APPROXIMATION OF SOLUTIONS TO RETARDED DIFFERENTIAL EQUATIONS WITH APPLICATIONS TO POPULATION DYNAMICS

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We consider a retarded differential equation with applications to population dynamics. We establish the convergence of a finite-dimensional approximations of a unique solution, the existence and uniqueness of which are also proved in the process.

1. Introduction

Consider the following partial differential equation with delay:

$$\begin{gathered}\frac{\partial u}{\partial t}(t,x)=\frac{\partial^2 u}{\partial x^2}(t,x)+f\big(t,x,u(t,x),u(t-r,x)\big),\quad t>0,\ x\in[0,1],\\ \frac{\partial u}{\partial x}(t,0)=0=\frac{\partial u}{\partial x}(t,1),\quad t\geq 0,\\ u(s,x)=h(s,x),\quad s\in[-r,0],\ x\in[0,1],\end{gathered}\tag{1.1}$$

where $f:[0,\infty)\times[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $h:[-r,0]\times[0,1]\to\mathbb{R}$ is a given function. The above problem for $f(t,x,u,v)=-d(x)u+b(x)v$ models a linear growth of a population in $[0,1]$, where $u(t,\cdot)$ is the population density at time t, and the term $\partial^2u/\partial x^2$ represents the internal migration. The continuous functions $d,b:[0,1]\to[0,\infty)$ represent space-dependent death and birth rates, respectively, and r is the delay due to pregnancy (cf. Engel and Nagel [10, page 434]).

We formulate (1.1) as the following retarded differential equation:

$$\begin{gathered}u'(t)+Au(t)=f\big(t,u(t),u(a(t))\big),\quad 0<t\leq T<\infty,\\ u(t)=h(t),\quad t\in[-r,0],\end{gathered}\tag{1.2}$$

in a Hilbert space H, where $-A$ is the infinitesimal generator of a C_0 semigroup $\{S(t):t\geq 0\}$ of bounded linear operators in H, $h\in\mathscr{C}_0:=C([-r,0];H)$ is a given function and the function a is defined from $[0,T]$ into $[-r,T]$ with the delay property $a(t)\leq t$ for $t\in[0,T]$. For (1.1), we may take $X=L^2[0,1]$ and $D(A)=\{u\in H^2[0,1]:u'(0)=u'(1)=0\}$ with $Au=-d^2u/dx^2$ for $u\in D(A)$. It is known that the semigroup $S(t)$ generated by $-A$ is analytic in H (cf. Engel and Nagel [10, page 454]).

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For $t \in [0,T]$, we will use the notation $\mathscr{C}_t := C([-r,t];H)$ for the Banach space of all continuous functions from $[-r,t]$ into H endowed with the supremum norm

$$\|\psi\|_t := \sup_{-r\le\eta\le t} \|\psi(\eta)\|, \quad \psi \in \mathscr{C}_t. \tag{1.3}$$

The linear case of (1.2) in which $f(t,\psi) = L\psi$, with a bounded linear operator $L : \mathscr{C}_T \to X$ is recently considered by Bátkai et al. [7] using the theory of perturbed Hille-Yosida operators. A particular semilinear case of (1.2) is considered by Alaoui [1].

For the earlier works on existence, uniqueness, and stability of various types of solutions of differential and functional differential equations, we refer to Bahuguna [2, 3], Balachandran and Chandrasekaran [6], Lin and Liu [13], and the references therein. The related results for the approximation of solutions may be found in [4, 5].

Initial studies concerning existence, uniqueness, and finite-time blowup of solutions for the equation

$$\begin{gathered} u'(t) + Au(t) = g(u(t)), \quad t \ge 0, \\ u(0) = \phi, \end{gathered} \tag{1.4}$$

have been considered by Segal [17], Murakami [15], and Heinz and von Wahl [12]. Bazley [8, 9] has considered the semilinear wave equation

$$\begin{gathered} u''(t) + Au(t) = g(u(t)), \quad t \ge 0, \\ u(0) = \phi, \qquad u'(0) = \psi, \end{gathered} \tag{1.5}$$

and has established the uniform convergence of approximations of solutions to (1.5) using the existence results of Heinz and von Wahl [12]. Göthel [11] has proved the convergence of approximations of solutions to (1.4), but assumed g to be defined on the whole of H. Based on the ideas of Bazley [8, 9], Miletta [14] has proved the convergence of approximations to solutions of (1.4). The existence, uniqueness, and continuation of classical solutions to (1.2) are considered by Bahuguna [3]. In the present work, we use the ideas of Miletta [14] and Bahuguna [2, 3] to establish the convergence of finite-dimensional approximations of the solutions to (1.2).

