Journal of Applied Mathematics and Stochastic Analysis, **16:2** (2003), 121-126.

COMPLETE CONVERGENCE FOR NEGATIVELY DEPENDENT RANDOM VARIABLES

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(Received February 2001; Revised January 2003)

In this paper, we study the complete convergence for the means $\frac{1}{n}\sum_{i=1}^{n} X_i$ and $\frac{1}{n^\alpha}\sum_{k=1}^{n} X_{nk}$ via. exponential bounds, where $\alpha > 0$ and $\{X_n,\ n \geq 1\}$ is a sequence of negatively dependent random variables and $\{X_{nk},\ 1 \leq k \leq n, n \geq 1\}$ is an array of rowwise pairwise negatively dependent random variables.
Key words: Complete Convergence, Negatively Dependent.
AMS (MOS) subject classification: 60E15, 60F15.

1 Introduction

Let $\{X_n,\ n \geq 1\}$ be a sequence of i.i.d., real random variables. Hsu and Rabbins [5] proved that if $E(X) = 0$ and $E(X^2) < \infty$, then the sequence $\frac{1}{n}\sum_{i=1}^{n} X_i$ converges to 0 completely. (i.e., the series $\sum_{n=1}^{\infty} P[|S_n| > n\varepsilon] < \infty$, converges for every $\varepsilon > 0$). Now let $\{X_n,\ n \geq 1\}$ be a sequence of negatively dependent real random variables. In this paper, we proved the complete convergence of the sequence $\frac{1}{n}\sum_{i=1}^{n} X_i$, via. exponential bounds. In addition if $\{X_{nk},\ 1 \leq k \leq n, n \geq 1\}$ is an array of rowwise pairwise negatively dependent random variables, we proved complete convergence of the sequence $\{\frac{1}{n^\alpha}\sum_{k=1}^{n} X_{nk},\ n \geq 1\}$ where $\alpha > 0$. To prove these theorems we need to the following definitions and lemmas.

Definition 1: The random variables $X_1, \cdots, X_n$ are pairwise negatively dependent if

$$P(X_i \leq x_i, X_j \leq x_j) \leq P(X_i \leq x_i)P(X_j \leq x_j), \tag{1.1}$$

for all $x_i, x_j \in IR$, $i \neq j$. It can be shown that (1.1) is equivalent to

$$P(X_i > x_i, X_j > x_j) \leq P(X_i > x_i)P(X_j > x_j), \tag{1.2}$$

for all $x_j, x_i \in IR$, $i \neq j$.

Definition 2: The random variables $X_1, \cdots, X_n$ are said to be negatively dependent (ND) if we have

$$P(\cap_{j=1}^n (X_j \leq x_j)) \leq \prod_{j=1}^n P(X_j \leq x_j), \tag{1.3}$$

and

$$P(\cap_{j=1}^n (X_j > x_j)) \leq \prod_{j=1}^n P(X_j > x_j), \tag{1.4}$$

for all $x_1, \cdots, x_n \in IR$. An infinite sequence $\{X_n,\ n \geq 1\}$ is said to be ND if every finite subset $\{X_1, \cdots, X_n\}$ is ND.

Conditions (1.3) and (1.4) are equivalent for $n = 2$. However Ebrahimi and Ghosh [4] show that these definitions do not agree for $n \geq 3$.

Definition 3: The sequence $\{X_n,\ n \geq 1\}$ of random variables converges to zero completely (denoted $lim_{n\to\infty} X_n = 0$ completely), if for every $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P[|X_n| > \varepsilon] < \infty. \tag{1.5}$$

Lemma 1: (Petrov [8]) *Let X be a random variable with $E(X) = 0$, $E(X^2) < \infty$, and suppose there exists a positive constant H such that for all $m \geq 2$*

$$|E(X^m)| \leq \frac{1}{2} m! \sigma^2 H^{m-2}, \tag{1.6}$$

then for every $|t| \leq \frac{1}{2H}$

$$Ee^{tX} \leq e^{t^2\sigma^2}.$$

Lemma 2: (Serfeling [9]) *Let X be a r.v. with $E(X) = \mu$. If $P[a \leq X \leq b] = 1$. Then for every real number $h > 0$,*

$$Ee^{h(X-\mu)} \leq e^{\frac{h^2(b-a)^2}{8}},$$

and

$$Ee^{h|X-\mu|} \leq 2e^{\frac{h^2(b-a)^2}{8}}.$$

The next three lemmas will be needed in the proofs of the strong law of large numbers in the next section [3].

