 0 & 0 \\ S_1^0\beta_1 & A_1 & \lambda I & \dots & 0 & 0 \\ 0 & S_1^0\beta_1 & A_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & S_1^0\beta_1 A_1 + \lambda I \end{bmatrix}$$

$$B_{33}^c = \begin{bmatrix} A_2 & \lambda I & 0 & \dots & 0 \\ 0 & A_2 & \lambda I & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \lambda I \\ 0 & 0 & 0 & \dots & A_2 + \lambda I \end{bmatrix} \tag{5}$$

with

$$A_1 = A_2 - (\lambda + \mu_{SF})I \text{ and } A_2 = I \otimes S_2 - \lambda I. \tag{6}$$

The Markov process for a *warm* k-out-of-n reliability system has the infinitesimal generator Q_w given by

$$Q_w = \begin{bmatrix} B_{11}^c & B_{12}^w & 0 \\ B_{21}^w & B_{22}^w & \mu_{SF}(I \otimes \beta_2) \\ 0 & I \otimes S_2^0 & B_{33}^w \end{bmatrix}, \tag{7}$$

where the B_{ij}'s are such that B_{12}^w and B_{21}^w are of the form B_{12}^c and B_{21}^c except that the dimensions of these matrices are, respectively, $Nm_3 \times nm_1$ and $nm_1 \times Nm_3$. The matrices B_{22}^w and B_{33}^w are given by

$$B_{22}^{w} = \begin{bmatrix} A_1 & \lambda I & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ S_1^0\beta_1 & A_1 & \lambda I & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & S_1^0\beta_1 & A_1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & S_1^0\beta_1 & A_3 & \theta I & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & S_1^0\beta_1 & A_3 & \theta I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & S_1^0\beta_1 & A_3+\theta I \end{bmatrix} \tag{8}$$

$$B_{33}^{w} = \begin{bmatrix} A_2 & \lambda I & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & A_2 & \lambda I & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & A_2 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & A_4 & \theta I & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & A_4 & \theta I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & A_4+\theta I \end{bmatrix}, \tag{9}$$

with

$$A_3 = S_1 - (\theta + \mu_{SF})I \text{ and } A_4 = I \otimes S_2 - \theta I. \tag{10}$$

The Markov process for a *hot* k-out-of-n reliability system has the infinitesimal generator Q_h given by

$$Q_h = \begin{bmatrix} B_{11}^{c} & B_{12}^{h} & 0 \\ B_{21}^{h} & B_{22}^{h} & \mu_{SF}(I \otimes \beta_2) \\ 0 & I \otimes S_2^0 & B_{33}^{h} \end{bmatrix}, \tag{11}$$

where $B_{12}^{h}, B_{21}^{h}, B_{22}^{h}$ and B_{33}^{h} are of the form $B_{12}^{w}, B_{21}^{w}, B_{22}^{w}$ and B_{33}^{w}, except that θ in these matrices is replaced by λ.

4. The Steady-State Analysis

In this section we will present the steady-state analysis of the reliability system discussed above.

4.1 The Steady-State Probability Vector for the *Cold* System

The steady-state equations are obtained by solving $xQ_c = 0$, and $xe = 1$. By partitioning $x = (u_0, u_1, \ldots, u_{N-1}, v_1, v_2, \ldots, v_{n-k+1}, w_1, w_2, \ldots, w_{n-k+1})$, where the vectors u_i are of dimension m_3, the vectors v_i of dimension m_1 and w_i of dimension $m_1 m_2$ are further partitioned into $w_i = (w_i(1), \ldots, w_i(m_1))$ of dimension m_2. The steady-state equations can be written in terms of matrices of lower dimension suitable for numerical implementation as follows:

$$u_0[S_3 + S_3^0\beta_3 - \lambda I] + v(1)S_1^0\beta_3 = 0, \tag{12}$$

$$\lambda u_{i-1} + u_i[S_3 - \lambda I] = 0, 1 \le i \le N-1, \tag{13}$$

$$u_1 S_3^0\beta_1 + v_1 A_1 + v_2 S_1^0\beta_1 + w_1[I \otimes S_2^0] = 0, \tag{14}$$

$$u_i S_3^0\beta_1 + \lambda v_{i-1} + v_i A_1 + v_{i+1} S_1^0\beta_1 + w_i[I \otimes S_2^0] = 0, \ 2 \le i \le N-1, \tag{15}$$

$$\lambda u_{N-1} e\beta_1 + \lambda v_{N-1} + v_N A_1 + v_{N+1} S_1^0\beta_1 + w_N[I \otimes S_2^0] = 0, \tag{16}$$

$$\lambda v_{i-1} + v_i A_1 + v_{i+1} S_1^0\beta_1 + w_i[I \otimes S_2^0] = 0, \ N+1 \le i \le n-k, \tag{17}$$

