PREVALENCE OF BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS WITH UNIQUE SOLUTION

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We prove that in the sense of Baire category, almost all backward stochastic differential equations (BSDEs) with bounded and continuous coefficient have the properties of existence and uniqueness of solutions as well as the continuous dependence of solutions on the coefficient and the L^2-convergence of their associated successive approximations.

1. Introduction

Let $(W_t)_{0\le t\le 1}$ be an r-dimensional Wiener process defined on a probability space $(\Omega,\mathcal{F}, P)$ and let $(\mathcal{F}_t)_{0\le t\le 1}$ denote the natural filtration of (W_t) such that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$. Let ξ be an $\mathcal{F}_1$-measurable d-dimensional square-integrable random variable. Let f be an $\mathbb{R}^d$-valued process defined on $\mathbb{R}_+\times\Omega\times\mathbb{R}^d\times\mathbb{R}^{d\times r}$ with values in $\mathbb{R}^d$ such that for all $(y,z)\in\mathbb{R}^d\times\mathbb{R}^{d\times r}$, the map $(t,\omega)\to f(t,\omega,y,z)$ is $\mathcal{F}_t$-progressively measurable. We consider the following backward stochastic differential equation (BSDE):

$(E^{f,\xi})$

$$Y_t=\xi+\int_t^1 f(s,Y_s,Z_s)\,ds-\int_t^1 Z_s\,dW_s \quad (0\le t\le 1). \tag{1.1}$$

Equation $(E^{f,\xi})$ is closely connected to stochastic optimal control (via the adjoint process in the formulation of the Pontryagin maximum principle [5]) and to certain nonlinear partial differential equations (via nonlinear Feynmann-Kac formula [19, 20]); it is also used in mathematical finance. The BSDEs are also studied for their mathematical interest since nice problems still remain open.

Linear BSDEs were introduced by Bismut [5]. Pardoux and Peng [18] were the first to consider general nonlinear BSDEs in the above form. They have proved that when the coefficient f is globally Lipschitz, then the BSDE $(E^{f,\xi})$ has a unique adapted and square-integrable solution. Moreover, the solution can be constructed by a successive approximations procedure, see also [8]. Since the paper [18], several works have attempted to weaken the Lipschitz condition on the coefficient f, see, for example, Mao

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[15], Hamadène [10], Lepeltier and San Martin [14], Dermoune et al. [7], and Kobylanski [12] and the references therein. When the coefficient f is merely continuous, the questions of existence (and sometimes the uniqueness) of solutions have been partially solved for one-dimensional equations, either by comparison techniques or by using a classical transformation which removes the drift, see, for example, [7, 12, 14]. Stronger conditions are required to obtain the uniqueness. Note that the techniques used in dimension one do not work in the multidimensional case. Moreover, with the above-quoted techniques, there is no information about the convergence of the Picard successive approximation and, probably, this approximation does not converge in these situations.

In the multidimensional case, the questions of existence and uniqueness of solutions still remain largely open. Up to our knowledge, except for the papers [2, 3, 15, 20], no results are known about when the coefficient is nonuniformly Lipschitz in the two variables (y,z). Moreover, in [15, 20], the assumptions imposed on the coefficient are global.

Our approach is quite topological but it allows the derivation of some precise examples. More precisely, we consider the set of multidimensional BSDEs, with bounded and continuous coefficients. We are then concerned with the prevalence, in the sense of Baire categories, of BSDEs which have the properties of existence and uniqueness as well as the stability of solutions and the L^2-convergence of their associated successive approximations.

Prevalence questions were studied in many areas of mathematics (see, e.g., [1, 4, 6, 9, 11, 13, 16, 21, 22, 23]) and seem to take their origin from an earlier paper of Orlicz [16], where it is shown that "most" ordinary differential equations with continuous coefficient have unique solutions. In the theory of stochastic differential equations (SDE), the first result in this direction is due to Skorokhod [23], where the author has used it also to study the dependence of weak solutions on a parameter. The method developed in [23] cannot be extended to BSDEs, since it needs the notion of solutions in the sense of law and unfortunately this notion is actually not clear in BSDE's theory.

In this paper, we give an analytic approach. We consider the space of bounded $\mathcal{F}_t$-progressively measurable processes $f(t,\omega,y,z)$ which are continuous in (y,z) for almost all (t,ω) and measurable in (t,ω) for all (y,z). We define an appropriate complete metric on it and then look at the prevalence, in the sense of Baire categories, of the set of all f such that

(1) the corresponding BSDE $(E^{f,\xi})$ has a unique solution;
(2) the approximate solutions, given by the successive approximations associated to $(E^{f,\xi})$, converge to the unique solution of $(E^{f,\xi})$;
(3) the solutions of equation $(E^{f,\xi})$ (when they exist) are continuous with respect to the coefficient f.

It is shown, by using the Baire categories theorem, that the set of coefficients f having the above three properties is a set of a second category of Baire. See Definition 2.2 below for the Baire category sets and Oxtoby's book [17] for more details on this subject. Since a set of the second category in a Baire space contains "almost all" the points of the space, it may be thought of as the topological analogue of the measure-theoretical concept of a set whose complement is of measure zero. Our results state that, in some sense, almost all BSDEs with bounded continuous coefficient have solutions which satisfy the above properties (1), (2), and (3). We do not impose any boundedness condition on the terminal data ξ which is assumed to be square-integrable only.

The paper is organized as follows. Section 2 introduces some notations and definitions. Section 3 is devoted to the continuous dependence of solutions with respect to the coefficient. In Section 4, we deal with the continuous dependence of the solutions with respect to the coefficient. Section 5 is devoted to the convergence of the Picard successive approximations.