2. Preliminaries and assumptions

Existence of a solution to (1.2) is closely associated with the existence of a function $u \in \mathscr{C}_{\widetilde{T}}$, $0 < \widetilde{T} \le T$ satisfying

$$u(t) = \begin{cases} h(t), & t \in [-r,0], \\ S(t)h(0) + \displaystyle\int_0^t S(t-s)f(s,\psi(s),\psi(a(s)))\,ds, & t \in [0,\widetilde{T}], \end{cases} \tag{2.1}$$

and such a function u is called a *mild solution* of (1.2) on $[-r,\widetilde{T}]$. A function $u \in \mathscr{C}_{\widetilde{T}}$ is called a *classical solution* of (1.2) on $[-r,\widetilde{T}]$ if $u \in C^1((0,\widetilde{T}];H)$ and u satisfies (1.2) on $[-r,\widetilde{T}]$.

We assume that in (1.2), the linear operator A satisfies the following hypothesis.

(H1) A is a closed, positive definite, selfadjoint linear operator from the domain $D(A) \subset H$ into H such that $D(A)$ is dense in H, A has the pure point spectrum

$$0 < \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots \tag{2.2}$$

and a corresponding complete orthonormal system of eigenfunctions $\{u_i\}$, that is,

$$Au_i = \lambda_i u_i, \qquad (u_i, u_j) = \delta_{ij}, \tag{2.3}$$

where $\delta_{ij} = 1$ if $i = j$ and zero otherwise.

If (H1) is satisfied, then the semigroup $S(t)$ generated by $-A$ is analytic in H. It follows that the fractional powers A^α of A for $0 \le \alpha \le 1$ are well defined from $D(A^\alpha) \subseteq H$ into H (cf. Pazy [16, pages 69–75]). $D(A^\alpha)$ is a Banach space endowed with the norm

$$\|x\|_\alpha = ||A^\alpha x||, \quad x \in D(A^\alpha). \tag{2.4}$$

For $t \in [0,T]$, we denote $\mathscr{C}_t^\alpha := C([-r,t];D(A^\alpha))$ endowed with the norm

$$\|\psi\|_{t,\alpha} := \sup_{-r \le \eta \le t} ||\psi(\eta)||_\alpha. \tag{2.5}$$

The nonlinear function f is assumed to satisfy the following hypotheses.

(H2) The function $h \in \mathscr{C}_0^\alpha$.

(H3) The map f is defined from $[0,\infty) \times D(A^\alpha) \times D(A^\alpha)$ into $D(A^\beta)$ for $0 < \beta \le \alpha < 1$ and there exists a nondecreasing function L_R from $[0,\infty)$ into $[0,\infty)$ depending on $R > 0$ such that

$$||f(t_1,u_1,v_1) - f(t_2,u_2,v_2)|| \le L_R(t)\big[|t-s|^\gamma + ||u_1 - u_2||_\alpha + ||v_1 - v_2||_\alpha\big], \tag{2.6}$$

for all $(t_i,u_i,v_i) \in [0,\infty) \times B_R(D(A^\alpha),h(0)) \times B_R(D(A^\alpha),h(a(0)))$, for $i = 1,2$, where $0 < \gamma < 1$, $B_R(Z,z_0) = \{z \in Z : \|z - z_0\|_Z \le R\}$ is the ball of radius R centered at z_0 in a Banach space Z with its norm $\|\cdot\|_Z$.

(H4) The function $a : [0,T] \to [-r,T]$ is continuous and satisfies the delay property $a(t) \le t$ for $t \in [0,T]$.