$$\lambda v_{n-k} + v_{n-k+1}[A_1 + \lambda I] + w_{n-k+1}[I \otimes S_2^0] = 0, \tag{18}$$

$$\mu_{SF} v_1[I \otimes \beta_2] + w_1 A_2 = 0, \tag{19}$$

$$\mu_{SF} v_i[I \otimes \beta_2] + \lambda w_{i-1} + w_i A_2 = 0, \ 2 \le i \le n-k, \tag{20}$$

$$\mu_{SF} v_{n-k+1}[I \otimes \beta_2] + \lambda w_{n-k} + w_{n-k+1}[A_2 + \lambda I] = 0. \tag{21}$$

4.2 The Steady-State Probability Vectors for the *Warm* and *Hot* Systems

The steady-state equations for the warm and hot systems have more equations than those of the *cold* system. Suppose that vector x is partitioned as $x = (u_0, u_1, \ldots, u_{N-1}, v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n)$ to account for the extra states resulting from components failing when the system is down. But first, note that equations (12) through (17), (19) and (20) are common to all three systems. Equations (18) and (21) are replaced with the following set of equations for *warm* and *hot* systems found by letting $\tau = \theta$ for the *warm* system equations and letting $\tau = \lambda$ and $\theta = \lambda$ in A_3 and A_4 for the *hot* system:

$$\lambda v_{n-k} + v_{n-k+1} A_3 + v_{n-k+2} S_1^0\beta_1 + w_{n-k+2}[I \otimes S_2^0] = 0, \tag{18a}$$

$$\tau v_{i-1} + v_i A_3 + v_{i+1} S_1^0\beta_1 + w_i[I \otimes S_2^0] = 0, \ n-k+2 \le i \le n-1, \tag{18b}$$

$$\tau v_{n-1} + v_n[A_3 + \tau I] + w_n[I \otimes S_2^0] = 0, \tag{18c}$$

$$\mu_{SF} v_{n-k+1}[I \otimes \beta_2] + \lambda w_{n-k} + w_{n-k+1} A_4 = 0, \tag{21a}$$

$$\mu_{SF} v_i [I \otimes \beta_2] + \tau w_{i-1} + w_i A_4 = 0, \; n-k+2 \leq i \leq n-1, \tag{21b}$$

$$\mu_{SF} v_n [I \otimes \beta_2] + \tau w_{n-1} + w_n [A_4 + \tau I] = 0. \tag{21c}$$

Before we state the main results, we need the following lemmas.

Lemma 1:

$$\mu_2 \sum_{i=1}^{n-k+1} w_i e = \mu_{SF} \sum_{i=1}^{n-k+1} v_i e \tag{22}$$

where $\mu_2^{-1} = -\beta_2 S_2^{-1} e$.

Proof: Adding equations (19), (20) and (21), the stated result follows immediately.

Lemma 2:

$$\lambda u_{N-1} e + \sum_{i=1}^{N-1} u_i S_3^0 = v_1 S_1^0. \tag{23}$$

Proof: This result follows immediately by adding the equations (12) and (13).

Lemma 3:

$$\sum_{i=1}^{n-k+1} v_i e = \frac{1}{\mu_1} \sum_{i=1}^{n-k+1} v_i S_1^0, \tag{24}$$

where $\mu_1^{-1} = -\beta_1 S_1^{-1} e$.

Proof: The stated result follows by adding equations (14) through (18), and using Lemma 2 with the fact that

$$\sum_{i=1}^{n-k+1} w_i (I \otimes S_2^0) = \mu_{SF} \sum_{i=1}^{n-k+1} v_i.$$

Let X denote the time to repair a failed component from the point when the component enters into service. We then have the following result.

Theorem 1: *The random variable* X *is of phase type with representation* (ζ, U) *of order* $(m_2+1)m_1$ *where*

$$\zeta = (\beta_1, 0) \text{ and } U = \begin{bmatrix} S_1 - \mu_{SF} I & \mu_{SF} I \otimes \beta_2 \\ I \otimes S_2^0 & I \otimes S_2 \end{bmatrix}.$$

Proof: This result follows immediately by noting that

(*a*) the initial phase of the repair is chosen according to the probability vector α;

(*b*) an absorption from this set of m_1 states corresponds to the completion of the repair without any failure on the part of the repairman;

(*c*) if the repairman fails during the repair time, the Markov chain enters into the set of $m_1 m_2$ states that represent the service time to fix the repairman.

