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FUNCTIONAL INTEGRO-DIFFERENTIAL STOCHASTIC EVOLUTION EQUATIONS IN HILBERT SPACE

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We investigate a class of abstract functional integro-differential stochastic evolution equations in a real separable Hilbert space. Global existence results concerning mild and periodic solutions are formulated under various growth and compactness conditions. Also, related convergence results are established and an example arising in the mathematical modeling of heat conduction is discussed to illustrate the abstract theory.
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1 Introduction

The purpose of this paper is to study the global existence and convergence properties of mild solutions to a class of abstract semi-linear functional stochastic integro-differential equations of the general form

$$x'(t) = Ax(t) + F(x)(t) + \int_0^t G(x)(s)dW(s), \quad 0 \le t \le T, \tag{1.1}$$

$$x(0) = h(x) + x_0,$$

in a real separable Hilbert space H. Here, $A : D(A) \subset H \to H$ is a linear (possibly unbounded) operator, $G : \mathcal{C}([0,T];H) \to C([0,T];\ L^2(\Omega; BL(K;H)))$ (where K is a

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real separable Hilbert space), $F : \mathcal{C}([0,T];H) \to L^p(0,T;L^2(\Omega;H))$ $(1 \leq p < \infty)$, W is a K-valued Wiener process with incremental covariance described by the nuclear operator Q, x_0 is an $\mathcal{F}_0$-measurable H-valued random variable independent of W, and $h : \mathcal{C}([0,T];H) \to L_0^2(\Omega;H)$.

The present work may be regarded as a direct attempt to extend recent results developed in [7, 10, 16, 18, 20] to a broader class of functional stochastic equations. The equations considered in the aforementioned papers can be viewed as special cases of (1.1) by making the appropriate identifications of F, G, and h. Moreover, we further extend these results by incorporating more general initial conditions. In particular, mild periodic solutions are obtained. To the authors' knowledge the results in this paper are new even in the case of a classical initial condition (i.e., when $h = 0$).

The deterministic version of (1.1) (and related equations) coupled with a classical initial condition has been studied extensively both when A is linear and when A is nonlinear. We refer the reader to [8, 30] and the references therein. Byszewski [13] introduced nonlocal initial conditions into such abstract initial-value problems and argued that the corresponding models more accurately describe the phenomena since more information was taken into account at the onset of the experiment, thereby reducing the ill effects incurred by a single (possibly erroneous) initial measurement. Since then, many authors have continued this work in several directions and established existence theories for first-order nonlinear evolution equations [2, 4, 29], second-order equations [7], delay equations [7, 28], Volterra integral and integro-differential equations [5, 25], and differential inclusions [1]. Concrete nonlocal parabolic and elliptic partial (integro-) differential equations arising in the mathematical modeling of various physical, biological, and ecological phenomena, as well as a discussion of the advantages of replacing the classical initial condition with a more general functional, can be found in [13, 21] and the references contained therein.

Stochastic differential equations (SDEs) in both finite and infinite dimensions have also received considerable attention. We refer the reader to [10, 32] for a thorough discussion in the finite dimensional setting, and [14, 19] for the infinite dimensional setting. A semi-group-theoretic development of a theory for the stochastic analogues of deterministic evolution equations is both powerful and beneficial since it enables one to investigate a broad class of stochastic partial differential equations within a unified context. SDEs are important from the viewpoint of applications since they incorporate (natural) randomness into the mathematical description of the phenomena, and, therefore, provide a more accurate description of it. Moreover, coupling such equations with a nonlocal initial condition strengthens the model even further.

The basic tools used in this paper include fixed-point techniques, the theory of (compact) linear semi-groups, results for probability measures, and methods and results for infinite dimensional SDEs. The results are important from the viewpoint of applications since they cover nonlocal generalizations of integro-differential SDEs arising in fields such as electromagnetic theory, population dynamics, and heat conduction in materials with memory [10, 17, 19, 32].

The outline of the paper is as follows. We review some basic facts about linear semi-groups, the theory of SDEs, and probability measures in Section 2. Then, Sections 3 and 4 are devoted to the development of our main existence results, while a discussion of various convergence results immediately follows in Section 5. Finally, the paper concludes with a discussion of a concrete nonlocal integro-partial SDE in Section 6.

2 Preliminaries

For further background of this section, we refer the reader to [9, 11, 12, 14, 15, 19, 23, 30, 32]. Throughout this manuscript, H and K denote real separable Hilbert spaces equipped with norms $\|\cdot\|_H$ and $\|\cdot\|_K$, respectively, and the space of bounded linear operators from K to H is denoted by $BL(K;H)$ (or simply $BL(H)$ if $K = H$). Also, for Banach spaces X and Y, the space of continuous functions from X into Y (equipped with the usual sup-norm) shall be denoted by $C(X;Y)$, while $L^p(0,T;X)$ shall represent the space of X-valued functions that are p-integrable on $[0,T]$.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a normal filtration $\{\mathcal{F}_t : 0 \leq t \leq T\}$ (i.e., a right-continuous, increasing family of sub σ-algebras of $\mathcal{F}$). An *H-valued random variable* is an $\mathcal{F}$-measurable function $X : \Omega \to H$ and a collection of random variables $S = \{X(t;\omega) : \Omega \to H | 0 \leq t \leq T\}$ is called a *stochastic process*. Henceforth, we shall suppress the dependence on $\omega \in \Omega$ and write $X(t)$ instead of $X(t;\omega)$ and $X : [0,T] \to H$ in place of S.

The collection of all strongly-measurable, square-integrable H-valued random variables, denoted by $L^2(\Omega;H)$, is a Banach space equipped with norm $\|X(\cdot)\|_{L^2(\Omega;H)} = (E\|X(\cdot;\omega)\|_H^2)^{1/2}$, where the *expectation*, E, is defined by $E(g) = \int_\Omega g(\omega)dP$. An important subspace is given by $L_0^2(\Omega;H) = \{f \in L^2(\Omega;H) : f \text{ is } \mathcal{F}_0\text{-measurable}\}$. Next, we define the space $\mathcal{C}([0,T];H)$ to be the set $\{v \in C([0,T];L^2(\Omega;H)) : v \text{ is } \mathcal{F}_t\text{-adapted}\}$. One can prove that this is a Banach space when equipped with the norm

$$\|v\|_{\mathcal{C}} = \sup_{0 \leq t \leq T} (E\|v(t)\|_H^2)^{1/2}. \tag{2.1}$$

Definition 2.1: A stochastic process $\{W(t) : t \geq 0\}$ in a real separable Hilbert space H is a *Wiener process* if for each $t \geq 0$,

(i) $W(t)$ has continuous sample paths and independent increments,

(ii) $W(t) \in L^2(\Omega;H)$ and $E(W(t)) = 0$,

(iii) $Cov(W(t) - W(s)) = (t-s)Q$, where $Q \in BL(K;H)$ is a nonnegative nuclear operator.

Consider the initial-value problem

$$x'(t) = Ax(t) + f(t) + g(t)W'(t), \quad 0 \leq t \leq T, \tag{2.2}$$

$$x(0) = x_0,$$

where $A : H \to H$ generates a C_0-semi-group $\{S(t) : t \geq 0\}$ on H, $f \in L^1(0,T;H)$, $g \in BL(K;H)$, W is a K-valued Wiener process with respect to $\{\mathcal{F}_t : 0 \leq t \leq T\}$, and $x_0 \in L_0^2(\Omega;H)$.

Definition 2.2: An $\mathcal{F}_t$-adapted stochastic process $x : [0,T] \to H$ is called a *mild solution* of (2.2) if $x(t)$ is measurable, for all $t \in [0,T]$, $\int_0^T \|x(s)\|_H^2 ds < \infty$ a.s. $[P]$, and

$$x(t) = S(t)x_0 + \int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)g(s)dW(s), \quad \text{a.s. } [P], \tag{2.3}$$

for all $0 \leq t \leq T$.

(The second integral in (2.3) is taken in the sense of Itó. A complete discussion of the construction of the Itó integral can be found in [14].) It is well-known that (2.2) has a unique mild solution $x \in \mathcal{C}([0,T];H)$, and if stronger regularity restrictions are imposed on the data, this solution is a strong solution (see [19, 20]).

The following alternative of the Leray-Schauder principle [24] plays a key role in Section 4.

Theorem 2.3: (Schaefer's Fixed Point Theorem [31]) *Let X be a Banach space and $\Phi : X \to X$ a continuous, compact map. Then, either the set $\xi(\Phi) = \{x \in X : \lambda x = \Phi x,$ for some $\lambda \geq 1\}$ is unbounded, or Φ has a fixed point.*

We conclude this section with some comments regarding probability measures. We refer the reader to [9, 11] for a more detailed discussion.

Let X be an H-valued random variable and let $\mathcal{P}(H)$ denote the set of all probability measures on H. The *probability measure P induced by X*, denoted P_X, is defined by $PoX^{-1} : \mathcal{B}(H) \to [0,1]$, where $\mathcal{B}(H)$ is the Borel class on H. A sequence $\{P_n\} \subset \mathcal{P}(H)$ is said to be *weakly convergent to P* if $\int_\Omega f dP_n \to \int_\Omega f dP$, for every bounded, continuous function $f : H \to I\!R$; in such case, we write $P_n \stackrel{w}{\to} P$. Next, a family $\{P_n\}$ is *tight* if for each $\epsilon > 0$, there exists a compact set K_ϵ such that $P_n(K_\epsilon) \geq 1 - \epsilon$, for all n. Prokhorov [11] established the equivalence of tightness and relative compactness of a family of probability measures. Consequently, the Arzelá-Ascoli Theorem can be used to establish tightness.