2. Notations and definitions

We denote by $\mathscr{E}$ the set of $(\mathbb{R}^d \times \mathbb{R}^{d\times r})$-valued processes (Y,Z) defined on $\mathbb{R}_+ \times \Omega$, which are $\mathscr{F}_t$-adapted and such that

$$||(Y,Z)||^2 = E\left(\sup_{0\le t\le 1} |Y(t)|^2 + \int_0^1 |Z(s)|^2 ds\right) < +\infty. \tag{2.1}$$

$(\mathscr{E}, \|\cdot\|)$ is a Banach space.

Definition 2.1. A solution of equation $(E^{f,\xi})$ is a pair (Y,Z) which belongs to the space $(\mathscr{E}, \|\cdot\|)$ and satisfies $(E^{f,\xi})$.

Throughout the paper, the solutions of equation $(E^{f,\xi})$ will be denoted by (Y^f, Z^f). For a given real number $M > 0$, we denote by $\mathscr{C}$ the set of functions $f(t,\omega,y,z)$, defined on $\mathbb{R}_+ \times \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times r}$ with values in $\mathbb{R}^d$, which are continuous in (y,z) for almost all (t,ω), measurable in (t,ω) for all (y,z) and such that $\operatorname{ess\,sup}_{(t,\omega,y,z)} |f(t,\omega,y,z)| \le M$. Let $\mathscr{L}$ be the subset of $\mathscr{C}$ consisting of functions f which are Lipschitz in (y,z).

Definition 2.2. A Baire space is a separated topological space in which all countable intersections of dense open subsets are dense also. Let T be a Baire space. A subset F of T is said to be meager (or a first-category set in the Baire sense) if it is contained in a countable union of closed nowhere dense subsets of T. The complement of a meager set is called a residual (or a second-category set).

3. Prevalence of existence and uniqueness

Since f is bounded, we can assume without loss of generality that $\xi = 0$. Indeed, let (Y,Z) be a solution of the BSDE $(E^{f,\xi})$. By Itô's representation theorem, there exists a predictable process Z' such that $E(\xi/\mathscr{F}_t) = \xi + \int_0^t Z'_s dW_s$. Thus, $Y_t - E(\xi/\mathscr{F}_t) = \int_t^1 f(s, Y_s, Z_s)ds - \int_t^1 (Z_s - Z'_s)dW_s$. Define $Y''_t := Y_t - E(\xi/\mathscr{F}_t)$, $Z''_t := Z_t - Z'_t$, and $f''(s,u,v) := f(s, u + E(\xi/\mathscr{F}_s), v + Z'_s)$. It is not difficult to see that (Y,Z) is a solution to the BSDE $(E^{(f,\xi)})$ if and only if (Y'', Z'') is a solution to the BSDE $(E^{(f'',0)})$.

As a consequence, the terminal condition will play no role in our situation. Hence, we consider BSDEs with $\xi = 0$.

We denote by $\mathscr{R}_e$ the set of processes $f \in \mathscr{C}$ for which equation $(E^{f,0})$ has a (not necessarily unique) solution and by $\mathscr{R}_1$ the subset of $\mathscr{C}$ which consists of all functions f for which equation $(E^{f,0})$ has a unique solution.

Theorem 3.1. *$\mathcal{R}_1$ is a residual set in the Baire space $(\mathcal{C},\rho)$.*

To prove this theorem we need some lemmas.

Lemma 3.2. *Endowed with the distance*

$$\rho(f,g)=\sum_{n=1}^{\infty}\left(\frac{1}{2^n}\right)\frac{\left(E\int_0^1\sup_{|y|,|z|\le n}|f(s,y,z)-g(s,y,z)|^2ds\right)^{1/2}}{1+\left(E\int_0^1\sup_{|y|,|z|\le n}|f(s,y,z)-g(s,y,z)|^2ds\right)^{1/2}},\tag{3.1}$$

$(\mathcal{C},\rho)$ is a complete metric space in which $\mathcal{L}$ is dense.

The above lemma can be proved by truncation and regularization.

Lemma 3.3. *Let f be an element of $\mathcal{L}$ and let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{R}_e$. Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of square-integrable random variables which are $\mathcal{F}_1$-measurable. Assume that*

$$\rho(f_n,f)\longrightarrow 0,\quad E(|\xi_n-\xi|^2)\longrightarrow 0,\quad \text{as } n\longrightarrow\infty.\tag{3.2}$$

Then (Y^{f_n},Z^{f_n}) converges to (Y^f,Z^f) in $(\mathcal{E},\|\cdot\|)$.

Proof. Without loss of generality, we may suppose that $\xi=\xi_n=0$ for each n. Let (Y^f,Z^f) (resp., (Y^{f_n},Z^{f_n})) be a solution of equation $(E^{f,0})$ (resp., $(E^{f_n,0})$). Itô's formula shows that

$$\begin{aligned}\left|Y_t^{f_n}-Y_t^f\right|^2&+\int_t^1\left|Z_s^{f_n}-Z_s^f\right|^2ds\\&=2\int_t^1\left(Y_s^{f_n}-Y_s^f\right)^*\left(f_n\left(s,Y_s^{f_n},Z_s^{f_n}\right)-f\left(s,Y_s^f,Z_s^f\right)\right)ds\\&\quad-\int_t^1\left(Y_s^{f_n}-Y_s^f\right)^*\left(Z_s^{f_n}-Z_s\right)dW_s,\end{aligned}\tag{3.3}$$

where $(Y_s^{f_n}-Y_s^f)^*$ denotes the transpose of the vector $(Y_s^{f_n}-Y_s^f)$. Let α be an arbitrary number in $\mathbb{R}_+^*$ and let L be the Lipschitz constant of the function f. For a given positive number N, let $A_n^N=\{(s,\omega);\ |Y_s^{f_n}|^2+|Z_s^{f_n}|^2+|Y_s^f|^2+|Z_s^f|^2\ge N^2\}$ and $\overline{A}_n^N=\Omega\setminus A_n^N$, and denote by $\mathcal{X}_E$ the indicator function of the set E. Using Young and Chebychev inequalities and the fact that f is uniformly Lipschitz, we get