Lemma 3: *Let $X_1, \cdots, X_n$ be ND random variables and $f_1, \cdots, f_n$ be a sequence of Borel functions which all are monotone increasing (or all are monotone decreasing), then $f_1(X_1), \cdots, f_n(X_n)$ are ND random variables.*

Lemma 4: *Let $X_1, \cdots, X_n$ be pairwise ND random variables, then*

$$E(X_iX_j) \leq E(X_i)E(X_j), \qquad \forall \quad i \neq j.$$

Lemma 5: *Let $X_1, \cdots, X_n$ be ND nonnegative random variables, then*

$$E[\prod_{j=1}^{n} X_j] \le \prod_{j=1}^{n} E[X_j].$$

2 Exponential Bounds and Complete Convergence

In this section, we obtained some exponential bounds for probability $P[|S_n| > x]$ for every $x > 0$ using Lemmas 1 and 2, and then we proved the complete convergence of the sequence $\{\frac{1}{n}\sum_{i=1}^{n} X_i\}$. We shall consider a sequence of ND random variables $\{X_n,\ n \ge 1\}$, with zero means and finite variances . We put

$$S_n = \sum_{k=1}^{n} X_k, \quad B_n = \sum_{k=1}^{n} \sigma_k^2.$$

Theorem 1: *Let $\{X_n,\ n \ge 1\}$ be a sequence of ND r.v.'s and suppose there exists a positive constant H such that for all $m \ge 2$ and $1 \le k \le n$,*

$$|E(X_k^m)| \le \frac{1}{2} m! \sigma_k^2 H^{m-2}, \tag{2.7}$$

if $\sum_{n=1}^{\infty} \exp[-\frac{n^2\varepsilon^2}{4B_n}] < \infty$, for every $\varepsilon > 0$, then

$$\frac{1}{n}\sum_{k=1}^{n} X_k \longrightarrow 0, \quad \text{completely}.$$

Proof: By Lemmas 1, 3, 5 and Markov's inequality for every $|t| \le \frac{1}{2H}$ we have

$$P[|S_n| \ge x] \le P[S_n \ge x] + P[-S_n \ge x] \le e^{-tx} E e^{tS_n} + e^{-tx} E e^{-tS_n}$$

$$\le e^{-tx} (\prod_{k=1}^{n} E e^{tX_k} + \prod_{k=1}^{n} E e^{-tX_k}) \le 2\exp[-tx + t^2 B_n].$$

Hence

$$P[|S_n| \ge x] \le 2\exp[-tx + t^2 B_n]. \tag{2.8}$$

With $h(t) = t^2 B_n - tx$ and $0 \le x \le \frac{B_n}{H}$, the equation $h'(t) = 0$ has the unique solution $t = \frac{x}{2B_n}$ which minimize $h(t)$. Hence

$$P[|S_n| \ge x] \le 2\exp[-\frac{x^2}{4B_n}] \quad \text{if} \quad 0 \le x \le \frac{B_n}{H}.$$

Let $a = \frac{B_{n^\star}}{H}$, where $n^\star$ is the first subscript so that $B_n > 0$. Then for every $0 < \varepsilon \le a$, and by the assumption

$$\sum_{n=1}^{\infty} P[\frac{|S_n|}{n} \ge \varepsilon] \le \sum_{n=1}^{\infty} 2\exp[-\frac{n^2\varepsilon^2}{4B_n}] < \infty,$$

and for each $\varepsilon' > a \geq \varepsilon > 0$, we have

$$[\sum_{n=1}^{\infty} P[\frac{|S_n|}{n} \geq \varepsilon'] \leq \sum_{n=1}^{\infty} P[\frac{|S_n|}{n} \geq \varepsilon] < \infty.$$

These complete the proof.