Definition 2.4: Let $P \in \mathcal{P}(H)$ and $0 \leq t_1 < t_2 < \ldots < t_k \leq T$. Define $\pi_{t_1,\ldots,t_k} : C([0,T];H) \to H^k$ by $\pi_{t_1,\ldots,t_k}(X) = (X(t_1),\ldots,X(t_k))$. The probability measures induced by $\pi_{t_1,\ldots,t_k}$ are the *finite dimensional joint distributions of P*.

Proposition 2.5: ([23], pg. 37) *If a sequence $\{X_n\}$ of H-valued random variables converges weakly to an H-valued random variable X in $L^2(\Omega;H)$, then the sequence of finite dimensional joint distributions corresponding to $\{P_{X_n}\}$ converges weakly to the finite dimensional joint distribution of P_X.*

Finally, the next theorem, in conjunction with Proposition 2.5, is the main tool in establishing a convergence result in Section 5.

Theorem 2.6: *Let $\{P_n\} \subset \mathcal{P}(H)$. If the sequence of finite dimensional joint distributions corresponding to $\{P_n\}$ converges weakly to the finite dimensional joint distribution of P and $\{P_n\}$is relatively compact, then $P_n \stackrel{w}{\to} P$.*

3 Existence Results - Lipschitz Case

Consider the initial-value problem (1.1) in a real separable Hilbert space H under the following assumptions:

(H1) The linear operator $A : D(A) \subset H \to H$ generates a C_0-semi-group $\{S(t) : t \geq 0\}$ on H,

(H2) $F : \mathcal{C}([0,T];H) \to L^p(0,T;L^2(\Omega;H))$ is such that there exists $M_F > 0$ for which

$$\|F(x) - F(y)\|_{L^p} \leq M_F \|x - y\|_{\mathcal{C}}, \quad \text{for all } x, y \in \mathcal{C}([0,T];H),$$

(H3) $G : \mathcal{C}([0,T];H) \to C([0,T];L^2(\Omega;BL(K;H)))$ $(= \mathcal{C}_{BL})$ is such that there exists $M_G > 0$ for which

$$\|G(x) - G(y)\|_{\mathcal{C}_{BL}} \leq M_G \|x - y\|_{\mathcal{C}}, \quad \text{for all } x, y \in \mathcal{C}([0,T];H),$$

(H4) $h : \mathcal{C}([0,T];H) \to L_0^2(\Omega;H)$ is such that there exists $M_h > 0$ for which

$$\|h(x) - h(y)\|_{L_0^2} \leq M_h \|x - y\|_{\mathcal{C}}, \quad \text{for all } \ x, y \in \mathcal{C}([0,T];H),$$

(H5) $x_0 \in L_0^2(\Omega;H)$.

Definition 3.1: A function $x \in \mathcal{C}([0,T];H)$ is a *mild solution* of (1.1) on $[0,T]$ if x satisfies Definition 2.2 with (2.3) replaced by

$$x(t) = S(t)(h(x)+x_0) + \int_0^t S(t-s)F(x)(s)ds + \int_0^t \int_0^s S(s-\tau)G(x)(\tau)dW(\tau)ds, \text{ a.s. } [P], \tag{3.1}$$

for all $0 \leq t \leq T$. (The Uniform Boundedness Principle and the strong continuity of $S(t)$ together guarantee the existence of a positive constant M_S such that $\|S(t)\|_{BL} \leq M_S$ for all $0 \leq t \leq T$.) Our first result is:

Theorem 3.2: *Assume that* (H1) - (H5) *hold. Then,* (1.1) *has a unique mild solution on* $[0,T]$*, if*

$$M_S[M_h + M_G T C_G + M_F T^{1/q}] < 1, \tag{3.2}$$

where $1 \leq p,\ q \leq \infty$ *are conjugate indices.*

Proof: Define the solution map $\mathcal{J} : \mathcal{C}([0,T];H) \to \mathcal{C}([0,T];H)$ by

$$(\mathcal{J}x)(t) = S(t)(h(x) + x_0) + \int_0^t S(t-s)F(x)(s)ds \tag{3.3}$$

$$+ \int_0^t \int_0^s S(s-\tau)G(x)(\tau)dW(\tau)ds, \ \ 0 \leq t \leq T.$$

The continuity of $\mathcal{J}$ is easily verified. Successive applications of Hölder's inequality yields

$$\left[E\|\int_0^t S(t-s)F(x)(s)ds\|_H^2\right]^{\frac{1}{2}} \leq T^{\frac{1}{2}} M_S \left[\int_0^T \|F(x)(s)\|_{L^2(\Omega;H)}^2 ds\right]^{\frac{1}{2}} \tag{3.4}$$

$$\leq T^{(p-1)/p} M_S \|F(x)\|_{L^p}.$$

Subsequently, an application of (H2), together with Minkowski's inequality, enables us to continue the string of inequalities in (3.4) to conclude that

$$\left[E\|\int_0^t S(t-s)F(x)(s)ds\|_H^2\right]^{\frac{1}{2}} \leq T^{\frac{1}{q}} M_S [M_F \|x\|_{\mathcal{C}} + \|F(0)\|_{L^p}]. \tag{3.5}$$

Taking the supremum over $[0,T]$ in (3.5) then implies that $\int_0^t S(t-s)F(x)(s)ds \in \mathcal{C}([0,T];H)$, for any $x \in \mathcal{C}([0,T];H)$. Further, for such x, $G(x)(s) \in BL(K;H)$ and $h(x) + x_0 \in L_0^2(\Omega;H)$ (by (H4) and (H5)). Consequently, one can argue as in [20] to conclude that $\mathcal{J}$ is a well-defined.

Next, we show that $\mathcal{J}$ is a strict contraction. Observe that for $x, y \in \mathcal{C}([0,T];H)$, we infer from (3.3) that

$$(\mathcal{J}x)(t) - (\mathcal{J}y)(t) = S(t)(h(x) - h(y)) + \int_0^t S(t-s)[F(x)(s) - F(y)(s)]ds$$

$$+\int_0^t \int_0^t S(s-\tau)[G(x)(\tau)-G(y)(\tau)]dW(\tau)ds, \;\; 0 \leq t \leq T. \tag{3.6}$$

For convenience, let I_1, I_2, and I_3 represent the first, second, and third terms, respectively, on the right-side of (3.6). Squaring both sides and taking the expectation in (3.6) yields, with the help of Young's inequality,

$$E\|(\mathcal{J}x)(t)-(\mathcal{J}y)(t)\|_H^2 \leq 4[E\|I_1\|_H^2 + E\|I_2\|_H^2 + E\|I_3\|_H^2]$$

and subsequently,

$$\|(\mathcal{J}x)(t)-(\mathcal{J}y)(t)\|_{\mathcal{C}} \leq 4[\|I_1\|_{\mathcal{C}} + \|I_2\|_{\mathcal{C}} + \|I_3\|_{\mathcal{C}}]. \tag{3.7}$$

Using reasoning similar to that which led to (3.4), one can show that

$$\|I_1\|_{\mathcal{C}} + \|I_2\|_{\mathcal{C}} \leq M_S[M_h + M_F T^{\frac{1}{q}}]\|x-y\|_{\mathcal{C}}. \tag{3.8}$$

Also, one can modify the argument of Proposition 1.9 in [20] to conclude that there exists a constant C_G (depending only on p, $Tr(Q)$, and T) such that

$$\|I_3\|_{\mathcal{C}} \leq M_S M_G C_G T\|x-y\|_{\mathcal{C}}. \tag{3.9}$$

Using (3.8) and (3.9) in (3.7) enables us to conclude that $\mathcal{J}$ is a strict contraction, provided that (3.2) is satisfied and thus, has a unique fixed point which coincides with a mild solution of (1.1). This completes the proof.

Next, we consider the following initial-value problem studied in [16].

$$x'(t) = Ax(t) + \int_0^t C(t,s)g(s,x(s))dW(s) + \int_0^t B(t,s)f_1(s,x(s))ds \tag{3.10}$$
$$+f_2(t,x(t)), \quad 0 \leq t \leq T,$$
$$x(0) = x_0,$$

where $\{B(t,s): 0 \leq t \leq s \leq T\} \bigcup \{C(t,s): 0 \leq t \leq s \leq T\} \subset BL(H)$, $g: [0,T] \times H \rightarrow BL(K;H)$, and $f_i: [0,T] \times H \rightarrow H (i=1,2)$ are given mappings satisfying the following conditions:

(H6) $f_i: [0,T] \times H \rightarrow H(i=1,2)$ is such that there exists $M_{f_i} > 0$ for which

$$\|f_i(t,x)-f_i(t,y)\|_H \leq M_{f_i}\|x-y\|_H, \quad \text{for all } t \in [0,T] \text{ and } x,y \in H,$$

(H7) $g: [0,T] \times H \rightarrow BL(K;H)$ is such that there exists $M_g > 0$ for which

$$\|g(t,x)-g(t,y)\|_{BL} \leq M_g\|x-y\|_H, \quad \text{for all } t \in [0,T] \text{ and } x,y \in H.$$

We recover Theorem 2.1 in [16] as the following corollary of Theorem 3.2.