$$\begin{aligned}E\left(\left|Y_t^{f_n}-Y_t^f\right|^2\right)&+E\int_t^1\left|Z_s^{f_n}-Z_s^f\right|^2ds\\&\le E\int_t^1\left|Y_s^{f_n}-Y_s^f\right|\left(\left|f_n\left(s,Y_s^{f_n},Z_s^{f_n}\right)-f\left(s,Y_s^f,Z_s^f\right)\right|\right)ds\\&\le 2\alpha^2E\int_t^1\left|Y_s^{f_n}-Y_s^f\right|^2ds\\&\quad+\frac{2}{\alpha^2}E\int_t^1\left|f_n\left(s,Y_s^{f_n},Z_s^{f_n}\right)-f\left(s,Y_s^f,Z_s^f\right)\right|^2\mathcal{X}_{A_n^N}ds\\&\quad+\frac{2}{\alpha^2}E\int_t^1\left(\left|f_n\left(s,Y_s^{f_n},Z_s^{f_n}\right)-f\left(s,Y_s^f,Z_s^f\right)\right|^2\right)\mathcal{X}_{\overline{A}_n^N(s)}ds\end{aligned}$$

$$
\begin{aligned}
&\le 2\alpha^2 E\int_t^1 \left| Y_s^{f_n} - Y_s^f \right|^2 ds \\
&\quad + \frac{8M^2}{\alpha^2}\frac{1}{N^2}E\int_t^1 \left(\left| Y_s^{f_n}\right|^2 + \left| Z_s^{f_n}\right|^2 + \left| Y_s^{f}\right|^2 + \left| Z_s^{f}\right|^2 \right) ds \\
&\quad + \frac{4}{\alpha^2}E\int_t^1 \left| f_n\left(s, Y_s^{f_n}, Z_s^{f_n}\right) - f\left(s, Y_s^{f_n}, Z_s^{f_n}\right)\right|^2 \mathcal{X}_{\overline{A}_n^N(s)} ds \\
&\quad + \frac{4}{\alpha^2}E\int_t^1 \left| f\left(s, Y_s^{f_n}, Z_s^{f_n}\right) - f\left(s, Y_s^{f}, Z_s^{f}\right)\right|^2 \mathcal{X}_{\overline{A}_n^N(s)} ds \\
&\le 2\alpha^2 E\int_t^1 \left| Y_s^{f_n} - Y_s^f \right|^2 ds + \frac{8M^2K}{\alpha^2}\frac{1}{N^2} + \frac{4}{\alpha^2}\left(\frac{2^N\rho(f_n,f)}{1-2^N\rho(f_n,f)}\right)^2 ds \\
&\quad + \frac{4}{\alpha^2}L^2 E\int_t^1 \left| Y_s^{f_n} - Y_s^f \right|^2 ds + \frac{4}{\alpha^2}L^2 E\int_t^1 \left| Z_s^{f_n} - Z_s^f \right|^2 ds,
\end{aligned}
\tag{3.4}
$$

where K is a constant which depends only on M.

We choose α such that $4L^2/\alpha^2 < 1$, then we use Gronwall lemma to get $E(|Y^{f_n}(t) - Y^f(t)|^2) \le [(4/\alpha^2)(2^N\rho(f_n,f)/1-2^N\rho(f_n,f))^2 + (8M^2K/\alpha^2)(1/N^2)]\exp(2\alpha^2+1)$. We use Burkholder-Davis-Gundy inequality to show that a positive constant $C = C(\alpha, L)$ exists such that

$$
\begin{aligned}
E\Big(\sup_{0\le t\le 1} \left| Y^{f_n}(t) - Y^f(t)\right|^2\Big) &\le C\left[\frac{4}{\alpha^2}\left(\frac{2^N\rho(f_n,f)}{1-2^N\rho(f_n,f)}\right)^2 + \frac{8M^2K}{\alpha^2}\frac{1}{N^2}\right]\exp(2\alpha^2+1), \\
E\int_0^1 \left| Z_s^{f_n} - Z_s^f \right|^2 ds &\le C\left[\frac{4}{\alpha^2}\left(\frac{2^N\rho(f_n,f)}{1-2^N\rho(f_n,f)}\right)^2 + \frac{8M^2K}{\alpha^2}\frac{1}{N^2}\right]\exp(2\alpha^2+1).
\end{aligned}
\tag{3.5}
$$

Lemma 3.3 follows by passing to the limit first on n and next on N. □

Now, we define the oscillation function $D_\delta : \mathcal{C} \to \mathbb{R}_+$ as follows:

$$
\begin{aligned}
D_\delta(f) &= \sup\left\{ d\left(\left(Y^{f_1}, Z^{f_1}\right), \left(Y^{f_2}, Z^{f_2}\right)\right);\ f_i \in \mathcal{L} \text{ and } \rho(f, f_i) < \delta \text{ for } i = 1,2 \right\}, \\
D_e(f) &= \lim_{\delta\to 0} D_\delta(f).
\end{aligned}
\tag{3.6}
$$

We then have the following lemma.