3. Approximate solutions and convergence

Let H_n denote the finite-dimensional subspace of H spanned by $\{u_0,u_1,\dots,u_n\}$ and let $P^n : H \to H_n$ be the corresponding projection operator for $n = 0,1,2,\dots$. Let $0 < t < \widetilde{T} \le T$ be such that

$$\begin{aligned} ||(S(t) - I)A^\alpha h(0)|| &\le \frac{R}{3}, \\ ||A^\alpha(h_n(0) - h(0))|| &\le \frac{R}{3}. \end{aligned} \tag{3.1}$$

Let $\bar{h}$ be the extension of h by the constant value $h(0)$ on $[0,T]$. We set

$$T_0 = \min\left\{\widetilde{T}, \left(\frac{(1-\alpha)R}{3M_0C_\alpha}\right)^{1/(1-\alpha)}, \left(\frac{3(1-\alpha)}{8L_R(T_0)C_\alpha}\right)^{1/(1-\alpha)}\right\}, \tag{3.2}$$

where $M_0 = [L_R(T_0)(2R+T^\gamma+4\|\bar{h}\|_{T_0,\alpha}) + \|f(0,h(0),h(a(0)))\|]$ and C_α is a positive constant such that $\|A^\alpha S(t)\| \le C_\alpha t^{-\alpha}$ for $t>0$.

We define

$$\begin{gathered} f_n : [0,T_0]\times H\times H \longrightarrow D(A), \\ f_n(t,u,v) = P^n f(t,P^n u, P^n v), \quad (t,u,v)\in[0,T_0]\times H\times H, \\ h_n : [-r,0] \longrightarrow D(A), \quad h_n(t) = P^n h(t),\ t\in[-r,0]. \end{gathered} \tag{3.3}$$

Let $A^\alpha : \mathscr{C}_t^\alpha \to \mathscr{C}_t$ be given by $(A^\alpha\psi)(s) = A^\alpha(\psi(s))$, $s\in[-r,t]$, $t\in[0,T_0]$. We define a map $F_n : B_R(\mathscr{C}_{T_0}, A^\alpha\bar{h}) \to \mathscr{C}_{T_0}$ as follows:

$$(F_n\psi)(t) = \begin{cases} A^\alpha h_n(t), & t\in[-r,0], \\ S(t)A^\alpha h_n(0) & \\ \quad + \displaystyle\int_0^t A^\alpha S(t-s) f_n(s, A^{-\alpha}\psi(s), A^{-\alpha}\psi(a(s)))\,ds, & t\in[0,T_0], \end{cases} \tag{3.4}$$

for $\psi\in B_R(\mathscr{C}_{T_0}, A^\alpha\bar{h})$.

Proposition 3.1. *For each $n\ge n_0$, where n_0 is large enough and $n,n_0\in\mathbb{N}$, there exists a unique $w_n\in B_R(\mathscr{C}_{T_0}, A^\alpha\bar{h})$ such that $F_n w_n = w_n$ on $[-r,T_0]$.*

Proof. First, we show that for any $\psi\in B_R(\mathscr{C}_{T_0}, A^\alpha\bar{h})$, $F_n\psi\in B_R(\mathscr{C}_{T_0}, A^\alpha\bar{h})$. For $t\in[-r,0]$,

$$(F_n\psi)(t) - A^\alpha\bar{h}(t) = A^\alpha(P^n - I)h(t) = A^\alpha(P^n-I)h(t) \longrightarrow 0, \quad \text{as } n\longrightarrow\infty. \tag{3.5}$$

Thus, for $n\ge n_0$, n_0 large enough, for $t\in[-r,0]$, we have

$$\|(F_n\psi)(t) - A^\alpha\bar{h}(t)\| \le R. \tag{3.6}$$

Now, for $t\in(0,T_0]$, we have

$$\begin{aligned} \|(F_n\psi)(t) - A^\alpha\bar{h}(t)\| \le{}& \|(S(t)-I)A^\alpha h(0)\| + \|A^\alpha(h_n(0)-h(0))\| \\ &+ \int_0^t \|A^\alpha S(t-s)\|\|f_n(s,A^{-\alpha}\psi(s),A^{-\alpha}\psi(a(s)))\|ds. \end{aligned} \tag{3.7}$$

For $s \in [0, T_0]$,

$$\begin{aligned} &\|f_n(s, A^{-\alpha}\psi(s), A^{-\alpha}\psi(a(s)))\| \\ &\quad \le \|f(s, P^n A^{-\alpha}\psi(s), P^n A^{-\alpha}\psi(a(s)))\| \\ &\quad \le \|f(s, P^n A^{-\alpha}\psi(s), P^n A^{-\alpha}\psi(a(s))) - f(s, P^n \bar{h}(s), P^n \bar{h}(a(s)))\| \\ &\qquad + \|f(s, P^n \bar{h}(s), P^n \bar{h}(a(s))) - f(0, h(0), h(a(0)))\| \\ &\qquad + \|f(0, h(0), h(a(0)))\| \\ &\quad \le L_R(T_0)(2R + T^\gamma + 4\|\bar{h}\|_{T_0,\alpha}) + \|f(0, h(0), h(a(0)))\|. \end{aligned} \tag{3.8}$$