Corollary: *The mean time spent in the repair facility by a failed component is given by*

$$\mu'_X = \mu'_1 [1 + \mu_{SF} \mu'_2], \tag{25}$$

where μ'_1 *is the mean repair time of a failed component and* μ'_2 *is the mean time to fix the failed server.*

Proof: The proof is obtained by using the fact that $\mu'_X = (\pi U^0)^{-1}$, where U^0 is

such that $Ue + U^0 = 0$, and π is the solution to $\pi(U + U^0\zeta) = 0$, and $\pi e = 1$. By partitioning $\pi = (\pi_1, \pi_2)$ and using the special structure of U, it can easily be verified that

$$\pi_2 = \mu_{SF}\pi_1(I \otimes \beta_2(-S_2)^{-1}) \text{ and } \pi_1 = c\beta_1(-S_1)^{-1}, \tag{26}$$

where c is the normalizing constant such that $\pi_1 e + \pi_2 e = 1$. The stated result follows immediately.

Remark: Note that the mean time spent in the repair facility depends only on the failure rate of the server, the mean service time of the unit, and the mean repair time of the server.

Let Y denote the time it takes a failed component to return to working condition for a cold system. Note that Y includes the time, if any, a failed component spends in the queue of the repair facility. We then have the following theorem.

Theorem 2: *The random variable Y is of phase type with representation (ξ, R) of order $Nm_3 + (n-k+1)[m_1 + m_2]$, where the vector $\xi = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{N-1}, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_{n-k+1}, \boldsymbol{0}, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_{n-k+1})$ is such that $\boldsymbol{a}_i = d\boldsymbol{u}_{i-1}$, $1 \le i \le N-1$, $\boldsymbol{b}_1 = d\boldsymbol{u}_{N-1}e$, $\boldsymbol{b}_i = d\boldsymbol{v}_i$, $2 \le i \le n-k+1$, and $\boldsymbol{c}_i = d\boldsymbol{w}_{i-1}$, for $2 \le i \le n-k+1$, with $d = [1 - \boldsymbol{v}_{n-k+1}e - \boldsymbol{w}_{n-k+1}e]^{-1}$. The matrix R is of the form*

$$R = \begin{bmatrix} R_{11} & R_{12} & 0 \\ 0 & R_{22} & \mu_{SF}(I \otimes \beta_2) \\ 0 & I \otimes S_2^0 & R_{33} \end{bmatrix},$$

where the entries of R are

$$R_{11} = \begin{bmatrix} S_3 - \lambda I & \lambda I & 0 & \ldots & 0 & 0 \\ 0 & S_3 - \lambda I & \lambda I & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & S_3 - \lambda I & \lambda I \\ 0 & 0 & 0 & \ldots & 0 & S_3 - \lambda I \end{bmatrix},$$

$$R_{12} = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \\ \lambda I & 0 & 0 & \ldots & 0 \end{bmatrix},$$

$$R_{22} = \begin{bmatrix} A_1 & \lambda I & 0 & \ldots & 0 & 0 \\ S_1^0\beta_1 & A_1 & \lambda I & \ldots & 0 & 0 \\ 0 & S_1^0\beta_1 & A_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & S_1^0\beta_1 & A_1 + \lambda I \end{bmatrix},$$

$$R_{33} = \begin{bmatrix} A_2 & \lambda I & 0 & \dots & 0 \\ 0 & A_2 & \lambda I & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \lambda I \\ 0 & 0 & 0 & \dots & A_2 + \lambda I \end{bmatrix}.$$

Proof: The theorem follows by noting that a failed component will see the server either

(*a*) on vacation with i $(1 \leq i \leq N)$ components (including the current one) waiting for repair (note that when $i = N$ the server will be recalled from vacation instantaneously), or

(*b*) busy with i $(1 \leq i \leq n-k+1)$ components (including the current one) in repair facility,

(*c*) is under repair with i $(1 \leq i \leq n-k+1)$ components (including the current one) in the repair facility.

Since failure times of the components are exponential, the probability vector $\boldsymbol{\xi}$ is just a scalar multiple of the relevant probabilities at an arbitrary time.

Let Z denote the time it takes the system to fail starting with all functioning components. First note that the probability density function of Z is identical for all three systems (cold, warm, hot). The following theorem shows that Z is of phase type.