Lemma 3.4. (i) *If f belongs to $\mathcal{L}$, then $D_e(f) = 0$.*

(ii) *The function D_e is upper semicontinuous on $\mathcal{L}$.*

Proof. Assertion (i) is a consequence of Lemma 3.3. We prove assertion (ii). Let (f_n) be a sequence in $\mathcal{C}$ converging to a limit f, which belongs to $\mathcal{L}$. Assume that $\lim_{n\to\infty} D_e(f_n) > 0$. Then there exist $\epsilon > 0$ and a subsequence (n_k) such that, for each k, there exist two sequences $(f^1_{n_k})$ and $(f^2_{n_k})$ in $\mathcal{L}$, which satisfy

$$\rho\left(f_{n_k}, f^1_{n_k}\right) < \frac{1}{n_k}, \qquad \rho\left(f_{n_k}, f^2_{n_k}\right) < \frac{1}{n_k}, \tag{3.7}$$

$$d\left(\left(Y^{f^1_{n_k}}, Z^{f^1_{n_k}}\right), \left(Y^{f^2_{n_k}}, Z^{f^2_{n_k}}\right)\right) > \epsilon. \tag{3.8}$$

Thus, (3.7) and Lemma 3.3 imply that $\lim_{k\to\infty} d((Y^{f^1_{n_k}}, Z^{f^1_{n_k}}), (Y^{f^2_{n_k}}, Z^{f^2_{n_k}})) = 0$. This contradicts (3.8). Assertion (ii) is proved. □

The following proposition gives a sufficient condition which ensures the existence of solutions to the BSDE $(E^{f,0})$.

Proposition 3.5. *If $D_e(f) = 0$ for an f in $\mathcal{C}$, then equation $(E^{f,0})$ has at least one solution in $\mathcal{E}$.*

Proof. Let $f \in \mathcal{C}$. Since $D_e(f) = 0$, then there exists a decreasing sequence of strictly positive numbers δ_n ($\delta_n \downarrow 0$) such that

$$\sup\left\{d((Y^{f_1}, Z^{f_1}), (Y^{f_2}, Z^{f_2}));\ f_i \in \mathcal{L} \text{ and } \rho(f, f_i) < \delta_n \text{ for } i = 1, 2\right\} < \frac{1}{n}. \tag{3.9}$$

But Lemma 3.2 implies that for each $n \in \mathbb{N}^*$, there exists $f_n \in \mathcal{L}$ such that $\rho(f_n, f) < \delta_n$. Since δ_n decreases, it follows from (3.9) that $d((Y^{f_n}, Z^{f_n}), (Y^{f_m}, Z^{f_m})) < \sup(1/m, 1/n)$. Hence $(Y^{f_n}, Z^{f_n})_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $(\mathcal{E}, \|\cdot\|)$. Let (Y, Z) be its limit. We will show that (Y, Z) satisfies equation $(E^{f,0})$. We immediately have

$$\lim_{n\to\infty} E\left(\sup_{0\le s\le 1} \left|Y^{f_n}(s) - Y(s)\right|^2\right) = 0, \tag{3.10}$$

$$\lim_{n\to\infty} E\int_0^1 \left|Z^{f_n}(s) - Z(s)\right|^2 ds = 0. \tag{3.11}$$

From (3.11) we get that, for each $t \in [0,1]$,

$$\lim_{n\to\infty} \int_t^1 Z^{f_n}(s)\,dW_s = \int_t^1 Z(s)\,dW_s \quad \text{in probability}. \tag{3.12}$$

Moreover, (3.10) and (3.11) imply that there exists a subsequence (n_k) such that

$$(Y^{f_{n_k}}, Z^{f_{n_k}}) \text{ converges to } (Y, Z) \qquad dP \times dt \text{ -a.e.} \tag{3.13}$$

It remains now to prove that, for each $t \in [0,1]$,

$$\lim_{n\to\infty} \int_t^1 f_{n_k}(s, Y^{f_{n_k}}(s), Z^{f_{n_k}}(s))\,ds = \int_t^1 f(s, Y(s), Z(s))\,ds \quad \text{in probability}. \tag{3.14}$$

Without loss of generality, we may assume that (3.13) holds without extracting subsequence. Let N be an arbitrary positive number. Since both f_n and f are bounded by M, we can show that

$$
\begin{aligned}
E&\left|\int_t^1 f_n(s,Y^{f_n}(s),Z^{f_n}(s))\,ds-\int_t^1 f(s,Y(s),Z(s))\,ds\right| \\
&\le E\int_t^1 |\, f_n(s,Y^{f_n}(s),Z^{f_n}(s))-f(s,Y^{f_n}(s),Z^{f_n}(s))\,|\,ds \\
&\quad +E\int_0^1 |\, f(s,Y^{f_n}(s),Z^{f_n}(s))-f(s,Y(s),Z(s))\,|\,ds \\
&\le E\int_0^1 \sup_{|y|,|z|\le N} |\, f_n(s,y,z)-f(s,y,z)\,|\,ds+\frac{2M}{N} \\
&\quad +E\int_0^1 |\, f(s,Y^{f_n}(s),Z^{f_n}(s))-f(s,Y(s),Z(s))\,|\,ds \\
&= I_1(n)+\frac{2M}{N}+I_2(n).
\end{aligned}
\tag{3.15}
$$

Lemma 3.2 shows that $\lim_{n\to\infty} I_1(n)=0$. On the other hand, since $f\in\mathscr{C}$, then (3.13) implies that $f(\cdot,Y^{f_n}(\cdot),Z^{f_n}(\cdot))$ converges to $f(\cdot,Y(\cdot),Z(\cdot))$, $dP\times ds$-a.e. Hence, the Lebesgue dominated convergence theorem shows that $\lim_{n\to\infty} I_2(n)=0$. Proposition 3.5 is proved. □

Proof of Theorem 3.1. Lemma 3.2 and assertions (i) and (ii) of Lemma 3.4 imply that, for each integer n, the set $\mathscr{G}_n=\{f\in\mathscr{C};\ D_e(f)<1/n\}$ is a dense open subset of $(\mathscr{C},\rho)$. Then, by the Baire categories theorem, the set $\mathscr{G}=\bigcap_{n\in\mathbb{N}^*}\mathscr{G}_n$ is a dense G_δ subset of the Baire space $(\mathscr{C},\rho)$. Moreover, if $f\in\mathscr{G}$, then Proposition 3.5 implies that the corresponding equation $(E^{f,0})$ has one solution. Hence, $\mathscr{G}\subset\mathscr{R}_e$. This implies that $\mathscr{R}_e$ is a residual subset in $(\mathscr{C},\rho)$.