It follows from the choice of T_0 that $F_n : B_R(\mathscr{C}_{T_0}, A^\alpha \bar{h}) \to B_R(\mathscr{C}_{T_0}, A^\alpha \bar{h})$ for n large enough. Now, we show that F_n is a strict contraction. For $\psi_1, \psi_2 \in B_R(\mathscr{C}_{T_0}, A^\alpha \bar{h})$, $(F_n\psi_1)(t) - (F_n\psi_2)(t) = 0$ on $[-r, 0]$ and for $t \in [0, T_0]$, we have

$$\|(F_n\psi_1)(t) - (F_n\psi_2)(t)\| \le 2L_R(T_0)C_\alpha \frac{T_0^{1-\alpha}}{1-\alpha}\|\psi_1 - \psi_2\|_{T_0} \le \frac{3}{4}\|\psi_1 - \psi_2\|_{T_0}. \tag{3.9}$$

Taking the supremum over $[-r, T_0]$, it follows that F_n is a strict contraction on $B_R(\mathscr{C}_{T_0}, A^\alpha \bar{h})$ and hence there exits a unique $w_n \in B_R(\mathscr{C}_{T_0}, A^\alpha \bar{h})$ with $w_n = F_n w_n$ on $[-r, T_0]$. This completes the proof of the proposition. □

Let $u_n = A^{-\alpha} w_n$. Then, $u_n \in B_R(\mathscr{C}_{T_0}^\alpha, \bar{h})$ and satisfies

$$u_n(t) = \begin{cases} h_n(t), & t \in [-r, 0], \\ S(t)h_n(0) & \\ \quad + \int_0^t S(t-s) f_n(s, u_n(s), u_n(a(s)))\, ds, & t \in [0, T_0]. \end{cases} \tag{3.10}$$

Remarks 3.2. The above solution $u_n(t)$ is known as the Faedo-Galerkin approximate solution of (1.2).

Collorary 3.3. *If $h(t) \in D(A)$ for all $t \in [-\tau, 0]$, then $w_n(t) \in D(A^\beta)$ for all $t \in [-\tau, T_0]$, where $0 \le \beta < 1$, $0 \le \alpha + \beta < 1$, and $w_n(t)$ is the solution of the integral equation (3.4).*

Proof. For any $g \in D(A^\beta)$ and $t \in [-\tau, 0]$, we have

$$|(A^\beta g, w_n(t))| \le \|g\| \|A^{\beta+\alpha} h_n(t)\|. \tag{3.11}$$

Now, for any $t \in (0, T_0]$, we have

$$\begin{aligned} (A^\beta g, w_n(t)) &= (g, A^{\beta+\alpha} S(t) h_n(0)) \\ &\quad + \int_0^t (g, A^{\beta+\alpha} S(t-s), f_n(s, A^{-\alpha} w_n(s), A^{-\alpha} w_n(a(s))))\, ds. \end{aligned} \tag{3.12}$$

The first term is bounded for $t \in (0, T]$ as

$$|(A^\beta g, S(t) h_n(0))| \le \|g\| M \|A^{\beta+\alpha} h(0)\|. \tag{3.13}$$

The second term is treated as follows:

$$\left|\left|\int_0^t \left(g, A^{\beta+\alpha}S(t-s), f_n(s,u_n(s),u_n(a(s)))\right)ds\right|\right| \leq M_0\|g\|C_{\beta+\alpha}\frac{T_0^{1-(\beta+\alpha)}}{1-(\beta+\alpha)}. \tag{3.14}$$

Hence the corollary is proved. □

Collorary 3.4. *If $h(t) \in D(A)$ for all $t \in [-\tau,0]$, then for any $t \in [-\tau,T_0]$, there exists a constant M_1, independent of n, such that*

$$||A^{\beta}w_n(t)|| \leq M_0 \tag{3.15}$$

for all $-\tau \leq t \leq T_0$ and $0 \leq \beta < 1$.