Corollary 3.3: *If* (H1), (H4) - (H7), *and* (3.2) *hold, then* (3.10) *has a unique mild solution on* $[0,T]$.

Proof: Define $F: \mathcal{C}([0,T];H) \rightarrow L^1(0,T;L^2(\Omega;H))$ and $G: \mathcal{C}([0,T];H) \rightarrow \mathcal{C}_{BL}$, respectively, by

$$F(x)(t) = \int_0^t B(t,s)f_1(s,x(s))ds + f_2(t,x(t)), \quad 0 \leq t \leq T,$$

$$G(x)(t) = C(t,s)g(t,x(t)), \quad 0 \leq t \leq s \leq T. \tag{3.11}$$

The Uniform Boundedness Principle guarantees the existence of positive constants M_B and M_C such that $\|B(t,s)\|_{BL} \leq M_B$ and $\|C(t,s)\|_{BL} \leq M_C$, for all $0 \leq t \leq s \leq T$. Standard computations involving properties of expectation and Hölder's inequality imply, with the help of (H6), that for all $x, y \in \mathcal{C}([0,T];H)$,

$$\|F(x) - F(y)\|_{L^1}$$

$$\leq 2\int_0^T \left[TM_B^2 \int_0^t E\|f_1(\tau, x(\tau)) - f_1(\tau, y(\tau))\|_H^2 d\tau + E\|f_2(s, x(s)) - f_2(s, y(s))\|_H^2\right]^{\frac{1}{2}}$$

$$\leq 2T[TM_B M_{f_1} + M_{f_2}]\|x - y\|_{\mathcal{C}}. \tag{3.12}$$

Similarly, (H7) enables us to infer that for all $x, y \in \mathcal{C}([0,T];H)$,

$$\|G(x) - G(y)\|_{\mathcal{C}_{BL}} \leq M_C M_g \|x - y\|_{\mathcal{C}}. \tag{3.13}$$

Thus, if we let $M_F = 2T[TM_B M_{f_1} + M_{f_2}]$ in (H2) and $M_G = M_C M_g$ in (H3), and take $h = 0$, we can conclude from Theorem 3.2 that (3.10) has a unique mild solution on $[0,T]$.

Remark 3.4:

(i) We also recover Theorem 3.3 in [22] as a corollary to Theorem 3.2 if we replace F and G in (3.11), respectively, by $F(x)(t) = f(t)$ and $G(x)(t) = C(t-s)x(t)$, for all $0 \leq t \leq s \leq T$, where C is a convolution-type kernel satisfying Assumptions 3.2 on page 361 in [22]. The result then follows from Corollary 3.3.

(ii) A result analogous to Corollary 3.3 regarding a delay version of (3.10) (obtained by replacing $g(s, x(s))$ by $g(s, x(s), x(\sigma(s)))$, where $\sigma : [0,T] \to [0,T]$ is a continuous, nondecreasing function) can be established by making slight modifications to the above argument. A related delay equation is discussed in [7] using compactness methods.

We conclude this section with a comment on a special case of (3.10), namely where $x_0 = 0$ and h is given by

$$h(x) = x(T), \quad \text{for all } x \in \mathcal{C}([0,T];H). \tag{3.14}$$

Clearly, h, as given by (3.14), satisfies (H7) with $M_g = 1$. Since $M_S \geq 1$, condition (3.2) does not hold for such h. To incorporate (3.14) into our theory, we consider that the functions f_i and g are defined instead on $\mathcal{C}((0,\infty);H)$ and satisfy (H6) and (H7), respectively, with $[0,T]$ replaced by $[0,\infty)$. Also, we take B and C to be convolution kernels in $L^1(0,\infty)$ of the type described in Remark 3.4(i). And finally, we assume that A generates a semi-group $\{S(t) : t \geq 0\}$ on H such that

(H8) There exist $M_S \geq 1$ and $\omega > 0$ such that $\|S(t)\|_{BL} \leq M_S e^{-\omega t}$, for all $t \geq 0$.

For conditions that ensure that (H8) holds, see [30], pg. 116. Using an approach similar to the one employed in [25], we can now prove that the following initial-value problem has a unique mild solution, provided T is sufficiently large.

$$x'(t) = Ax(t) + \int_0^t C(t-s)g(s, x(s))dW(s) + \int_0^t B(t-s)f_1(s, x(s))ds \tag{3.15}$$

$$+f_2(t,x(t)), \quad 0 \le t \le T,$$
$$x(0) = x(T).$$

Theorem 3.5: *Suppose* (H1) *and* (H8) *hold, and that* f_i, g, B, *and* C *are as described above. If also*

(H9) $M_S \exp\left[-\omega T + M_S(M_{f_1}\|B\|_{L^1(0,\infty)} + M_{f_2} + M_g\|C\|_{L^1(0,\infty)})\right] < 1$, *then* (3.15) *has a unique mild solution on* $[0,T]$.

Proof: Arguing as in [22], it follows that for each fixed $T > 0$ and each $y \in L_0^2(\Omega; H)$, the initial-value problem (3.15) (with y in place of $x(T)$) has a unique mild solution x_y on $[0,T]$ given by

$$x_y(t) = S(t)y + \int_0^t \int_0^s S(s-\tau)B(s-u)f_1(\tau,x(\tau))d\tau ds + \int_0^t S(t-s)f_2(s,x(s))ds \quad (3.16)$$
$$+\int_0^t \int_0^s S(s-\tau)C(s-\tau)g(\tau,x(\tau))dW(\tau)ds, \quad 0 \le t \le T.$$

On account of (H8), and the assumptions imposed on f_i, g, B, and C, (3.16) yields

$$\|x_y(t) - x_z(t)\|_H \le M_S e^{-\omega t}\|y-z\|_H + M_S(M_{f_1}\|B\|_{L^1(0,\infty)} + M_{f_2} + M_g\|C\|_{L^1(0,\infty)}) \quad (3.17)$$
$$\cdot \int_0^t e^{-\omega(t-s)}\|x_y(s) - x_z(s)\|_H ds, \quad 0 \le t \le T.$$

Now, using a Gronwall-type inequality in (3.17) (cf. [25], Lemma 4.2), we arrive at

$$\|x_y(T) - x_z(T)\|_H \le M_S \exp[-\omega T + M_S(M_{f_1}\|B\|_{L^1(0,\infty)} + M_{f_2} + M_g\|C\|_{L^1(0,\infty)})] \cdot \|y-z\|_H,$$

for all $y, z \in L_0^2(\Omega; H)$, and subsequently,

$$\|x_y(T) - x_z(T)\|_{L^2(\Omega;H)} \le M_S \exp[-\omega T + M_S(M_{f_1}\|B\|_{L^1(0,\infty)} + M_{f_2} + M_g\|C\|_{L^1(0,\infty)})] \quad (3.18)$$
$$\cdot \|y-z\|_{L_0^2}.$$

Define $Q_T : L^2(\Omega; H) \to L^2(\Omega; H)$ by $Q_T(y) = u_y(T)$. Observe that (3.18) and (H9) imply that Q_T is a strict contraction on $L^2(\Omega; H)$, for sufficiently large T. Thus, for T chosen such that (H9) is satisfied, Q_T has a unique fixed point y_0. The corresponding function $u = u_{y_0}$ is the desired mild solution of (3.15).

4 Existence Results - Compactness Case

We now develop existence results for (1.1) in which the Lipschitz conditions on F, G, and h are replaced by sublinear growth conditions. This is done at the expense of a compactness restriction on the semi-group. Precisely, we use the following assumptions instead:

(H10) A generates a compact C_0-semi-group $\{S(t) : t \ge 0\}$ on H,

(H11) $F : \mathcal{C}([0,T]; H) \to L^p(0,T; L^2(\Omega; H))$ is a continuous map for which there exists positive constants c_1 and c_2 such that $\|F(x)\|_{L^p} \le c_1\|x\|_{\mathcal{C}} + c_2$, for all $x \in \mathcal{C}([0,T]; H)$, (H12) $G : \mathcal{C}([0,T];H) \to \mathcal{C}_{BL}$ is a continuous map for which there exists $d_1 > 0$ and $d_2 \in L^2(0,T;\mathbb{R}^+)$ such that $\|G(x)\|_{\mathcal{C}_{BL}} \le d_1\|x\|_{\mathcal{C}} + d_2(\cdot)$, for all $x \in \mathcal{C}([0,T];H)$,

(H13) $h : \mathcal{C}([0,T];H) \to L_0^2(\Omega;H)$ is a continuous, compact map for which there exists positive constants e_1 and e_2 such that $\|h(x)\|_{L_0^2} \le e_1\|x\|_{\mathcal{C}} + e_2$, for all $x \in \mathcal{C}([0,T];H)$.

We begin by establishing certain compactness properties of the mappings $\Phi_1 : L^p(0,T;L^2(\Omega,H)) \to \mathcal{C}([0,T];H)$ and $\Phi_2 : \mathcal{C}_{BL} \to \mathcal{C}([0,T];H)$ defined, respectively, by

$$\Phi_1(v)(t) = \int_0^t S(t-s)v(s)ds, \quad 0 \le t \le T, \tag{4.1}$$

$$\Phi_2(v)(t) = \int_0^t \int_0^s S(s-\tau)v(\tau)dW(\tau)ds, \quad 0 \le s \le t \le T. \tag{4.2}$$

The well-definedness of these two mappings follows from an application of Lebesgue's Dominated Convergence Theorem.