Theorem 3: *The random variable Z is of phase type with representation $(\boldsymbol{\eta}, L)$ of order $Nm_3 + (n-k)[m_1 + m_2]$, where $\boldsymbol{\eta} = (\boldsymbol{\beta}_3, \mathbf{0})$ and the matrix L is of the form*

$$L = \begin{bmatrix} L_{11} & L_{12} & 0 \\ L_{21} & L_{22} & \mu_{SF}(I \otimes \beta_2) \\ 0 & I \otimes S_2^0 & L_{33} \end{bmatrix},$$

where $L_{11} = B_{11}^c$ and L_{12} is of the form B_{12}^c with dimension $Nm_3 \times (n-k)m_1$ and L_{21} is of the form B_{21}^c with dimension $(n-k)m_1 \times Nm_3$, and

$$L_{22} = \begin{bmatrix} A_1 & \lambda I & 0 & \dots & 0 & 0 \\ S_1^0\beta_1 & A_1 & \lambda I & \dots & 0 & 0 \\ 0 & S_1^0\beta_1 & A_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & S_1^0\beta_1 & A_1 \end{bmatrix},$$

$$L_{33} = \begin{bmatrix} A_2 & \lambda I & 0 & \dots & 0 \\ 0 & A_2 & \lambda I & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \lambda I \\ 0 & 0 & 0 & \dots & A_2 \end{bmatrix}.$$

Proof: The results follow immediately by noting that the set of states $(\boldsymbol{n}-\boldsymbol{k}+\mathbf{1})$ and $(\boldsymbol{n}-\boldsymbol{k}+\mathbf{1})'$ correspond to the system failing, and then, combining these into one absorbing state.

Suppose V_1 denotes the vacation duration of the server. Then we have

Theorem 4: *The random variable V_1 is of phase type with representation (φ, B_{11}^c) of order Nm_3, where the vector $\varphi = (\beta_3, \mathbf{0})$.*

Proof: This follows by noting the vacation period of the server starts when all components are in functioning state and returns according to (N,T)-policy.

Remark: Note that, as is expected, the random variable V_1 depends only on the components' failure rate λ and the server's vacation time T.

Corollary: *The mean vacation period of the server is given by*

$$\mu_V' = \lambda \frac{\sum_{i=0}^{N-1} u_i e}{u_0 e}, \tag{27}$$

where the steady-state probabilities, u_i, $0 \le i \le N-1$, are as given in (12)-(14).

Proof: The result is obtained by using the fact that $\mu_V' = (\pi B^0)^{-1}$, where B^0 is such that $B_{11}^c e + B^0 = 0$, and π is the solution to $\pi(B_{12}^c + B^0\varphi) = 0$, and $\pi e = 1$. Partitioning $\pi = (\pi_0, \pi_1, \ldots, \pi_{N-1})$, and noting that $\pi_i = du_i$, $0 \le i \le N-1$ with d is such that $\pi e = 1$, the stated result follows immediately.

Suppose that V_2 denotes the non-vacation duration of the server. Note that this duration includes the time the server is busy fixing failed components as well as the time the server is being repaired. Then we have:

Theorem 5: *The random variable V_2 is of phase type with representation $(\psi \otimes \beta_1, M)$ of order $(n-k+1)\,[m_1+m_2]$, where the vector $\psi = (\psi_1, \ldots, \psi_N, 0, \ldots, 0)$ is given by*

$$\psi_i = du_i S_3^0,\ 1 \le i \le N-1,\ \text{and}\ \psi_N = du_{N-1}e,$$

where d is the normalizing constant and the matrix M is

$$M = \begin{bmatrix} B_{22}^c & \mu_{SF}(I \otimes \beta_2) \\ I \otimes S_2^0 & B_{33}^c \end{bmatrix},$$

where B_{22}^c and B_{33}^c are as given in (4) *and* (5).

Proof: This result follows by noting that the non-vacation period of the server starts with at least one failed component waiting for repair and with the return of the server under (N,T)-policy.

The following interesting result gives an expression for the fraction of time the server is under repair.

Theorem 6: *The fraction of time the server is under repair is given by*

$$P(\text{server is under repair}) = \frac{p_{NV}\mu_{SF}}{\mu_{SF}+\mu_2}, \tag{28}$$

where μ_2 is the repair rate of the failed server, and p_{NV} is the probability that the server is not on vacation where

$$p_{NV} = 1 - \sum_{i=0}^{N-1} u_i e.$$

Proof: This follows immediately from Lemma 1.

Remark: The result (28) can be intuitively explained as follows. First note that the failure of the server may occur only during a non-vacation period. Since a failed server is immediately started on a phase type service, the fraction of time the server is under repair is equal to the product of the probability that the server is not on vacation and the probability of the server being busy in an M/PH/1 queue with no waiting room. The arrival rate is given by μ_{SF} and the service time is of phase type with representation (β_2, S_2).