Corollary 3.4 is a consequence of Corollary 3.3.

Proposition 3.5. *The sequence $\{u_n\} \subset \mathcal{C}_{T_0}$ is a Cauchy sequence and therefore converges to a function $u \in \mathcal{C}_{T_0}$ if the assumptions (H1)–(H4) hold.*

Proof. From Proposition 3.1 we have (3.10). With $u_n = A^{-\alpha}w_n$, (3.10) becomes

$$w_n(t) = \begin{cases} A^{\alpha}h_n(t), & t \in [-r,0], \\ S(t)A^{\alpha}h_n(0) & \\ +\displaystyle\int_0^t A^{\alpha}S(t-s)f_n(s,A^{-\alpha}u_n(s),A^{-\alpha}u_n(a(s)))ds, & t \in [0,T_0]. \end{cases} \tag{3.16}$$

For $n \geq m \geq n_0$, where n_0 is large enough, $n,m,n_0 \in \mathbb{N}$, $t \in [-r,0]$, we have

$$\begin{aligned} ||w_n(t) - w_m(t)|| &\leq ||h_n(t) - h_m(t)||_{\alpha} \\ &\leq ||(P^n - P^m)h(t)||_{\alpha} \longrightarrow 0 \quad \text{as } m \longrightarrow \infty. \end{aligned} \tag{3.17}$$

For $t \in (0,T_0]$ and n, m, and n_0 as above, we have

$$\begin{aligned} ||w_n(t) - w_m(t)|| \leq{}& ||(P^n - P^m)S(t)A^{\alpha}h(0)|| \\ &+ \int_0^t ||A^{\alpha}S(t-s)[f_n(s,A^{-\alpha}w_n(s),A^{-\alpha}w_n(a(s))) \\ &\qquad - f_m(s,A^{-\alpha}w_m(s),A^{-\alpha}w_m(a(s)))]||ds. \end{aligned} \tag{3.18}$$

Now, using Corollaries 3.3 and 3.4, we have

$$\begin{aligned} &||f_n(s,A^{-\alpha}w_n(s),A^{-\alpha}w_n(a(s)) - f_m(s,A^{-\alpha}w_m(s),A^{-\alpha}w_m(a(s))))|| \\ &\quad\leq ||(P^n - P^m)f(s,P^mA^{-\alpha}w_m(s),P^mA^{-\alpha}w_m(a(s)))|| \\ &\qquad + L_R(T_0)[||(P^n - P^m)w_m(s)|| + ||(P^m - P^m)w_m(a(s))||] \\ &\qquad + 2L_R(T_0)||w_n - w_m||_s \\ &\quad\leq C_1 + C_2\frac{1}{\lambda_m^{\beta}} + C_2||w_n - w_m||_s \end{aligned} \tag{3.19}$$

for some positive constants C_1 and C_2, where $C_1 = \|(p^n - P^m)\|[L_R(T_0)(2R + T^\gamma + 4\|\bar{h}\|_{T_0,\alpha}) + \|f(0,h(0),h(a(0)))\|]$ and $C_2 = 2L_R(T_0)$. Thus, we have the following estimate:

$$\begin{aligned} ||w_n(t) - w_m(t)|| \leq C_0||(P^n - P^m)A^\alpha h(0)|| + \frac{C_1 C_\alpha T_0^{1-\alpha}}{(1-\alpha)} + \frac{C_2 C_\alpha T_0^{1-\alpha}}{(1-\alpha)\lambda_m^\beta} \\ + C_2 C_\alpha \int_0^t (t-s)^{-\alpha}||w_n - w_m||_s ds, \end{aligned} \tag{3.20}$$

where $C_0 = Me^{\omega T}$. Since $\|w_n - w_m\| = \|h_n - h_m\|_\alpha$ on $[-r,0]$, we have

$$\begin{aligned} ||w_n - w_m||_t \leq ||h_n - h_m||_\alpha + C_0||(P^n - P^m)A^\alpha h(0)|| + \frac{C_1 C_\alpha T_0^{1-\alpha}}{(1-\alpha)} \\ + \frac{C_2 C_\alpha T_0^{1-\alpha}}{(1-\alpha)\lambda_m^\beta} + C_2 C_\alpha \int_0^t (t-s)^{-\alpha}||w_n - w_m||_s ds. \end{aligned} \tag{3.21}$$

Application of Gronwall's inequality gives the required result. This completes the proof of the proposition. □

With the help of Propositions 3.1 and 3.5, we may state the following existence, uniqueness, and convergence result.