Lemma 4.1: *Assume that $\{S(t) : 0 \le t \le T\}$ is a compact semi-group on H. Then,*

(i) *Φ_1 maps uniformly integrable sets in $L^1(0,T;L^2(\Omega,H))$ into precompact subsets of $\mathcal{C}([0,T];H)$. Further, Φ_1 is a compact map from $L^p(0,T;L^2(\Omega;H))$ into $\mathcal{C}([0,T];H)$, for $p > 1$,*

(ii) *Φ_2 is a compact map from $\mathcal{C}_{BL}$ into $\mathcal{C}([0,T];H)$.*

Proof: Part (i) is essentially a stochastic analog of Lemma 3.1 in [3] (where $S(t)$ plays the role of the resolvent operator) and its proof follows similarly by making the natural modifications. We shall only sketch the proof of (ii).

Let $K_r = \{v \in \mathcal{C}_{BL} : \|v\|_{\mathcal{C}_{BL}} \le r\}$. We shall show that $\Phi_2(K_r)$ is equicontinuous at each $t \in [0,T]$ and $\Phi_2(K_r)(t)$ is precompact in $L^2(\Omega;H)$, for each $t \in [0,T]$. To this end, observe that for $0 < t_1 \le t_2 \le T$ and $v \in K_r$, we have

$$\begin{aligned}\|\Phi_2(v)(t_2) - \Phi_2(v)(t_1)\|_{L^2(\Omega;H)} &\le \left[TE\int_{t_1}^{t_2}\int_0^s \|S(s-\tau)v(\tau)\|_{BL}^2 d\tau ds\right]^{\frac{1}{2}} \\ &\le M_S T^{1/2}\left[\int_{t_1}^{t_2}\int_0^s \|v\|_{\mathcal{C}_{BL}}^2 d\tau ds\right]^{\frac{1}{2}} \\ &\le M_S T\|v\|_{\mathcal{C}_{BL}}(t_2-t_1)^{1/2}.\end{aligned} \tag{4.3}$$

Observe that the right-side of (4.3) tends to zero as $t_2 \to t_1$, uniformly for $v \in K_r$. A similar argument works for $t = 0$, thereby verifying the equicontinuity.

Next, note that the precompactness of $\Phi_2(K_r)(0) = \{0\}$ is trivial. Let $0 < t \le T$, $0 < \epsilon < t$, and define by $\Phi_2^\epsilon : \mathcal{C}_{BL} \to \mathcal{C}([0,T];H)$ by

$$\Phi_2^\epsilon(v)(t) = \int_0^{t-\epsilon}\int_0^s S(s-\tau)v(\tau)dW(\tau)ds, \quad 0 \le t \le s \le T.$$

We claim that $K_2(\epsilon;t) = \{\Phi_2^\epsilon(v)(t) : v \in K_r\}$ is precompact in $L^2(\Omega;H)$. Indeed, observe that

$$\|\Phi_2(v)(t) - \Phi_2^\epsilon(v)(t)\|_{L^2(\Omega;H)} \tag{4.4}$$

$$\leq M_S T^{1/2}\left[\int_{t-\epsilon}^{t}\int_0^s \|v\|_{\mathcal{C}_{BL}}^2 d\tau ds\right]^{\frac{1}{2}} \leq M_S T r \epsilon^{1/2},\ \ 0<\epsilon<t.$$

Since the right-side of (4.4) can be made arbitrarily small, uniformly for $v \in K_r$, we conclude that $\Phi_2(K_r)(t)$ is totally bounded. This, combined with the work above, yields the precompactness, and the proof is complete.

Theorem 4.2: *Assume that* (H5) *and* (H10) - (H13) *are satisfied. Then,* (1.1) *has at least one mild solution on* $[0,T]$ *provided that*

(H14) $2M_S[e_1 + \sqrt{2}T^{3/2}d_1 + T^{1/q}c_1] < 1.$

Proof: We use Schaefer's theorem to prove that $\mathcal{J}$ (as defined in (3.3)) has a fixed point.

The well-definedness of $\mathcal{J}$ under (H10) - (H13) can be established using reasoning similar to that employed in the proof of Theorem 3.2. To verify the continuity of $\mathcal{J}$, let $\{v_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{C}([0,T];H)$ such that $v_n \to v$ as $n \to \infty$. Standard computations yield

$$\|\mathcal{J}(v_n)-\mathcal{J}(v)\|_{\mathcal{C}} \leq 2M_S\Bigg[\|h(v_n)-h(v)\|_{L_0^2}$$

$$+\left(E\left(\int_0^T \|F(v_n)(s)-F(v)(s)\|_H ds\right)^2\right)^{1/2}$$

$$+\left(E\left(\int_0^T\int_0^T \|G(v_n)(\tau)-G(v)(\tau)\|_{BL} dW(\tau) ds\right)^2\right)^{1/2}\Bigg] \tag{4.5}$$

$$\leq 2M_S[\|h(v_n)-h(v)\|_{L_0^2} + T^{1/q}\|F(v_n)-F(v)\|_{L^p} + T^{3/2}\|G(v_n)-G(v)\|_{\mathcal{C}_{BL}}].$$

The continuity of F, G, and h ensure that the right-side of (4.5) goes to zero as $n \to \infty$, thereby verifying the continuity of $\mathcal{J}$.

Next, we show that the set $\xi(\mathcal{J})$, as defined in Theorem 2.3 with $\mathcal{C}([0,T];H)$ in place of X, is bounded. Let $v \in \xi(\mathcal{J})$ and observe that the Hölder and Young inequalities (with (H12)) yield

$$T^{1/2}\left(E\int_0^T\int_0^T \|G(v)(\tau)\|_{BL}^2 d\tau\ ds\right)^{1/2} \leq \sqrt{2}T^{3/2}d_1\|v\|_{\mathcal{C}} + \sqrt{2}T^{1/2}\|d_2\|_{L^2(0,T)}. \tag{4.6}$$

Also, arguing as in (3.4), we obtain (with the help of (H14))

$$T^{1/2}\left(E\int_0^T \|F(v)(s)\|_H^2 ds\right)^{1/2} \leq T^{1/q}(c_1\|v\|_{\mathcal{C}} + c_2). \tag{4.7}$$

Hence, (4.6) and (4.7), in conjunction with (H13), enable us to conclude that for all $v \in \xi(\mathcal{J})$ and $0 \leq t \leq T$, we have

$$\lambda\|v\|_{\mathcal{C}} \leq 2M_S\Big[e_1\|v\|_{\mathcal{C}} + e_2 + \|x_0\|_{L_0^2} + \sqrt{2}T^{3/2}d_1\|v\|_{\mathcal{C}} + \sqrt{2}T^{1/2}\|d_2\|_{L^2(0,T)} \tag{4.8}$$

$$+T^{1/q}(c_1\|v\|_{\mathcal{C}} + c_2)\Big].$$

Taking into account that $\lambda \geq 1$ and (H14), we conclude from (4.7) that $\|v\|_{\mathcal{C}} \leq \eta$, where η is a constant independent of v and λ. So, $\xi(\mathcal{J})$ is bounded.

To apply Schaefer's theorem, we must finally show that $\mathcal{J}$ is compact. To this end, let $r > 0$ and define $K_r = \{v \in \mathcal{C}([0,T];H) : \|v\|_{\mathcal{C}} \leq r\}$. Using the notation of (4.1) and (4.2), we can express (3.3) as

$$\mathcal{J}(v) = S(\cdot)(h(v) + x_0) + \Phi_1(F(v)(\cdot)) + \Phi_2(G(v)(\cdot)),\ v \in \mathcal{C}([0,T];H). \tag{4.9}$$

We shall prove that $\mathcal{J}(K_r)$ is precompact in $\mathcal{C}([0,T];H)$. First, the facts that $\{F(v) : v \in K_r\}$ and $\{G(v) : v \in K_r\}$ are bounded subsets of $L^p(0,T;L^2(\Omega;H))$ and $\mathcal{C}_{BL}$, respectively (cf. (H11) and (H12)), it follows from Lemma 4.1 that the set $\{\Phi_1(F(v)) + \Phi_2(G(v)) : v \in K_r\}$ is precompact in $\mathcal{C}([0,T];H)$. It remains to establish the precompactness of $\{S(\cdot)(h(v) + x_0) : v \in K_r\}$. Since $\{S(\cdot)(x_0) : v \in K_r\} = \{S(\cdot)(x_0)\}$ is trivially precompact, we need only focus on $\{S(\cdot)(h(v)) : v \in K_r\}$. By (H13), the set $L = \{h(v) : v \in K_r\}$ is precompact in $L^2_0(\Omega;H)$. Let $\tilde{L} = S(\cdot)L(\subset \mathcal{C}([0,T];H))$ and $\epsilon > 0$. The precompactness of L in $L^2_0(\Omega;H)$ guarantees the existence of $\{x_1,\ldots,x_n\} \subset L$ such that $L \subset \bigcup_{i=1}^n B(x_i, \epsilon/M_S)$, where $B(x_i,\epsilon/M_S)$ is the ball in $L^2_0(\Omega;H)$ with radius ϵ/M_S and center x_i. Then, $\tilde{L} \subset \bigcup_{i=1}^n S(\cdot)B(x_i,\epsilon/M_S)$. Let $\tilde{x}_i = S(\cdot)x_i \in \mathcal{C}([0,T];H)$ and $\tilde{B}_i = \{y \in \mathcal{C}([0,T];H) : \|y - \tilde{x}_i\|_{\mathcal{C}} < \epsilon\}$. For $z \in \tilde{L}$, there exists $\Psi \in L$ such that $z \in S(\cdot)\Psi$. Since $\Psi \in L$, there is an $i \in \{1,\ldots,n\}$ such that $\|\Psi - x_i\|_{L^2_0} < \epsilon/M_S$. Observe that $\|z - \tilde{x}_i\|_{\mathcal{C}} = \|S(\cdot)\Psi - S(\cdot)x_i\|_{\mathcal{C}} \leq M_S\|\xi - x_i\|_{L^2_0} < \epsilon$. It then follows that $\tilde{L} \subset \bigcup_{i=1}^n \tilde{B}_i$ and hence, $\tilde{L}$ is totally bounded. Thus, $\tilde{L}$ is precompact in $\mathcal{C}([0,T];H)$. Hence, Schaefer's theorem implies that $\mathcal{J}$ has at least one fixed point $x \in \mathcal{C}([0,T];H)$ which is a mild solution to (1.1).