Theorem 7: *The probability mass function* $\{a_i\}$ *of the number of failed components waiting for repair at the time when the vacationing server returns to the system, and the conditional probability mass function* $\{b_i\}$ *of the number of failed components waiting for repair given that the server returning from vacation finds at least one component in the repair facility are given by*

$$a_i = gG^i S_3^0,\ 0 \le i \le N-1, a_N = \lambda g G^{N-1} e, \qquad b_i = da_i,\ 1 \le i \le N, \tag{29}$$

where

$$g = \beta_3(\lambda I - S_3)^{-1},\ G = \lambda(\lambda I - S_3)^{-1},\ \text{and } d = (\beta_3 G e)^{-1}. \tag{30}$$

Proof: First note that $u_i S_3^0 dt$, $1 \le i \le N$ gives the steady-state probability that in a small time interval of width dt, the server finishes the vacation to find i components waiting in the repair facility. The stated result follows from equations (12) through (14) and Lemma 2.

Remark: Note that the probabilities in (29), as is to be expected, depend only on the components' failure rate and the distribution of the server's vacation period.

5. System Performance Measures

In this section we give a number of system performance measures useful in qualitative interpretation of the model under study. These measures are for a *cold* system and similar measures are obtained for the other two systems. The details are omitted. However, in the numerical example section we will compare these measures for all three systems.

(*a*) The fraction of time the server is on vacation is given by $\sum_{i=0}^{N-1} u_i e$.

(*b*) The fraction of the time the server is under repair is $\sum_{i=1}^{n-k+1} w_i e$. Note that this measure is given in terms of μ_{SF} and μ_2 from equation (28).

(*c*) The probability mass function of the number of components under repair is given by

$$f_R(i) = \begin{cases} u_0 e, & i = 0, \\ u_i e + v_i e + w_i e, & 1 \le i \le N-1 \\ v_i e + w_i e, & N \le i \le n-k+1. \end{cases}$$

The mean and the standard deviation of this probability function can also be computed.

(*d*) The fraction of time the system is down is given by $u_{n-k+1}e + w_{n-k+1}e$.

(*e*) The mean time spent in the repair facility by a failed component is given by (25).

(*f*) The mean vacation duration of the server is given by (27).

(*g*) The mean non-vacation period of the server is calculated as $(\psi \otimes \beta_1)(-M)^{-1}e$.

(*h*) The mean number of components waiting for repair when the server returns from a vacation and the conditional mean number of components waiting for repair given that the server returning from vacation sees at least one component in the repair facility can be computed from the probability functions given in (29).

6. Numerical Examples

In this section we discuss some interesting numerical examples and present an optimization problem. A Fortran code was developed and tested on a large number of examples using a Pentium II PC. Particular cases, which reduce to simple models and internal accuracy checks, such as the one outlined in Section 2 and Lemma 1, are used to confirm the correctness of the code. Recall that $\mu_i' = \beta_i(-S_i)^{-1}e$, $1 \le i \le 3$, is the mean of the PH-distribution with representation (β_i, S_i).

For illustrative purpose, we consider the following three PH-distributions for repair times of the components, for vacation duration of the server, and for the fixing times of the failed server:

Exponential (10) (**EX**),

Erlang with 10 phases and with parameter 10 (**ER**), and

Hyperexponential: 0.90 EXP(100) + 0.09EXP(1) + 0.01EXP(0.1) (**HE**).

The above distributions are normalized so that $\mu_1' = 0.5$, $\mu_2' = 0.125$, and $\mu_3' = 0.25$. The other parameters are taken to be $n = 10$, $k = 4$, $\lambda = 1$, $\theta = 0.3$, and $\mu_{SF} = 1$. We consider 27 possible combinations for the reliability model under study by choosing one of these three distributions for the repair times of the components, one for the fixing times of the failed server and one for the server's vacation period. The five performance measures

(*a*) the mean time to system failure,

(*b*) the mean vacation period of the server,

(*c*) the mean non-vacation period of the server,

(*d*) the probability the server is on vacation and

(*e*) the mean number of components under repair

are plotted in Figures 1 through 16 as functions of N varying from 1 through $n - k = 6$. The two performance measures

(*f*) the probability mass function of the number of failed components seen by the server returning from vacation, and

(*g*) the conditional probability mass function of the number of failed components seen by the returning server given that at least one failed component in the repair facility,

are plotted in Figures 17 through 18. On examination of these figures, we notice the following observations.

The Mean Time to System Failure (Figures 1 through 3):

- As expected the mean time to system failure decreases as N increases for all cases.
- By fixing the two distributions as **ER** and varying the other with one of the three distributions (**ER**, **EX**, or **HE**), we notice that this measure tends to decrease with increasing variability. In Figure 1, we plot this performance measure by varying the server's vacation period.

The Mean Vacation Period of the Server (Figure 4):

- Recall that the mean vacation period of the server depends only on λ and the distribution of the vacation period. As expected, this measure is identical for all three distributions when $N = 1$. However, for other values of N this measure increases with increasing variance of the vacation period and also with N.
- For hyperexponential case, this measure varies significantly as N is varied. However, for the other two cases, when $N \geq 2$, the change in this measure appears to be insignificant.