To prove that $\mathscr{R}_1$ is residual, we define the function $D_u:\mathscr{G}\to\mathbb{R}_+$ as follows: $D_u(f)=\sup\{d((Y_1^f,Z_1^f),(Y_2^f,Z_2^f));(Y_i^f,Z_i^f)$ is a solution to equation $(E^{f,0})$, $i=1,2\}$ and for each $n\in\mathbb{N}^*$, we put $\overline{\mathscr{G}}_n=\{f\in\mathscr{G};\ D_u(f)<1/n\}$. By using Lemma 3.3, we see, as in the proof of Lemma 3.4(ii), that the function D_u is upper semicontinuous on $\mathscr{L}$. This implies that each $\overline{\mathscr{G}}_n$ contains the intersection of $\mathscr{G}$ and a dense open subset of $(\mathscr{C},\rho)$. Thus, the set $\overline{\mathscr{G}}=\bigcap_{n\in\mathbb{N}^*}\overline{\mathscr{G}}_n$ contains a dense G_δ subset of the Baire space $(\mathscr{C},\rho)$. Hence, it is residual in $(\mathscr{C},\rho)$. Finally, if $f\in\overline{\mathscr{G}}$, then the corresponding equation $(E^{f,0})$ has a unique solution. Thus, $\overline{\mathscr{G}}\subset\mathscr{R}_1$. Theorem 3.1 follows. □

Examples. In this section, we give two examples of BSDEs with nonuniformly Lipschitz coefficient and for which the existence and uniqueness hold. The proof follows as a direct application of Proposition 3.5. This shows that the sufficient condition given by the oscillation function $D_e(f)=0$ is not only theoretical but can also be applied to concrete cases.

Corollary 3.6. *Let $f \in \mathscr{C}$ and let ξ be a square-integrable random variable. Assume moreover that f satisfies the following hypothesis:*

(H1) *for every $N \in \mathbb{N}^*$, there exists a constant $L_N > 0$ such that $|f(t,\omega,y,z) - f(t,\omega,y',z')| \le L_N(|y-y'| + |z-z'|)$, P-a.s., a.e. $t \in [0,1]$, and for all y, y', z, z' such that $|y| \le N$, $|y'| \le N$, $|z| \le N$, $|z'| \le N$.*

If $L_N = \mathbb{O}(\sqrt{\log N})$, then equation $(E^{f,0})$ has a unique solution.

Proof. Let $\delta > 0$ and $f_1, f_2 \in \mathscr{L}$ be such that $\rho(f, f_i) < \delta$, $i = 1,2$. Arguing as in the proof of Lemma 3.3, we show that there exists a constant $C = C(M,\xi) > 0$ such that

$$E\Big(\sup_{0\le t\le 1}\Big|Y_t^{f_1} - Y_t^{f_2}\Big|^2\Big) \le C\bigg[\frac{2^N\rho^2(f_1,f)}{1-2^N\rho^2(f_1,f)} + \frac{2^N\rho^2(f_2,f)}{1-2^N\rho^2(f_2,f)} + \frac{1}{(L_N^2)N^2}\bigg]\exp\big(2L_N^2\big) \tag{3.16}$$

for each N such that $N > 1$ and $\rho(f, f_i) < 1/2^N$.

Since $\rho(f_i, f) < \delta$ and $L_N = \mathbb{O}(\sqrt{\log N})$, we deduce that

$$D_\delta(f) \le C\bigg[\frac{2^N\delta^2}{1-2^N\delta^2} + \frac{1}{L_N^2}\bigg]. \tag{3.17}$$

Letting $\delta \to 0$ and $N \to \infty$, we deduce that $D_e(f) = 0$. Corollary 3.6 is proved. □

Corollary 3.7. *Let $f \in \mathscr{C}$ and let ξ be a square-integrable random variable. Assume moreover that f satisfies the following hypotheses:*

(H2) *for every $N \in \mathbb{N}$, there exists a constant $\mu_N \in \mathbb{R}$ such that $\langle y-y', f(t,\omega,y,z) - f(t,\omega,y',z)\rangle \le \mu_N|y-y'|^2$, P-a.s., a.e. $t \in [0,1]$, and for all y, y', z such that $|y| \le N$, $|y'| \le N$, $|z| \le N$;*

(H3) *for every $N \in \mathbb{N}$, there exists a constant $L_N > 0$ such that $|f(t,\omega,y,z) - f(t,\omega,y,z')| \le L_N|z-z'|$, P-a.s., a.e. $t \in [0,1]$, and for all y, z, z' such that $|y| \le N$, $|z| \le N$, $|z'| \le N$.*

If $\mu_N^+ + L_N^2 = \mathbb{O}(\log N)$, then $(E^{(f,\xi)})$ has a unique solution.

Proof. Let $\delta > 0$ and $f_1, f_2 \in \mathscr{L}$ be such that $\rho(f, f_i) < \delta$, $i = 1,2$. Arguing as in the proof of Lemma 3.3, we show that there exists a constant $C = C(M,\xi) > 0$ such that

$$E\Big(\sup_{0\le t\le 1}\Big|Y_t^{f_1} - Y_t^{f_2}\Big|^2\Big) \le C\bigg[\rho_N^2(f_1 - f) + \rho_N^2(f_2 - f) + \frac{1}{(2\mu_N^+ + L_N^2)N^2}\bigg]\exp\big(2\mu_N^+ + L_N^2\big) \tag{3.18}$$

for each N such that $N > 1$ and $\rho(f, f_i) < 1/2^N$.