Next, we state a corollary regarding (3.10) under the following assumptions on f_i and g:

(H15) $f_i : [0,T] \times H \to H (i = 1,2)$ satisfies

- (i) $f_i(t,\cdot) : H \to H$ is continuous, for almost all $t \in [0,T]$,
- (ii) $f_i(\cdot,x) : [0,T] \to H$ is strongly $\mathcal{F}_t$-measurable, for all $x \in H$,
- (iii) there exist positive constants $a_{i,1}$ and $a_{i,2}$ such that $\|f_i(t,x)\|_H \leq a_{i,1}\|x\|_H + a_{i,2}$ for almost all $t \in [0,T]$ and for all $x \in H$,

(H16) $g : [0,T] \times H \to BL(K;H)$ satisfies

- (i) $g(t,\cdot) : H \to BL(K;H)$ is continuous, for almost all $t \in [0,T]$,
- (ii) $g(\cdot,x) : [0,T] \to BL(K;H)$ is strongly $\mathcal{F}_t$-measurable, for all $x \in H$,
- (iii) there exist positive constants b_1 and b_2 such that $\|g(t,x)\|_{BL} \leq b_1\|x\|_H + b_2$ for almost all $t \in [0,T]$ and for all $x \in H$.

Corollary 4.3: *If* (H5), (H10), *and* (H13)–(H16) *are satisfied, then (3.10) has at least one mild solution on* $[0,T]$.

Proof: An argument similar to the one used in [34], (Chapter 26, pg. 561) can be used to show that (H15) and (H16) guarantee that the mappings $F : \mathcal{C}([0,T];H) \to L^1(0,T;L^2(\Omega;H))$ and $G : \mathcal{C}([0,T];H) \to \mathcal{C}_{BL}$ defined in (3.11) are well-defined and continuous. Further, routine calculations show that F and G satisfy (H11) and (H12), respectively, with $c_1 = 2T(a_{1,1}M_BT^{3/2} + a_{2,1})$, $c_2 = 2T(a_{1,2}M_BT^{3/2} + a_{2,2})$, $d_1 = 2M_Cb_1T$, and $d_2 = 2M_Cb_2T$. Consequently, (3.10) has at least one mild solution by Theorem 4.2.

We can formulate a stronger version of Corollary 4.3 by replacing assumption (H15) and (H16), respectively, by

(H17) $f_i : [0,T] \times H \to H (i = 1,2)$ satisfies (H15) (i) and (ii), as well as

- (i) For each $k \in I\!N$, there exists $g_{i,k} \in L^1(0,T;I\!R^+)$ such that for almost all $t \in (0,T)$, $\sup_{\|x\|_H \le k} E\|f_i(t,x)\|_H^2 \le g_{i,k}(t)$,
- (ii) $\underline{\lim}_{k\to\infty} k^{-2}\int_0^T g_{i,k}(s)ds = \alpha_i < \infty$,

(H18) $g : [0,T] \times H \to BL(K;H)$ satisfies (H16) (i) and (ii), as well as

- (i) For each $k \in I\!N$ there exists $j_k \in L^1(0,T;I\!R^+)$ such that for almost all $t \in (0,T)$, $\sup_{\|x\|_H \le k} E\|g(t,x)\|_{BL}^2 \le j_k(t)$,
- (ii) $\underline{\lim}_{k\to\infty} k^{-2}\int_0^T j_k(s)ds = \beta < \infty$.

Comparable conditions appear in [7, 33].

Proposition 4.4: *Assume that* (H5), (H10), (H13), (H17), *and* (H18) *are satisfied. If, in addition,*

(H19) $4M_S[e_1 + T^{1/2}(\alpha_2^{1/2} + M_BT^3\alpha_1^{1/2} + M_CT^2\beta^{1/2})] < 1$, *then* (3.10) *has at least one mild solution on* $[0,T]$.

Proof: We use Schauder's fixed-point theorem [24] to argue that $\mathcal{J}$ (as defined in (3.3) with F and G given by (3.11) has a fixed point. The continuity and compactness follow by making slight changes to the proof of Theorem 4.1. For $n \in I\!N$, let $B_n = \{x \in \mathcal{C}([0,T];H) : \|x\|_{\mathcal{C}} \le n\}$. It remains to show that there exists an $n \in I\!N$ such that $\mathcal{J}(B_n) \subset B_n$.

Suppose, by way of contradiction, that for each $k \in I\!N$, there exists $u_k \in B_k$ such that $\mathcal{J}(u_k) \notin B_k$. Then,

$$1 \le \underline{\lim}_{k\to\infty} k^{-1}\|\mathcal{J}(u_k)\|_{\mathcal{C}}. \tag{4.10}$$

Observe that

$$\begin{aligned} &\|\mathcal{J}(u_k)\|_{\mathcal{C}} \\ &\le 4M_S\left[\|h(u_k)\|_{L_0^2} + \|x_0\|_{L_0^2} + T^{1/2}\left(T^{1/2}M_B\left(\int_0^T\int_0^T E\|f_1(\tau,u_k(\tau))\|_H^2 d\tau\, ds\right)^{1/2}\right.\right. \\ &\left.\left. + \left(\int_0^T E\|f_2(s,u_k(s))\|_H^2 ds\right)^{1/2} + T^{1/2}M_C\left(\int_0^T\int_0^T E\|g(\tau,u_k(\tau))\|_{BL}^2 d\tau\, ds\right)^{1/2}\right)\right]. \end{aligned} \tag{4.11}$$

Note that for each $k \in I\!N$, $u_k \in B_k$ and hence, $\|u_k(s)\|_H \le k$, for all $0 \le s \le T$. So, by (H17) and (H18), there exist $g_{i,k}(i = 1,2)$, $j_k \in L^1(0,T;I\!R^+)$ such that for almost all $0 \le s \le T$

$$E\|f_i(s,u_k(s))\|_H^2 \le g_{i,k}(s), \ \ (i = 1,2), \tag{4.12}$$
$$E\|g(s,u_k(s))\|_{BL}^2 \le j_k(s).$$

Using (4.12) in (4.11) yields (with the help of (H13))

$$\|\mathcal{J}(u_k)\|_{\mathcal{C}} \le 4M_S \left[\left[e_1 \|u_k\|_{\mathcal{C}} + e_2 + \|x_0\|_{L_0^2} \right] + T^{1/2} \left(\int_0^T g_{2,k}(s)ds \right)^{1/2} \right.$$

$$\left. + M_B T^{3/2} \left(\int_0^T g_{1,k}(s)ds \right)^{1/2} + M_C T^{1/2} \left(\int_0^T j_k(s)ds \right)^{1/2} \right].$$

and subsequently,

$$\underline{\lim}_{k\to\infty} k^{-1} \|\mathcal{J}(u_k)\|_{\mathcal{C}} \le 4M_S \underline{\lim}_{k\to\infty} \left[e_1 k^{-1} \|u_k\|_{\mathcal{C}} + (e_2 + \|x_0\|_{L_0^2}) k^{-1} \right.$$

$$+4M_S T^{1/2} (k^{-2} \int_0^T g_{2,k}(s)ds)^{1/2}$$

$$\left. + M_B T^{3/2} \left(k^{-2} \int_0^T g_{1,k}(s)ds \right)^{1/2} + 4M_S M_C T \left(k^{-2} \int_0^T j_k(s)ds \right)^{1/2} \right]$$

$$\le 4M_S [e_1 + T^{1/2} (\alpha_2^{1/2} + M_B T^3 \alpha_1^{1/2} + M_C T^2 \beta^{1/2})]$$

$$< 1 \ \text{(by (H19))},$$

contradicting (4.10). Consequently, there is an $n_0 \in I\!N$ such that $\mathcal{J}(B_{n_0}) \subset B_{n_0}$. Thus, Schauder's fixed point theorem guarantees the existence of $x \in B_{n_0}$ such that $\mathcal{J}(x) = x$, which is the mild solution that we seek.

Remark: An inspection of the proof shows that (H13) can be weakened slightly in that instead of imposing the sublinear growth restriction on h, we need only assume that $\lim_{\|x\|_{\mathcal{C}}\to\infty} \|h(x)\|_{L_0^2} / \|x\|_{\mathcal{C}} = \zeta < \infty$.