Remark 1: In particular if $B_n = O(n^{\alpha}),\ 0 < \alpha < 2,$ then series $\sum_{n=1}^{\infty} \exp[-\frac{n^2\varepsilon^2}{4B_n}]$ converges.

Remark 2: If the random variables $X_1, X_2, \cdots, X_n$ are ND r.v.'s with zero means and uniformly bounded, that is if there exists a positive constant c such that

$$P[|X_k| \leq c] = 1, \qquad k \geq 1$$

then for all integers $m \geq 2$ we have

$$|E(X_k^m)| \leq c^{m-2}\sigma_k^2.$$

Thus Condition (1.6) in Lemma 1 is satisfied with $H = c$. Hence if $\sum_{n=1}^{\infty} \exp[-\frac{n^2\varepsilon^2}{4B_n}] < \infty$, for every $\varepsilon > 0$, then

$$\frac{1}{n}\sum_{k=1}^{n} X_k \longrightarrow 0, \quad \text{completely.}$$

Theorem 2: *Let* $\{X_n,\ n \geq 1\}$ *be a sequence of ND random variables and* $c_n = \max\{ess\sup \frac{|X_k|}{\sqrt{B_n}}, 1 \leq k \leq n\}$. *If* $\sum_{n=1}^{\infty} \exp[-\frac{2n\varepsilon^2}{c_n^2 B_n}] < \infty$, *then for each* $\varepsilon > 0$,

$$\frac{1}{n}\sum_{k=1}^{n} X_k \longrightarrow 0, \quad \text{completely.}$$

Proof: By Lemmas 2, 3, 5 and Markov's inequality for every $t > 0$, we have

$$P[|S_n| > \varepsilon] \leq P[S_n > \varepsilon] + P[-S_n > \varepsilon] \leq e^{-\frac{t\varepsilon}{\sqrt{B_n}}} Ee^{\frac{tS_n}{\sqrt{B_n}}} + e^{-\frac{t\varepsilon}{\sqrt{B_n}}} Ee^{\frac{-tS_n}{\sqrt{B_n}}}$$

$$\leq e^{-\frac{t\varepsilon}{\sqrt{B_n}}}\{\prod_{k=1}^{n}(Ee^{\frac{tX_k}{\sqrt{B_n}}} + Ee^{\frac{-tX_k}{\sqrt{B_n}}})\} \leq 2\exp[-\frac{t\varepsilon}{\sqrt{B_n}} + \frac{nt^2c_n^2}{2}].$$

Thus, for $t = \frac{\varepsilon}{nc_n^2\sqrt{B_n}}$ we have

$$P[|S_n| > \varepsilon] \leq 2\exp[-\frac{\varepsilon^2}{2nc_n^2 B_n}],$$

and by the assumption we have

$$\sum_{n=1}^{\infty} P[\frac{|S_n|}{n} > \varepsilon] \leq 2\sum_{n=1}^{\infty} \exp[-\frac{n\varepsilon^2}{2c_n^2 B_n}] < \infty$$

which completes the proof.

Remark 3: In particular if $c_n^2 B_n = O(n^{\alpha}),\ 0 < \alpha < 1,$ then series $\sum_{n=1}^{\infty} \exp[-\frac{n\varepsilon^2}{2c_n B_n}]$ converges.

3 Strong Limit Theorem for arrays

Let $\{X_{nk},\ 1 \le k \le n,\ n \ge 1\}$ be an array of rowwise pairwise ND random variables with

$$E[X_{nk}] = 0, \quad \sigma_{nk}^2 = E[X_{nk}^2], \quad 1 \le k \le n,\ n \ge 1.$$

We consider the means $\xi_n = \frac{1}{n^\alpha}\sum_{k=1}^n X_{nk},\ n \ge 1$ where α is a fixed positive real number. Since $X_{nk},\ 1 \le k \le n$ are pairwise ND random variables, by Lemma 4 we can write

$$E[\xi_n^2] \le \frac{1}{n^{2\alpha}}\sum_{k=1}^n \sigma_{nk}^2, \tag{3.9}$$

because

$$E[\xi_n^2] = E[\frac{1}{n^\alpha}\sum_{k=1}^n X_{nk}]^2 = \frac{1}{n^{2\alpha}}\sum_{k=1}^n\sum_{j=1}^n E[X_{nk}X_{nj}]$$

$$= \frac{1}{n^{2\alpha}}[\sum_{k=1}^n E[X_{nk}^2] + \sum\sum_{k\neq j} E[X_{nk}X_{nj}]] \le \frac{1}{n^{2\alpha}}\sum_{k=1}^n \sigma_{nk}^2.$$