The Mean Non-Vacation Period of the Server (Figures 5-10):

- By varying the repair times of the components with one of the three distributions (**ER**, **EX**, or **HE**) and fixing the other two with **ER**, we notice that the mean non-vacation period of the server appears to decrease with increasing variability (see Figure 5). Furthermore, the hyperexponential distribution appears to yield smaller means for all values of N compared to the other two cases.
- By varying the fixing times of the failed server with one of the three distributions (**ER**, **EX**, or **HE**), and fixing the other two with **ER**, w notice that this measure decreases with increasing variability (see Figure 6). Going from $N = 1$ to $N = 2$, there is a significant jump in the mean for all three distributions. For other values of N, there appears to be insignificant change.
- By varying the vacation period with one of the three distributions (**ER**, **EX**, or **HE**) and fixing the other two as **ER**, we notice that this measure increases with increasing variability (see Figure 7). This observation is in direct contrast with the previous two observations.
- Suppose we use **HE** instead of **ER** to fix the other two distributions and vary the other with one of **ER**, **EX**, or **HE**. While the behavior (see Figures 8 and 10) of this measure is very similar to the earlier ones in the cases when the repair times of the components and when the vacation period the server is varied, an interesting observation (see Figure 9) is seen in the case when the fixing times of the failed server is varied. Here, we note that this measure appears to be independent of the distribution.

The Probability that the Server is on Vacation (Figures 11-13):

- By varying the repair times of the components with one of the three distributions (**ER**, **EX**, or **HE**), and fixing the other two as **ER**, we notice that the probability that the server is on vacation appears to increase with increasing variability (see Figure 11). Furthermore, this measure appears to be independent of the values of N. The same observation holds when the fixing time of the failed server is varied and the other two fixed to be **ER** (see Figure 12).

- In the case when the vacation period of the server is varied with one of the the three distributions (**ER**, **EX**, or **HE**) and fixing the other two as **ER**, we notice that this measure increases with increasing variance. In the **HE** case, we notice that the probability increases as N increases.

The Mean Number of Components Under Repair (Figures 14-16):

- here, we look at the three systems: *cold*, *warm*, and *hot* by fixing two of the three distributions as **ER** and the other with **HE**. When the vacation period is **HE**, the mean number of components under repair appears to be almost the same for all three systems (see Figure 16); however, we see a significant difference in this measure for the other two cases (see Figures 14 and 15). As expected, this measure is largest for *hot* system and smallest for the *cold* system.

The Probability Functions of the Number of Failed Components (Figures 17-18):

- In Figure 17, the probability mass function of the number of failed components seen by the server returning from vacation, is plotted for the three distributions for the case when $N = 6$. We notice that the probability that the server sees no component waiting for repair is highest for the **HE** case. This can be intuitively explained as follows. When the vacation period is **HE**, the server will have a large number of small vacation periods followed by a long vacation period. During the shorter vacation periods, the server is more likely to see no components waiting for repair.
- In Figure 18, the conditional probability mass function of the number of failed components seen by the returning server, conditional on the fact that at least one failed component is waiting for repair, is plotted for the case when $N = 6$. The figure indicates, as expected, that an **HE** vacation is more likely to see larger components waiting for repair than the other two distributions.

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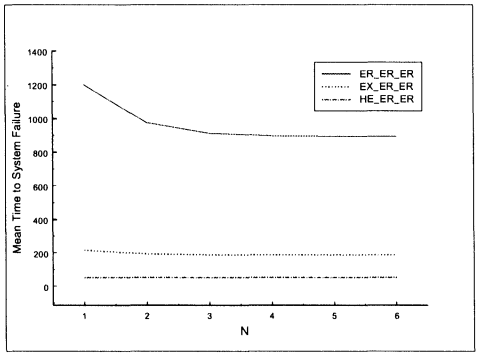


Figure 1

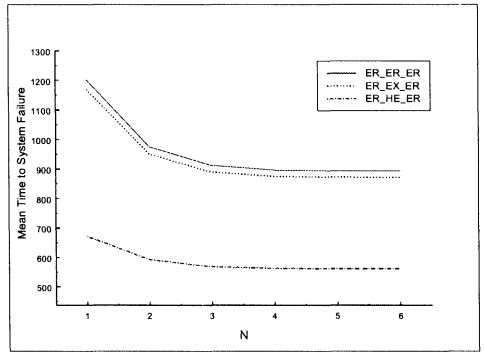
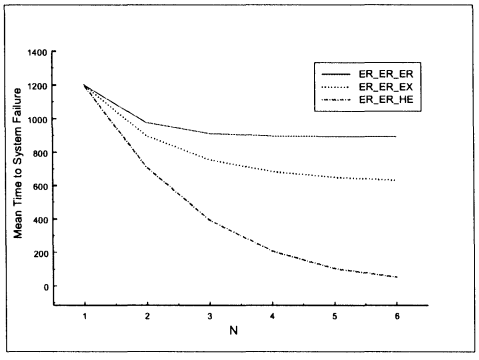