5 Convergence Results

Throughout this section we assume that A, F, G, and h satisfy (H1)—(H4) and that (3.2) holds. For each $n \in I\!N$, consider a linear operator $A_n : D(A_n)(= D(A)) \to H$ and mappings $F_n : \mathcal{C}([0,T];H) \to L^p(0,T;L^2(\Omega;H))$, $G_n : \mathcal{C}([0,T];H) \to \mathcal{C}_{BL}$, and $h_n : \mathcal{C}([0,T];H) \to L_0^2(\Omega;H)$ satisfying the following conditions:

(H20) A_n generates a C_0-semi-group $\{S_n(t) : t \ge 0\}$ such that $\|S_n(t)\|_{BL} \le M_S e^{\alpha t}$, for some $\alpha > 0$ (independent of n), for each $n \in I\!N$, and $A_n x \to Ax$ strongly as $n \to \infty$, for each $x \in D(A)$,

(H21) (i) $\|F_n(x) - F_n(y)\|_{L^p} \le M_F \|x - y\|_{\mathcal{C}}$, for all $x, y \in \mathcal{C}([0,T];H)$,

(ii) $F_n(x) \stackrel{L^p}{\to} F(x)$ as $n \to \infty$, for all $x \in \mathcal{C}([0,T];H)$,

(H22) (i) $\|G_n(x) - G_n(y)\|_{\mathcal{C}_{BL}} \le M_G \|x - y\|_{\mathcal{C}}$, for all $x, y \in \mathcal{C}([0,T];H)$,

(ii) $G_n(x) \stackrel{\mathcal{C}_{BL}}{\to} G(x)$ as $n \to \infty$, for all $x \in \mathcal{C}([0,T];H)$,

(H23) (i) $\|h_n(x) - h_n(y)\|_{L_0^2} \le M_h \|x - y\|_{\mathcal{C}}$, for all $x, y \in \mathcal{C}([0,T];H)$,

(ii) $h_n(x) \overset{L_0^2}{\to} h(x)$ as $n \to \infty$, for all $x \in \mathcal{C}([0,T];H)$.

(Here, the constants M_S, M_F, M_G, and M_h are the same ones appearing in (H1)–(H4) and so, are independent of n.)

Let x be the mild solution to (1.1) as guaranteed by Theorem 3.2. By virtue of (H6), (H20), (H21)(i), (H22)(i), and (H23)(i), Theorem 3.2 implies that, for each $n \in I\!N$, the problem

$$x_n'(t) = A_n x_n(t) + F_n(x_n)(t) + \int_0^t G_n(x_n)(s)dW(s), \;\; 0 \le t \le T, \tag{5.1}$$

$$x_n(0) = h_n(x_n) + x_0,$$

has a unique mild solution $x_n \in \mathcal{C}([0,T];H)$.

Consider the following initial-value problem:

$$y_n'(t) = A_n y_n(t) + F_n(x)(t) + \int_0^t G_n(x)(s)dW(s), \;\; 0 \le t \le T, \tag{5.2}$$

$$y_n(0) = h_n(x) + x_0.$$

Since $h_n(x) + x_0$ is a fixed element of $L_0^2(\Omega;H)$, a standard argument (see Ch. 7 in [14]) guarantees the existence of a unique mild solution y_n of (5.2). We need the following lemma.

Lemma 5.1: *If (H20)–(H23) hold, then* $y_n \overset{\mathcal{C}}{\to} x$ *as* $n \to \infty$.

Proof: Using (H20) in conjunction with Theorem 4.1 in [19], pg. 46, we infer that $S_n(t)z \to S(t)z$ strongly as $n \to \infty$, for all $z \in H$, uniformly in $t \in [0,T]$. Observe that

$$\|y_n(t) - x(t)\|_H \le \|S_n(t)(h_n(x) - h(x)) + (S_n(t) - S(t))h(x)\|_H$$

$$+ \int_0^t \|S_n(t-s)(F_n(x)(s) - F(x)(s))\|_H ds + \int_0^t \|(S_n(t-s) - S(t-s))F(x)(s)\|_H ds$$

$$+ \| \int_0^t \int_0^s [(S_n(s-\tau)(G_n(x)(\tau) - G(x)(\tau))) + (S_n(s-\tau) - S(s-\tau))G(x)(\tau)] dW(\tau) ds\|_H.$$

A standard argument invoking the Trotter-Kato Theorem [30] can be used, invoking (H21)(ii)–(H23)(ii), to complete the proof.

We now state the first of our two main convergence results. A comparable theorem for a nonlinear deterministic evolution equation is discussed in [2].

Theorem 5.2: *Assume that (H1)–(H6), (3.2), and (H20)–(H23) are satisfied. Then,* $x_n \overset{\mathcal{C}}{\to} x$, *provided* $8\overline{M_S}[M_h + T^{1/q}M_F + T^{5/2}M_G] < 1$, *where* $\overline{M_S} = M_S e^{\alpha T}$.

Proof: Let y_n be the mild solution of (5.2). Observe that

$$\|x_n(t) - x(t)\|_H^2 \le 4[\|x_n(t) - y_n(t)\|_H^2 + \|y_n(t) - x(t)\|_H^2]$$

$$\le 4\left\{16\left[\|S_n(t)(h_n(x_n) + x_0 - h_n(x) - x_0)\|_H^2\right.\right.$$

$$+ \left(\int_0^T \|S_n(T-s)(F_n(x_n)(s) - F_n(x)(s))\|_H ds\right)^2$$

$$+\left(\int_0^T \|\int_0^T S_n(T-\tau)(G_n(x_n)(\tau)-G_n(x)(\tau))dW(\tau)\|_H ds\right)^2\Bigg] + \|y_n(t)-x(t)\|_H^2\Bigg\}.$$

Now, taking the expectation, followed by taking square roots, yields after some computation

$$\|x_n(t)-x(t)\|_{L^2(\Omega;H)} \le 2\left\{4\left[E\|S_n(t)(h_n(x_n)-h_n(x))\|_{L^2(\Omega;H)}\right.\right. \tag{5.3}$$

$$+T^{1/2}\left(\int_0^T E\|S_n(T-s)(F_n(x_n)(s)-F_n(x)(s))\|_H^2 ds\right)^{1/2}$$

$$\left.+T^{1/2}\left(\int_0^T E\|\int_0^T S_n(T-\tau)(G_n(x_n)(\tau)-G_n(x)(\tau))dW(\tau)\|_H^2 ds\right)^{1/2}\right]$$

$$\left.+\|y_n(t)-x(t)\|_{L^2(\Omega;H)}\right\}.$$

For convenience, we relabel the first three terms on the right-side of (5.3) as I_1, I_2 and I_3, respectively, and estimate each separately below.

First, note that (H23) immediately yields

$$I_1 \le \overline{M_S}\|h_n(x_n)-h_n(x)\|_{L_0^2} \le \overline{M_S}M_h\|x_n-x\|_{\mathcal{C}}. \tag{5.4}$$

Next, (H21) yields, with the help of Hölder's inequality,

$$I_2 \le T^{1/q}\overline{M_S}\|F_n(x_n)-F_n(x)\|_{L^p} \le T^{1/q}\overline{M_S}M_F\|x_n-x\|_{\mathcal{C}}. \tag{5.5}$$

Finally, using (H22), we obtain

$$I_3 \le T^{1/2}\overline{M_S}\left(\int_0^T\int_0^T \|G_n(x_n)(\tau)-G_n(x)(\tau)\|^2_{L^2(\Omega;BL(K;H))}d\tau\ ds\right)^{1/2} \tag{5.6}$$

$$\le T^{5/2}\overline{M_S}\|G_n(x_n)-G_n(x)\|_{\mathcal{C}_{BL}}$$

$$\le T^{5/2}\overline{M_S}M_G\|x_n-x\|_{\mathcal{C}}.$$

Using (5.4)–(5.6) in (5.3) yields, after taking supremum over $[0,T]$,

$$1/2(1-8\overline{M_S}[M_h+T^{1/q}M_F+T^{5/2}M_G])\|x_n-x\|_{\mathcal{C}} \le \|y_n-x\|_{\mathcal{C}}. \tag{5.7}$$

In view of (H20)–(H23), and the fact that $1-8\overline{M_S}[M_h+T^{1/q}M_F+T^{5/2}M_G]>0$, we can apply Lemma 5.1 to conclude from (5.7) that $x_n \xrightarrow{\mathcal{C}} x$ as $n\to\infty$.

Now, let P_x and P_{x_n} denote the probability measures on $\mathcal{C}([0,T];H)$ induced by the mild solutions x and x_n of (1.1) and (5.1), respectively. Using Theorem 5.2, we can prove that $P_{x_n} \xrightarrow{w} P_x$ as $n\to\infty$, for a certain subclass of perturbations. Precisely, we have

Theorem 5.3: *Let $p \ge 4$ and assume that $S_n(\cdot)A_n$ is a bounded operator, for each $n \in \mathbb{N}$. Then, $P_{x_n} \xrightarrow{w} P_x$ as $n\to\infty$, provided that*

(H24) $1-\overline{M_S}^2[M_h^2+T^{2/q}\overline{M_F}^2+C_G^3T^3M_G^2]>0.$

Proof: We shall employ a standard argument involving Theorem 2.6 similar to the one used in [22].