Theorem 3: *Let $\{X_{nk},\ 1 \le k \le n,\ n \ge 1\}$ be an array of rowwise pairwise ND random variables with $E[X_{nk}] = 0$. If for some $\alpha > 0$*

$$\sum_{n=1}^\infty \frac{1}{n^{2\alpha}}\sum_{k=1}^n \sigma_{nk}^2 < \infty,$$

then

$$\frac{1}{n^\alpha}\sum_{k=1}^n X_{nk} \longrightarrow 0 \quad \text{completely.}$$

Proof: By Chebyshev's inequality and (3.9), we have

$$P[|\xi_n| > \varepsilon] \le \frac{1}{\varepsilon^2}E[\xi_n^2] \le \frac{1}{\varepsilon^2 n^{2\alpha}}\sum_{k=1}^n \sigma_{nk}^2.$$

Since for some $\alpha > 0$

$$\sum_{n=1}^\infty P[|\xi_n| > \varepsilon] \le \sum_{n=1}^\infty \frac{1}{\varepsilon^2 n^{2\alpha}}\sum_{k=1}^n \sigma_{nk}^2 < \infty,$$

by Definition 3

$$\frac{1}{n^\alpha}\sum_{k=1}^n X_{nk} \longrightarrow 0 \quad \text{completely.}$$

Corollary 1: *Under assumptions of Theorem 3, let $\sigma_{nk} \le \sigma_{kk},\ k \ge 1,\ n \ge (k+1)$. If $\sum_{k=1}^\infty \frac{\sigma_{kk}^2}{k^{2\alpha-1}} < \infty$ for some $\alpha > \frac{1}{2}$ then*

$$\frac{1}{n^\alpha}\sum_{k=1}^n X_{nk} \longrightarrow 0 \quad \text{completely.}$$

Proof: We have

$$\sum_{n=1}^{\infty} E[\xi_n^2] \leq \sum_{n=1}^{\infty} \frac{1}{n^{2\alpha}} \sum_{k=1}^{n} \sigma_{nk}^2 \leq \sum_{n=1}^{\infty} \frac{1}{n^{2\alpha}} \sum_{k=1}^{n} \sigma_{kk}^2$$

$$= \sum_{k=1}^{\infty} \sigma_{kk}^2 \sum_{n=k}^{\infty} \frac{1}{n^{2\alpha}} = O(1) \sum_{k=1}^{\infty} \frac{\sigma_{kk}^2}{k^{2\alpha-1}},$$

then

$$\sum_{n=1}^{\infty} P[|\xi_n| > \varepsilon] \leq O(1) \sum_{k=1}^{\infty} \frac{\sigma_{kk}^2}{k^{2\alpha-1}} < \infty,$$

which completes the proof.

Remark 4: The weaker condition $\sum_{k=1}^{\infty} \frac{\sigma_{kk}^2}{k^{2\alpha}} < \infty$, for every $\alpha > 0$, implies only the complete convergence of subsequence $\{\xi_{2^p}, p = 0, 1, 2, ...\}$, since

$$\sum_{p=0}^{\infty} E[\xi_{2^p}^2] \leq \sum_{p=0}^{\infty} \frac{1}{2^{2\alpha p}} \sum_{k=1}^{2^p} \sigma_{kk}^2$$

$$= \sum_{k=1}^{\infty} \sigma_{kk}^2 \sum_{p:2^p \geq k} \frac{1}{2^{2\alpha p}} = O(1) \sum_{k=1}^{\infty} \frac{\sigma_{kk}^2}{k^{2\alpha}}.$$

Acknowledgments

The authors would like to thank the referee for his careful reading of the manuscript and for many valuable suggestions which improved the presentation of the paper.

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