Since $\rho(f_i, f) < \delta$ and $\mu_N^+ + L_N^2 = \mathbb{O}(\log N)$, we deduce that

$$D_\delta(f) \leq 2C\left[\frac{2^N\delta^2}{1-2^N\delta^2} + \frac{1}{2\mu_N^+ + L_N^2}\right]. \tag{3.19}$$

Letting $\delta \to 0$ and $N \to \infty$, we deduce that $D_e(f) = 0$. Corollary 3.7 is proved. □

4. Continuous dependence on the coefficient

For a given $f \in \mathscr{C}$, we denote by $Sf = (Y^f, Z^f)$ the solution of $(E^{f,0})$ when it exists.

THEOREM 4.1. *There exists a second-category set $\mathscr{R}_2$ such that the map $S : \mathscr{R}_2 \to \mathscr{E}$, given by $Sf = (Y^f, Z^f)$, is well defined and continuous at each point of $\mathscr{R}_2$.*

Proof. We will show that S is continuous on $\overline{\mathscr{G}}$ (the dense G_δ set which has been defined in the proof of Theorem 3.1). Suppose the contrary. Then there exist $f \in \overline{\mathscr{G}}$, $\varepsilon > 0$, and a sequence $(f_p) \subset \overline{\mathscr{G}}$ such that

$$\lim_{p\to\infty} \rho(f_p, f) = 0, \quad d(Sf_p, Sf) \geq \varepsilon, \quad \text{for each } p. \tag{4.1}$$

Fix $n \in \mathbb{N}$ such that $\varepsilon < 1/n$. Since $\overline{\mathscr{G}} \subset \mathscr{G}$, then there exist a decreasing sequence of strictly positive numbers δ_n ($\delta_n \downarrow 0$) and a sequence of functions $g_n \in \mathscr{L}$ such that

$$\rho(g_n, f) < \delta_n, \qquad d(Sg_n, Sf) < \frac{1}{n}. \tag{4.2}$$

We choose p large enough so as to have $\rho(f_p, f) < \delta_n - \rho(g_n, f)$, then we use (4.2) to obtain $\rho(f_p, g_n) < \delta_n$. Therefore, $d(Sf_p, Sg_n) < 1/n$. Thus, $d(Sf_p, Sf) \leq d(Sf_p, Sg_n) + d(Sg_n, Sf) < 1/n + 1/n < (2/3)\varepsilon$, which contradicts (4.1). Theorem 4.1 is proved. □

5. The Picard successive approximations

For a given $f \in \mathscr{C}$, we denote by (Y_n^f, Z_n^f) the sequence of processes defined by the following equation:

(E_n^f)

$$Y_0^f(t) = Z_0^f(t) = 0, \quad Y_{n+1}^f(t) = \int_t^1 f\left(s, Y_n^f(s), Z_n^f(s)\right) ds - \int_t^1 Z_{n+1}^f(s)\, dW_s. \tag{5.1}$$

Itô's representation theorem shows that the sequence (Y_n^f, Z_n^f) is well defined for each n. Let $\mathscr{R}_3$ be the subset of $\mathscr{C}$ of all those $f \in \mathscr{C}$ such that the corresponding sequence (Y_n^f, Z_n^f), defined by $(E_n^{f,0})$, converges in $(\mathscr{E}, \|\cdot\|)$ to a solution (Y^f, Z^f) of equation $(E^{f,0})$.

THEOREM 5.1. *The set $\mathscr{R}_3$ is residual in $(\mathscr{C}, \rho)$.*

To prove this theorem, we need the following lemma which is the analogue of the previous Lemma 3.3.

Lemma 5.2. *Let f be an element of $\mathscr{L}$ and let $(f_p)_{p\in\mathbb{N}}$ be a sequence in $\mathscr{R}_3$. Denote by (Y_n^f, Z_n^f) (resp., $(Y_n^{f_p}, Z_n^{f_p})$) the sequence defined by equation (E_n^f) (resp., $(E_n^{f_p})$). Assume that $\rho(f_p, f) \to 0$ as $p \to \infty$. Then $\lim_{p\to\infty} \sup_{n\in\mathbb{N}} \|(Y_n^{f_p}, Z_n^{f_p}) - (Y_n^f, Z_n^f)\| = 0$.*

Proof. Let (Y_n^f, Z_n^f) (resp., $(Y_n^{f_p}, Z_n^{f_p})$) be a solution of equation (E_n^f) (resp., $(E_n^{f_p})$). Itô's formula shows that

$$\begin{aligned} &E\left(\left|Y_{n+1}^{f_p}(t) - Y_{n+1}^f(t)\right|^2\right) + E\int_t^1 \left|Z_{n+1}^{f_p}(s) - Z_{n+1}^f(s)\right|^2 ds \\ &\quad = 2E\int_t^1 \left(Y_{n+1}^{f_p}(s) - Y_{n+1}^f(s)\right)^* \left(f_p\left(s, Y_n^{f_p}(s), Z_n^{f_p}(s)\right) - f\left(s, Y_n^f(s), Z_n^f(s)\right)\right) ds, \end{aligned} \tag{5.2}$$

where $(Y_{n+1}^{f_p}(s) - Y_{n+1}^f(s))^*$ denotes the transpose of the vector $(Y_{n+1}^{f_p}(s) - Y_{n+1}^f(s))$. Let α be an arbitrary number in $\mathbb{R}_+^*$ and let L be the Lipschitz constant of the function f. For a given positive number N, let $A_{n,p}^N = \{(s,\omega);\ |Y_n^{f_p}(s)|^2 + |Z_n^{f_p}(s)|^2 + |Y_n^f(s)|^2 + |Z_n^f(s)|^2 \geq N^2\}$ and $\overline{A}_{n,p}^N = \Omega \setminus A_{n,p}^N$, and denote by $\mathscr{X}_E$ the indicator function of the set E. Arguing as in the proof of Lemma 3.3, we obtain the following inequalities:

$$\begin{aligned} &E\left(\left|Y_{n+1}^{f_p}(t) - Y_{n+1}^f(t)\right|^2\right) + E\int_t^1 \left|Z_{n+1}^{f_p}(s) - Z_{n+1}^f(s)\right|^2 ds \\ &\quad \leq 2\alpha^2 E\int_t^1 \left|Y_{n+1}^{f_p}(s) - Y_{n+1}^f(s)\right|^2 ds \\ &\qquad + \frac{2}{\alpha^2} E\int_t^1 \left|f_p\left(s, Y_n^{f_p}(s), Z_n^{f_p}(s)\right) - f\left(s, Y_n^f(s), Z_n^f(s)\right)\right|^2 \mathscr{X}_{A_{n,p}^N} ds \\ &\qquad + \frac{2}{\alpha^2} E\int_t^1 \left(\left|f_p\left(s, Y_n^{f_p}(s), Z_n^{f_p}(s)\right) - f\left(s, Y_n^f(s), Z_n^f(s)\right)\right|^2\right) \mathscr{X}_{\overline{A}_{n,p}^N} ds \\ &\quad \leq 2\alpha^2 E\int_t^1 \left|Y_{n+1}^{f_p}(s) - Y_{n+1}^f(s)\right|^2 ds \\ &\qquad + \frac{8M^2}{\alpha^2}\frac{1}{N^2} E\int_t^1 \left(\left|Y_n^{f_p}(s)\right|^2 + \left|Z_n^{f_p}(s)\right|^2 + \left|Y_n^f(s)\right|^2 + \left|Z_n^f(s)\right|^2\right) ds \\ &\qquad + \frac{4}{\alpha^2} E\int_t^1 \left|f_p\left(s, Y_n^{f_p}(s), Z_n^{f_p}(s)\right) - f\left(s, Y_n^{f_p}(s), Z_n^{f_p}(s)\right)\right|^2 \mathscr{X}_{\overline{A}_{n,p}^N} ds \\ &\qquad + \frac{4}{\alpha^2} E\int_t^1 \left|f\left(s, Y_n^{f_p}(s), Z_n^{f_p}(s)\right) - f\left(s, Y_n^f(s), Z_n^f(s)\right)\right|^2 \mathscr{X}_{\overline{A}_{n,p}^N} ds \\ &\quad \leq 2\alpha^2 E\int_t^1 \left|Y_{n+1}^{f_p}(s) - Y_{n+1}^f(s)\right|^2 ds \\ &\qquad + \frac{8M^2K}{\alpha^2}\frac{1}{N^2} + \frac{4}{\alpha^2}\left(\frac{2^N\rho(f_p, f)}{1 - 2^N\rho(f_p, f)}\right)^2 ds \\ &\qquad + \frac{4}{\alpha^2} L^2 E\int_t^1 \left|Y_n^{f_p}(s) - Y_n^f(s)\right|^2 ds + \frac{4}{\alpha^2} L^2 E\int_t^1 \left|Z_n^{f_p}(s) - Z_n^f(s)\right|^2 ds, \end{aligned} \tag{5.3}$$

where K is a constant which depends only on M.

We put $\varphi_n^p(t) = \sup_{t\le u\le 1} E(|Y_n^{f_p}(u) - Y_n^f(u)|^2) + E\int_t^1 |Z_n^{f_p}(s) - Z_n^f(s)|^2 ds$, then we have

$$\varphi_{n+1}^p(t) \le 2\alpha^2 \int_t^1 \varphi_{n+1}^p(s)ds + \frac{4L^2}{\alpha^2}\varphi_n^p(t) + \frac{8M^2K}{\alpha^2}\frac{1}{N^2} + \frac{4}{\alpha^2}\left(\frac{2^N\rho(f_p,f)}{1-2^N\rho(f_p,f)}\right)^2 \tag{5.4}$$

and Gronwall lemma implies that

$$\varphi_{n+1}^p(t) \le \frac{4L^2}{\alpha^2}\varphi_n^p(t)\exp 2\alpha^2(1-t) + \left[\frac{8M^2K}{\alpha^2}\frac{1}{N^2} + \frac{4}{\alpha^2}\left(\frac{2^N\rho(f_p,f)}{1-2^N\rho(f_p,f)}\right)^2\right]\exp 2\alpha^2. \tag{5.5}$$

If we choose $\alpha^2 = 12L^2$ and t sufficiently close to 1 so as to have $\exp(24L^2(1-t)) \le 3/2$, we obtain

$$\varphi_{n+1}^p(t) \le \frac{1}{2}\varphi_n^p(t) + \left[\frac{8M^2K}{12L^2}\frac{1}{N^2} + \frac{4}{12L^2}\left(\frac{2^N\rho(f_p,f)}{1-2^N\rho(f_p,f)}\right)^2\right]\exp\left(24L^2\right). \tag{5.6}$$

Since $\varphi_0^p(t) = 0$ for each t and p, we deduce that

$$\sup_{n\in\mathbb{N}}\varphi_n^p(t) \le \left[\frac{8M^2K}{12L^2}\frac{1}{N^2} + \frac{4}{12L^2}\left(\frac{2^N\rho(f_p,f)}{1-2^N\rho(f_p,f)}\right)^2\right]\exp\left(24L^2\right). \tag{5.7}$$

We successively pass to the limit in p and N to get $\lim_{p\to\infty}\sup_{n\in\mathbb{N}}\varphi_n^p(t) = 0$ for each t such that $\exp(24L^2(1-t)) \le 3/2$. Iterating this procedure on the subintervals $[t_i, t_{i+1}]$ such that $\exp(24L^2(t_{i+1}-t_i)) \le 3/2$, we obtain $\lim_{p\to\infty}\sup_{n\in\mathbb{N}}\varphi_n^p(t) = 0$ for each $t \in [0,1]$. To finish the proof, we use Burkholder-Davis-Gundy inequality. Lemma 5.2 is proved. □