Figure 2



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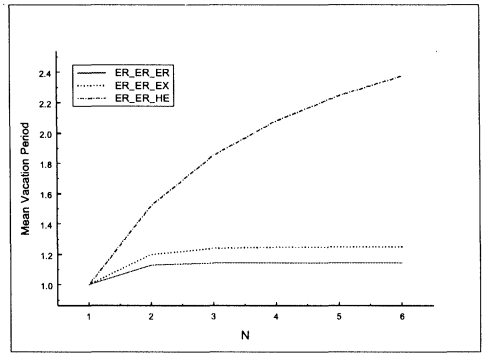


Figure 4

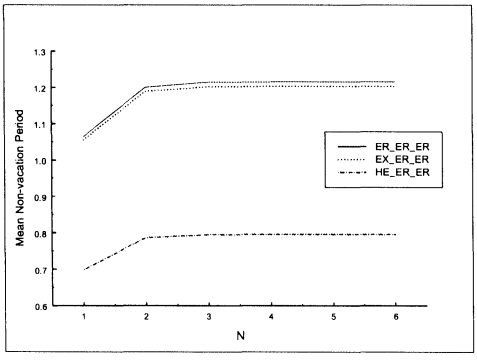
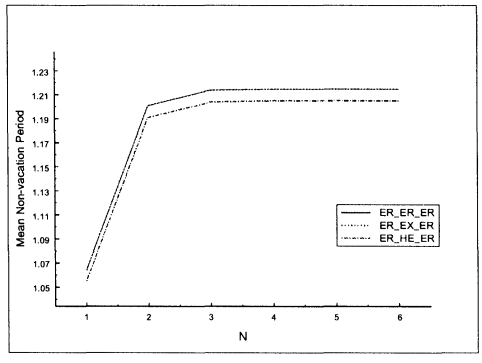


Figure 5



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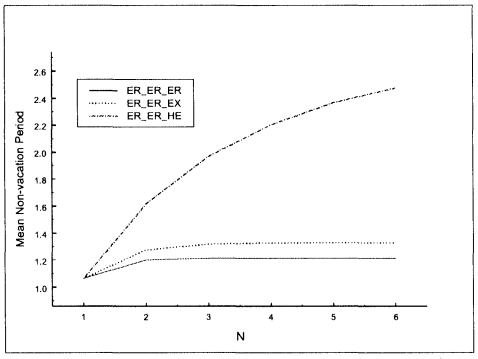


Figure 7

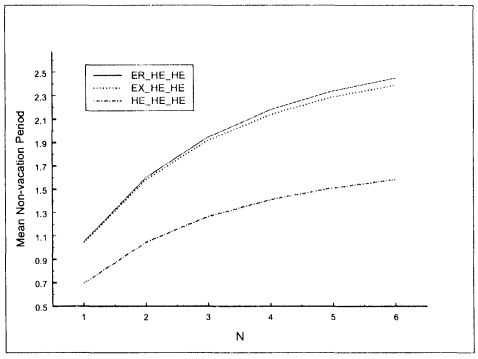


Figure 8

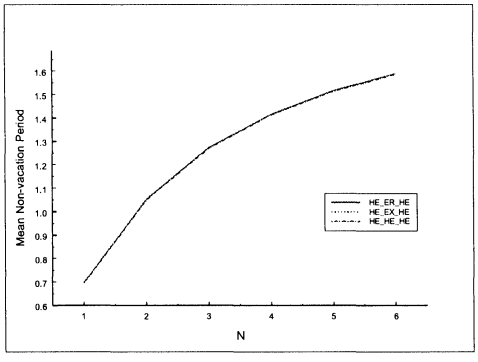
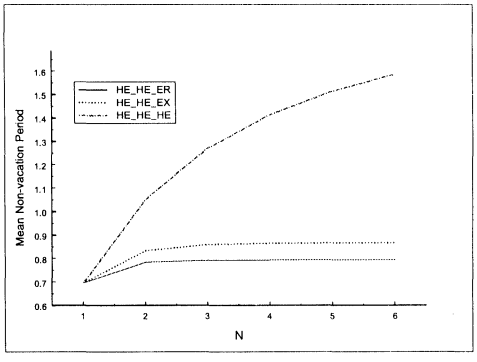