We begin by showing $\{P_{x_n}\}_{n=1}^{\infty}$ is relatively compact in $\mathcal{C}([0,T];H)$ by appealing to the Arzelá-Ascoli theorem. To this end, we shall first show that there exists $\eta > 0$ such that

$$\sup_{n\in \mathbb{N}} \sup_{0\le t\le T} \|x_n(t)\|_{L^2(\Omega;H)} = \eta < \infty. \tag{5.8}$$

Note that x_n is given by

$$x_n(t) = S_n(t)(h_n(x_n)+x_0) + \int_0^t S_n(t-s)F_n(x_n)(s)ds \tag{5.9}$$

$$+ \int_0^t \int_0^s S_n(s-\tau)G_n(x_n)(\tau)dW(\tau)ds, \;\; 0\le t\le T.$$

Since $h_n(0) \overset{L_0^2}{\to} h(0)$, there exists $\overline{M_h} > 0$ (independent of n) such that $\|h_n(0)\|_{L_0^2} \le \overline{M_h}$, for all n. Using this fact, together with (H20) and (H23)(i), we arrive at

$$\|S_n(t)(h_n(x_n)+x_0)\|^2_{L^2(\Omega;H)} \le \overline{M_S}^2 M_h^2 \|x_n\|^2_{\mathcal{C}} + \overline{M_S}^2[\overline{M_h}^2 + \|x_0\|^2_{L_0^2}]. \tag{5.10}$$

Likewise, (H21)(ii) and (H22)(ii) guarantee that there exist $\overline{M_F}$, $\overline{M_G} > 0$ such that $\|F_n(0)\|_{L^p} \le \overline{M_F}$ and $\|G_n(0)\|_{\mathcal{C}_{BL}} \le \overline{M_G}$, for all n, so that a standard argument now yields

$$E\|\int_0^t S_n(t-s)F_n(x_n)(s)ds\|_H^2 \le T^{2/q}\overline{M_S}^2[M_F^2\|x_n\|^2_{\mathcal{C}} + \overline{M_F}^2] \tag{5.11}$$

and

$$E\|\int_0^t \int_0^s S_n(s-\tau)G_n(x_n)(\tau)dW(\tau)ds\|_H^2 \le T^3\overline{M_S}^2 C_G^2[M_G^2\|x_n\|^2_{\mathcal{C}} + \overline{M_G}^2]. \tag{5.12}$$

Combining the estimates (5.10)–(5.12) and rearranging terms, we can now conclude from (5.9) that (5.8) holds due to (H24) and the fact that all constants in (5.10)–(5.12) are independent of n.

Next, we establish the equicontinuity by showing $E\|x_n(t)-x_n(s)\|_H^4 \to 0$ as $(t-s)\to 0$, for all $0\le s\le t\le T$, uniformly for all $n\in \mathbb{N}$. We estimate each term of the representation formula for $x_n(t)-x_n(s)$ (cf. (5.9)) separately. Employing Theorem 2.4(d) in [30] and taking into account (H20), (H23), and the uniform boundedness of $S_n(\cdot)A_n$, we conclude that

$$E\|[S_n(t)-S_n(s)](h_n(x_n)+x_0)\|_H^4 \le T^{4/3}\int_s^t E\|S_n(\tau)A_n(h_n(x_n)+x_0)\|_H^4 d\tau \tag{5.13}$$

$$\le T^{4/3}\{M_{SA}[M_h^2\|x_n\|^2_{\mathcal{C}} + \overline{M_h}^2] + \overline{M_S}^2\|x_0\|^2_{L_2^0}\}(t-s)^2,$$

where $M_{SA} = \sup_{n\in \mathbb{N}} \|S_n(\cdot)A_n\|_{BL}$. Next, note that

$$\int_0^t S_n(t-\tau)F_n(x_n)(\tau)d\tau - \int_0^s S_n(s-\tau)F_n(x_n)(\tau)d\tau \tag{5.14}$$

$$= \int_0^s [S_n(t-\tau) - S_n(s-\tau)]F_n(x_n)(\tau)d\tau$$
$$+ \int_s^t S_n(t-\tau)F_n(x_n)(\tau)d\tau.$$

Estimating each of the two integrals on the right-side of (5.14) separately yields, from the boundedness of $S_n(\cdot)A_n$, (H20), and (H21)(i), that

$$E\| \int_0^s [S_n(t-\tau) - S_n(s-\tau)]F_n(x_n)(\tau)d\tau\|_H^4 \tag{5.15}$$
$$\leq T^{8/3} \int_0^s \int_{s-\tau}^{t-\tau} E\|S_n(w)A_nF_n(x_n)(w)\|_H^4 dwd\tau$$
$$\leq M_{SA}^4 T^{11/3}[M_F^4\|x_n\|_{\mathcal{C}}^4 + \overline{M_F}^4](t-s)^{(p-4)/p},$$

and similarly,

$$E\| \int_s^t S_n(t-\tau)F_n(x_n)(\tau)d\tau\|_H^4 \leq T^2\overline{M_S}^4 \int_s^t \|F_n(x_n)(\tau)\|_{L^2(\Omega;H)}^4 d\tau \tag{5.16}$$
$$\leq T^2\overline{M_S}^4[M_F^4\|x_n\|_{\mathcal{C}}^4 + \overline{M_F}^4](t-s)^{(p-4)/p}.$$

Regarding the difference of the stochastic integrals, note that Fubini's theorem, together with basic integral properties, enables us to write

$$\int_0^t \int_0^\tau S_n(\tau-\theta)G_n(x_n)(\theta)dW(\tau)d\theta - \int_0^s \int_0^\tau S_n(\tau-\theta)G_n(x_n)(\theta)dW(\tau)d\theta \tag{5.17}$$
$$= \int_0^s \int_s^\tau [S_n(t-\theta) - S_n(s-\theta)]G_n(x_n)(\tau)d\theta dW(\tau)$$
$$+ \left[\int_0^s \int_s^t + \int_s^t \int_\tau^s + \int_s^t \int_s^t\right] S_n(t-\theta)G_n(x_n)(\tau)d\theta dW(\tau).$$

Arguing as above, we see that

$$E\| \int_0^s \int_s^\tau [S_n(t-\theta) - S_n(s-\theta)]G_n(x_n)(\tau)d\theta dW(\tau)\|_H^4$$
$$\leq \int_0^s T^{4/3}E\| \int_s^\tau \int_0^{t-s} S_n(\mu+s-\theta)A_nG_n(x_n)(\tau)dW(\tau)d\theta\|_H^4 d\mu$$
$$\leq T^{4/3}M_{SA} \int_0^s \int_\tau^s \int_0^{t-s} \|G_n(x_n)(\tau)\|_{BL}^4 d\tau d\theta d\mu \tag{5.18}$$
$$\leq T^{4/3}M_{SA}[M_G^4\|x_n\|_{\mathcal{C}}^4 + \overline{M_G}^4](t-s)^4,$$

and that

$$E\| \left[\int_0^s \int_s^t + \int_s^t \int_\tau^s + \int_s^t \int_s^t\right] S_n(t-\theta)G_n(x_n)(\tau)d\theta dW(\tau)\|_H^4 \tag{5.19}$$

$$\leq 3T^2\overline{M_S}^4[M_G^4\|x_n\|_{\mathcal{C}}^4 + \overline{M_G}^4](t-s)^4.$$

Invoking (5.8) in (5.13), (5.15), (5.16), (5.18), and (5.19) enables us to conclude that, in fact, $E\|x_n(t) - x_n(s)\|_H^4 \to 0$ as $(t-s) \to 0$, uniformly for $0 \leq s \leq t \leq T$ and $n \in \mathbb{N}$, as desired. Thus, the family $\{P_{x_n}\}_{n=1}^{\infty}$ is relatively compact in $\mathcal{C}([0,T];H)$ and hence, tight (by Prokorhov's theorem [11]).

To finish the proof, we remark that Theorem 5.2 implies that the finite-dimensional joint distributions of P_{x_n} converge weakly to those of P (cf. Proposition 2.5). Hence, Theorem 2.6 ensures that $P_{x_n} \xrightarrow{w} P_x$ as $n \to \infty$.

Remark: For the classical version of (5.1) (i.e., when $h_n = 0$, for all n), a Gronwall-type argument can be used to establish the uniform boundedness (in $\mathcal{C}([0,T];H)$) of $\{x_n\}_{n=1}^{\infty}$ and, in such case, condition (H24) can be dropped.

6 Example

Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\mathcal{D}$ and consider the initial-boundary value problem

$$x_t(t,z) = \Delta_z x(t,z) + \int_0^T a(t,s) f_1\left(s, x(s,z), \int_0^s k(s,\tau,x(\tau,z))d\tau\right) ds \tag{6.1}$$

$$+ \int_0^T b(t,s) f_2(s, x(s,z)) dW(s), \text{ a.e. on } (0,T)\times\mathcal{D},$$

$$x(0,z) = \sum_{i=1}^n g_i(z) x(t_i,z) + \int_0^T c(s) f_3(s,x(s,z))ds, \text{ a.e. on } \mathcal{D},$$

$$x(t,z) = 0, \quad \text{a.e. on } (0,T)\times\partial\mathcal{D},$$

where $0 \leq t_1 < t_2 < \ldots < t_n \leq T$ are given and W is an $L^2(\mathcal{D})$-valued Wiener process (see [14] for examples). We consider (6.1) under the following conditions on the data:

(H25) $f_1 : [0,T]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ satisfies the Carathеódory conditions (i.e., measurable in (t,x) and continuous in the third variable), as well as

- (i) $f_i(\cdot,0,0) \in L^2(0,T)$,
- (ii) $|f_1(t,x_1,y_1) - f_1(t,x_2,y_2)| \leq M_{f_1}[|x_1 + x_2| + |y_1 - y_2|]$, for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$ and almost all $t \in (0,T)$, for some $M_{f_1} > 0$,

(H26) $f_2 : [0,T]\times\mathbb{R} \to BL(L^2(\mathcal{D}))$ satisfies the Carathеódory conditions (cf. H(12) (i), (ii)), as well as

- (i) $f_2(\cdot,0) \in L^2(0,T)$,
- (ii) $|f_2(t,x) - f_2(t,y)|_{BL(H)} \leq M_{f_2}|x-y|$, for all $x,y \in \mathbb{R}$ and almost all $t \in (0,T)$, for some $M_{f_2} > 0$.