Proof of Theorem 5.1. Let $f' \in \mathcal{L}$ and $k \in \mathbb{N}^*$. By Lemma 5.2, there exists $\delta(f',k) > 0$ such that, for every $f \in \mathcal{C}$ satisfying $\rho(f',f) < \delta(f',k)$, the inequality $\|(Y_n^{f'}, Z_n^{f'}) - (Y_n^f, Z_n^f)\| < 1/k$ holds. By Lemma 3.2 and the Baire categories theorem, the set $\mathcal{G}_1 = \bigcap_k \bigcup_{f'\in\mathcal{L}}\{f \in \mathcal{C};\ \rho(f',f) < \delta(f',k)\}$ is a dense G_δ subset in the Baire space $(\mathcal{C},\rho)$. We will prove that for each $f \in \mathcal{G}_1$, the sequence (Y_n^f, Z_n^f) defined by (E_n^f) converges, in $(\mathcal{E}, \|\cdot\|)$, to a solution of equation $(E^{f,0})$. Let $f \in \mathcal{G}_1$ and $\varepsilon > 0$. We use Lemma 5.2 and the fact that the sequence $(Y_n^{f'}, Z_n^{f'})$ converges for $f' \in \mathcal{L}$ to show that a positive number N_0 exists such that, for any $n, m \ge N_0$, the following inequality holds:

$$\begin{aligned} \left\|\left(Y_n^f, Z_n^f\right) - \left(Y_m^f, Z_m^f\right)\right\| &\le \left\|\left(Y_n^f, Z_n^f\right) - \left(Y_n^{f'}, Z_n^{f'}\right)\right\| + \left\|\left(Y_n^{f'}, Z_n^{f'}\right) - \left(Y_m^{f'}, Z_m^{f'}\right)\right\| \\ &\quad + \left\|\left(Y_m^{f'}, Z_m^{f'}\right) - \left(Y_m^f, Z_m^f\right)\right\| < 3\varepsilon. \end{aligned} \tag{5.8}$$

Hence, (Y_n^f, Z_n^f) is a Cauchy sequence in the Banach space $(\mathcal{E}, \|\cdot\|)$, and so its convergence follows. Let (Y,Z) be its limit. We will show that (Y,Z) satisfies equation $(E^{f,0})$.

Since (Y_n^f, Z_n^f) converges to (Y,Z) in the space $(\mathscr{E}, \|\cdot\|)$, we immediately have

$$\lim_{n\to\infty} E\Big(\sup_{0\le t\le 1}\Big|Y_{n+1}^f(t) - Y(t)\Big|^2\Big) = 0, \qquad \lim_{n\to\infty} E\int_0^1 \Big|Z_{n+1}^f(s) - Z(s)\Big|^2 ds = 0. \tag{5.9}$$

We prove that $\lim_{n\to\infty} E\int_0^1 |f(s,Y_n^f(s),Z_n^f(s)) - f(s,Y(s),Z(s))|^2 ds = 0$. Since f is bounded by M, we have $E(|Y_n^f(t)|^2) \le M^2$, and then by Fatou's lemma, we obtain $E(|Y_n^f(t) - Y(t)|^2) \le 2M^2$. Hence, by (5.9), the sequence (Y_n^f, Z_n^f) converges to (Y,Z) in $L^2([0,1]\times\Omega)$. Since f is bounded and continuous, then $\lim_{n\to\infty} E\int_o^1 |f(s,Y_n^f(s),Z_n^f(s)) - f(s,Y(s),Z(s))|^2 ds = 0$. Theorem 5.1 is proved. □

Remark 5.3. The prevalence of the continuity of the solution with respect to the coefficient can also be proved via the Picard successive approximations.

Indeed, let $\overline{\mathscr{G}}$ and $\mathscr{G}_1$ be the residual sets defined in the proofs of Theorems 3.1 and 5.1, respectively. Put $\mathscr{G}_2 := \overline{\mathscr{G}} \bigcap \mathscr{G}_1$ and let $S : \mathscr{R}_3 \mapsto \mathscr{E}$ be given by $Sf = (Y^f, Z^f)$. We will prove that S is continuous in $\mathscr{G}_2$, the residual set defined in the proof of Theorem 5.1. Assume the contrary holds. Then there exist $f \in \mathscr{G}_2$, $\varepsilon > 0$, and a sequence $(f_p) \subset \mathscr{G}_2$ such that

$$\lim_{p\to\infty} \rho(f_p, f) = 0, \quad ||Sf_p - Sf|| \ge \varepsilon, \quad \text{for each } p. \tag{5.10}$$

Let $k \in \mathbb{N}^*$ such that $1/k < \varepsilon/4$. Since $f \in \mathscr{G}_2 \subset \mathscr{G}_1$, there exists a sequence $(g_k) \subset \mathscr{L}$ such that

$$\rho(g_k, f) < \delta(g_k, k). \tag{5.11}$$

Hence, by Lemma 5.2, we have $\|(Y_n^{g_k}, Z_n^{g_k}) - (Y_n^f, Z_n^f)\| < 1/k$ for each $n \in \mathbb{N}^*$. Passing to the limit on n, we show by using Theorem 5.1 that

$$||Sg_k - Sf|| \le \frac{1}{k}. \tag{5.12}$$

We choose p large enough such that $\rho(f_p, f) < \delta(g_k, k) - \rho(g_k, f)$, then we use (3.8) and the triangular inequality to get $\rho(f_p, g_k) \le \delta(g_k, k)$. Hence, by using Lemma 3.3, we obtain $\|(Y_n^{g_k}, Z_n^{g_k}) - (Y_n^{f_p}, Z_n^{f_p})\| < 1/k$ for each $n \in \mathbb{N}^*$. We pass to the limit on n and use Theorem 5.1 to get $\|Sg_k - Sf_p\| \le 1/k$. Thus, $\|Sf_p - Sf\| \le \|Sf_p - Sg_k\| + \|Sg_k - Sf\| \le 2/k < \varepsilon/2$, which contradicts (3.7).

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