Figure 9



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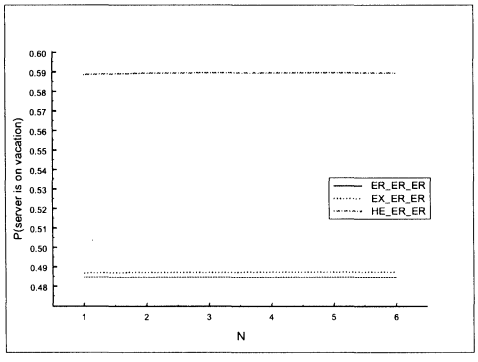


Figure 11

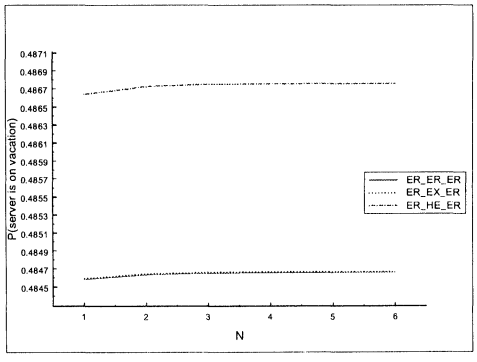


Figure 12

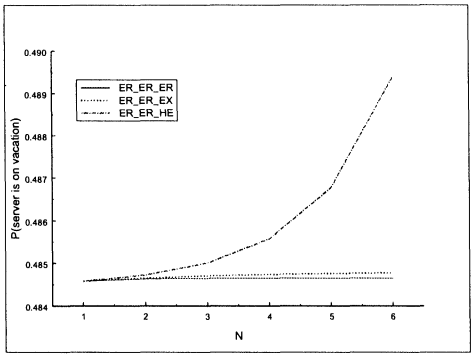
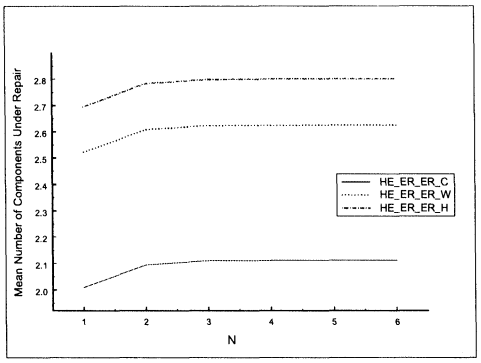


Figure 13



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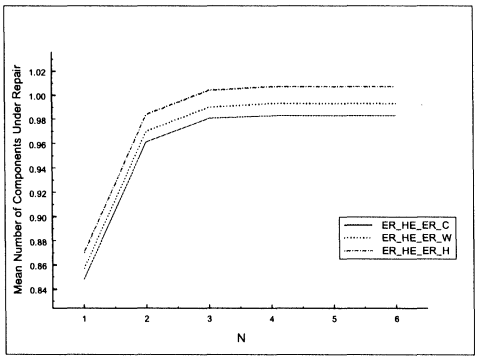

Figure 15

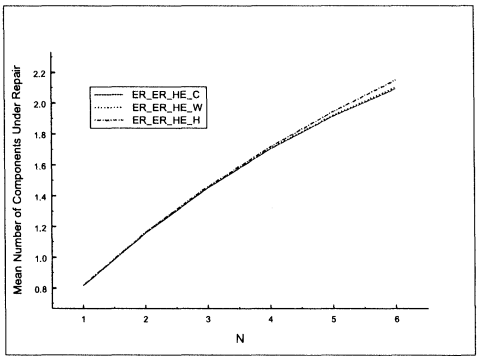

Figure 16

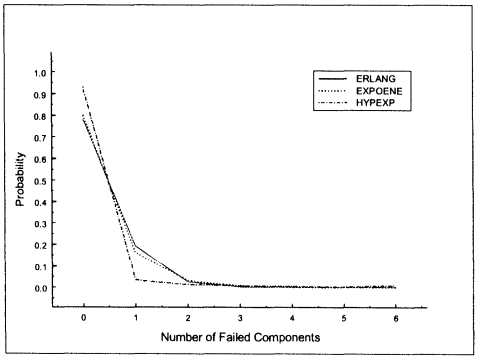

Figure 17

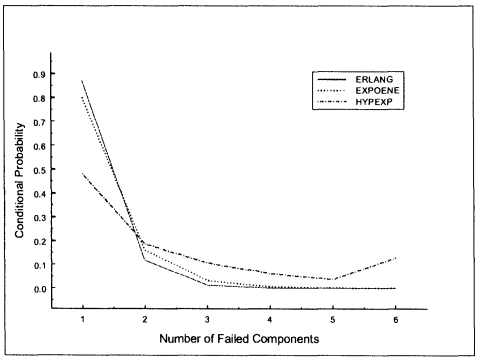

Figure 18