(H27) $f_3 : [0,T]\times\mathbb{R} \to \mathbb{R}$ satisfies the Carathеódory conditions (cf. H(12) (i), (ii)), as well as

- (i) $f_3(\cdot,0) \in L^2(0,T)$,

(ii) $|f_3(t,x) - f_3(t,y)| \leq M_{f_3}|x-y|$, for all $x, y \in I\!R$ and almost all $t \in (0,T)$, for some $M_{f_3} > 0$,

(H28) $a \in L^2((0,T)^2)$,

(H29) $b \in L^\infty((0,T)^2)$,

(H30) $c \in L^2(0,T)$,

(H31) $k : U \times I\!R \to I\!R$, where $U = \{(s,t) : 0 < s < t < T\}$ satisfies

$$|k(t,s,x_1) - k(t,s,x_2)| \leq M_k|x_1 - x_2|,$$

for all $x_1, x_2 \in I\!R$, and almost all $(s,t) \in U$,

(H32) $g_i \in L^2(\mathcal{D})$, $i = 1, \ldots, n$.

Let $H = K = L^2(\mathcal{D})$ and set

$$A = \Delta_z, \quad D(A) = H^2(\mathcal{D}) \cup H_0^1(\mathcal{D}). \tag{6.2}$$

It is well-known that A generates a C_0-semigroup on (see [30], Chapter 7). Next, define $F : \mathcal{C}([0,T];H) \to L^2(0,T;L^2(\Omega;H))$, $G : \mathcal{C}([0,T];H) \to \mathcal{C}_{BL}$, and $h : \mathcal{C}([0,T];H) \to L_0^2(\Omega;H)$, respectively, by

$$F(x)(t,\cdot) = \int_0^T a(t,s) f_1\left(s, x(s,\cdot), \int_0^s k(s,\tau,x(\tau,\cdot))d\tau\right) ds, \tag{6.3}$$

$$G(x)(s,\cdot) = a(t,s) f_2(s, x(s,\cdot)), \tag{6.4}$$

$$h(x)(\cdot) = x(0,z) = \sum_{i=1}^n g_i(\cdot) x(t_i,\cdot) + \int_0^T c(s) f_3(s, x(s,\cdot)) ds. \tag{6.5}$$

One can use (H25)–(H32) to verify that F, G, and h satisfy (H2)–(H4), respectively, with

$$M_F = 2M_{f_1} T |a|_{L^2((0,T)^2)} (1 + M_{k_1} T^3)^{1/2}, \tag{6.6}$$

$$M_G = |b|_{L^\infty((0,T)^2)} M_{f_2}, \tag{6.7}$$

$$M_h = 2\left(\sum_{i=1}^n \|g_i\|_{L^2(\mathcal{D})} + M_{f_3} \sqrt{m(\mathcal{D})} |G|_{L^2(0,T)}\right) \tag{6.8}$$

where $m(\mathcal{D})$ is the Lebesgue product measure on $\mathcal{D}$. Thus, (6.1) can be rewritten in the form (1.1) in H, with A, F, G, and h given by (6.2)–(6.5) so that, once (3.2) holds, an application of Theorem 3.2 immediately yields

Theorem 6.1: *Assume* (*H*25)–(*H*32) *are satisfied. If, in addition,* (3.2) *holds (with* M_F*,* M_G*, and* M_h *and given by* (6.6)–(6.8)*), then* (6.1) *has a unique mild solution* $x \in C([0,T]; L^2(\Omega; L^2(\mathcal{D})))$.

References

[1] Ahmed, N.U., Differential inclusions on Banach spaces with nonlocal state constraints, *Nonlinear Funct. Anal. & Appl.* **6**:3 (2001), 395–409.

[2] Aizicovici, S. and Gao, Y., Functional differential equations with nonlocal initial conditions, *J. Appl. Math. Stochastic Anal.* **10** (1997), 145–156.

[3] Aizicovici, S. and Hannsgen, K.B., Local existence for abstract semilinear Volterra integrodifferential equations, *J. Integral Eqns. Appl.* **5**:3 (1993), 299–313.

[4] Aizicovici, S. and McKibben, M., Existence results for a class of abstract nonlocal Cauchy problems, *Nonlinear Anal.* **39**:5 (2000), 649–668.

[5] Aizicovici, S. and McKibben, M., Semilinear Volterra integrodifferential equations with nonlocal initial conditions, *Abstract & Appl. Anal.* **4**:2 (1999), 127–139.

[6] Altman, M., *Contractors and Contractor Directions, Theory and Applications*, Marcel-Dekker, New York 1978.

[7] Balasubramaniam, P. and Ntouyas, S.K., Global existence for semilinear stochastic delay evolution equations with nonlocal conditions, *Soochow Jour. of Math.* **27**:3 (July 2001), 331–342.

[8] Barbu, V., *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff, Leyden 1976.

[9] Bergström, H., *Weak Convergence of Measures*, Academic Press, New York 1982.

[10] Bharucha-Reid, A.T., *Random Integral Equations*, Academic Press, New York 1972.

[11] Billingsley, P., *Weak Convergence of Measures: Applications in Probability*, SIAM, Bristol 1971.

[12] Breiman, L., *Probability*, SIAM, Philadelphia 1992.

[13] Byszewski, L., Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, *J. Math. Anal. & Appl.* **162** (1991), 494–505.

[14] DaPrato, G. and Zabczyk, J., *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge 1992.

[15] Dunford, N. and Schwarz, J.T., *Linear Operators*, Part I, Wiley Interscience, New York 1958.

[16] Govindan, T.E., Autonomous semilinear stochastic Volterra integrodifferential equations in Hilbert spaces, *Dyn. Sys. Appl.* **3** (1994), 51–74.

[17] Govindan, T.E. and Joshi, M.C., Stability and optimal control of stochastic functional differential equations with memory, *Numer. Func. Anal. Optim.* **13** (1992), 249–265.

[18] Govindan, T.E., Stability of stochastic differential equations in a Banach space, In: *Mathematical Theory of Control* Lecture Notes in Pure and Applied Mathematics **142**, Marcel-Dekker, New York 1992.

[19] Grecksch, W. and Tudor, C., *Stochastic Evolution Equations: A Hilbert Space Approach*, Akademic Verlag, Berlin 1995.

[20] Ichikawa, A., Stability of semilinear evolution equations, *J. Math. Anal. Appl.* **90** (1982), 12–44.

[21] Jackson, D., Existence and uniqueness of solutions to semilinear nonlocal parabolic equations, *J. Math. Anal. Appl.* **172** (1993), 256–265.

[22] Kannan, D. and Bharucha-Reid, A.T., On a stochastic integrodifferential evolution equation of Volterra type, *J. Integral Equations* **10** (1985), 351–379.

[23] Kunita, H., *Stochastic Flows and Stochastic Differential Equations*, Cambridge University Press, United Kingdom 1990.

[24] Leray, J. and Schauder, J., Topologie et equation fonctionelles, *Ann. Sci. Ecole Norm. Sys.* **51** (1934), 45–78.

[25] Lin, Y.P. and Liu, J.H., Semilinear integrodifferential equations with nonlocal Cauchy problems, *Nonlinear Anal.* **26**:5, (1996), 1023–1033.

[26] Londen, S.O. and Nohel, J.A., Nonlinear Volterra integrodifferential equation occurring in heat flow, *J. Integral Equations* **6** (1984), 11–50.

[27] Ntouyas, S.K. and Tsamatos, P.Ch., Global existence for semilinear evolution equations with nonlocal conditions, *J. Math. Anal. Appl.* **210** (1997), 679–687.

[28] Ntouyas, S.K. and Tsamatos, P.Ch., Global existence for semilinear evolution integro-differential equations with delay and nonlocal conditions, *Appl. Anal.* **64** (1997), 99–105.

[29] McKibben, M., *Existence Results for Nonlinear Functional Differential Equations*, Ph.D. dissertation, Ohio University, Athens, OH 1999.

[30] Pazy, A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York 1983.

[31] Schaefer, H., Über die methode der a priori shranken, *Math. Annal.* **129** (1955), 415–416.

[32] Sobczyk, K., *Stochastic Differential Equations with Applications to Physics and Engineering*, Klüwer Academic Publishers, London 1991.

[33] Ward, J.R., Boundary value problems for differential equations in Banach space, *J. Math. Anal. Appl.* **70** (1979), 589–598.

[34] Zeidler, E., *Nonlinear Functional Analysis and its Applications II/B: Nonlinear Monotone Operators*, Springer, New York 1990.