THEOREM 3.6. *Suppose that (H1)–(H4) hold. Then, there exist unique functions* $u_n \in C([-r,T_0];H_n)$ *and* $u \in C([-r,T_0];H)$ *satisfying (3.10) and*

$$u(t) = \begin{cases} h(t), & t \in [-r,0], \\ S(t)h(0) & \\ + \displaystyle\int_0^t S(t-s) f(s,u(s),u(a(s)))ds, & t \in [0,T_0], \end{cases} \tag{3.22}$$

such that $u_n \to u$ *in* $C([-r,T_0];H)$ *as* $n \to \infty$, *where* $h_n(t) = P^n h(t)$ *and* $f_n(t,u,v) = P^n f(t,P^n u,P^n v)$.

4. Regularity

The functions u_n and u in Theorem 3.6 satisfying (3.10) and (3.22) may be called approximate mild solution and mild solution of (1.2) on $[-\tau,T_0]$, respectively. In this section, we establish the regularity of the mild solution u of (1.2) under an additional assumption of Hölder continuity of the function a on $[0,T]$. We note that if a is Lipschitz continuous on $[0,T]$, then it is also Hölder continuous on $[0,T]$. We establish the following regularity result.

THEOREM 4.1. *Suppose that (H1)–(H4) hold and, in addition, suppose that* $a:[0,T] \to [-r,T]$ *is Hölder continuous, that is, there exist constants* $0 < \delta < 1$ *and* $L_a \geq 0$ *such that*

$$|a(t) - a(s)| \leq L_a|t-s|^\delta. \tag{4.1}$$

Then, the mild solution u *given by (3.10) of (1.2) is a unique classical solution of (1.2) on* $[-r,T_0]$.

We prove that u is in fact a unique classical solution. For this, we first prove that the mild solution

$$u(t)=\begin{cases} h(t), & t\in[-r,0],\\ S(t)h(0)+\int_0^t S(t-s)f(s,u(s),u(a(s)))\,ds, & t\in[0,T_0], \end{cases} \tag{4.2}$$

is locally Hölder continuous on $(0,\widetilde{T}]$. Let $v(t)=A^\alpha u(t)$. Then,

$$v(t)=\begin{cases} A^\alpha h(t), & t\in[-r,0],\\ S(t)A^\alpha h(0)+\int_0^t S(t-s)A^\alpha f(s,A^{-\alpha}v(s),A^{-\alpha}v(a(s)))\,ds, & t\in[0,T_0]. \end{cases} \tag{4.3}$$

Let

$$N=\sup_{t\in[0,\widetilde{T}]}\|f(t,A^{-\alpha}v(t),A^{-\alpha}v(a(t)))\|. \tag{4.4}$$

It is known that (cf. [16, page 197]) for every β with $0<\beta<1-\alpha$ and every $0<h<1$, we have

$$\|(S(h)-I)A^\alpha S(t-s)\|\le C_\beta h^\beta\|A^{\alpha+\beta}S(t-s)\|\le Ch^\beta(t-s)^{-\alpha+\beta},\quad 0<s<t. \tag{4.5}$$

For $0<t<t+h\le T_0$, we have

$$\begin{aligned} \|v(t+h)-v(t)\|\le{}& \|(S(h)-I)A^\alpha S(t)\chi(0)\|\\ &+\int_0^t\|(S(h)-I)A^\alpha S(t-s)f(s,A^{-\alpha}v(s),A^{-\alpha}v(a(s)))\|ds\\ &+\int_t^{t+h}\|A^\alpha S(t+h-s)f(s,A^{-\alpha}v(s),A^{-\alpha}v(a(s)))\|ds. \end{aligned} \tag{4.6}$$

Using (4.5), we get

$$\|(S(h)-I)A^\alpha S(t)\chi(0)\|\le Ct^{-(\alpha+\beta)}h^\beta\|\chi(0)\|\le M_1h^\beta, \tag{4.7}$$

where M_1 depends on t and $M_1\to\infty$ as $t\to 0$. Now,

$$\begin{aligned} &\int_0^t\|(S(h)-I)A^\alpha S(t-s)f(s,A^{-\alpha}v(s),A^{-\alpha}v(a(s)))\|ds\\ &\quad\le CNh^\beta\int_0^t(t-s)^{-(\alpha+\beta)}ds\\ &\quad\le M_2h^\beta, \end{aligned} \tag{4.8}$$

where M_2 is independent of t. For the last integral in (4.6), we have

$$\begin{aligned}\int_t^{t+h} &||A^\alpha S(t+h-s)f(s,A^{-\alpha}v(s),A^{-\alpha}v(a(s)))||ds \\ &\le NC_\alpha \int_t^{t+h}(t+h-s)^{-\alpha}ds \\ &\le \frac{NC_\alpha}{1-\alpha}h^{1-\alpha} \\ &\le M_3 h^\beta,\end{aligned} \tag{4.9}$$

where M_3 is also independent of t. The above estimates imply that

$$||v(t)-v(s)|| \le L_v|t-s|^\beta, \quad |t-s|<1,\ 0<s,\ t\le T_0. \tag{4.10}$$

For any $0<s<t\le T_0$, with $t-s\ge 1$, we insert $t_1<t_2<\cdots<t_n$ between s and t such that $1/2\le t_{i+1}-t_i<1$ for $i=1,2,\dots,n-1$ and $t-t_n<1$. Clearly, $n\le 2T_0\le 2T$. Then, for $0<s<t\le T_0$, with $t-s\ge 1$, we have

$$\begin{aligned}||v(t)-v(s)|| &\le ||v(t)-v(t_n)||+\sum_{i=1}^{n-1}||v(t_{i+1})-v(t_i)||+||v(t_1)-v(s)|| \\ &\le L_v\left[(t-t_n)^\beta+\sum_{i=1}^{n-1}(t_{i+1}-t_i)^\beta+(t_1-s)^\beta\right] \\ &\le (2T+1)L_v|t-s|^\beta=\widetilde{L}_v|t-s|^\beta,\end{aligned} \tag{4.11}$$

where $\widetilde{L}_v=(2T+1)L_v$. Now, for $0<s,\ t\le T_0$, we have

$$\begin{aligned}||f(t,&A^{-\alpha}v(t),A^{-\alpha}v(a(t)))-f(s,A^{-\alpha}v(s),A^{-\alpha}v(a(s)))|| \\ &\le L_f(R)[|t-s|^\gamma+||v(t)-v(s)||+||v(a(t))-v(a(s))||] \\ &\le L_f(R)[|t-s|^\gamma+\widetilde{L}_v|t-s|^\beta+\widetilde{L}_v|a(t)-a(s)|^\beta] \\ &\le L_f(R)[|t-s|^\gamma+\widetilde{L}_v|t-s|^\beta+\widetilde{L}_vL_a^\beta|t-s|^{\delta\cdot\beta}] \\ &\le L_f(R)(1+\widetilde{L}_v+\widetilde{L}_vL_a^\beta)|t-s|^{\max\{\gamma,\beta,\delta\cdot\beta\}},\end{aligned} \tag{4.12}$$

which shows that the function $t\mapsto f(t,A^{-\alpha}v(t),A^{-\alpha}v(a(t)))$ is locally Hölder continuous on $(0,T_0]$.

Now, consider the initial value problem

$$\frac{dw(t)}{dt}+Aw(t)=f(t,A^{-\alpha}v(t),A^{-\alpha}v(a(t))), \qquad w(0)=\chi(0). \tag{4.13}$$

By [16, Corollary 4.3.3], (4.13) has a unique solution $w \in C^1((0,T_0];H)$ given by

$$w(t) = S(t)\chi(0) + \int_0^t S(t-s) f\big(s, A^{-\alpha}v(s), A^{-\alpha}v(a(s))\big)\,ds. \tag{4.14}$$

For $t > 0$, each term on the right of (4.14) is in $D(A) \subseteq D(A^\alpha)$, we may apply A^α on w to get

$$A^\alpha w(t) = S(t)A^\alpha\chi(0) + \int_0^t A^\alpha S(t-s) f\big(s, A^{-\alpha}v(s), A^{-\alpha}v(a(s))\big)\,ds. \tag{4.15}$$

The right-hand side of (4.15) is equal to $v(t)$ and therefore $w(t) = u(t)$ on $[0,T_0]$. Thus, $u \in C^1((0,T_0];H)$ and hence u is a classical solution of (1.2). This completes the proof of the theorem.

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