CHOVER-TYPE LAWS OF THE ITERATED LOGARITHM FOR WEIGHTED SUMS OF ρ^*-MIXING SEQUENCES

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To derive a Baum-Katz-type result, we establish a Chover-type law of the iterated logarithm for the weighted sums of ρ^*-mixing and identically distributed random variables with a distribution in the domain of a stable law. Our result obtained not only generalizes the main results of Peng and Qi (2003) and Qi and Cheng (1996) to ρ^*-mixing sequences of random variables, but also improves them.

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1. Introduction

Let $\{X_i,\ i \geq 1\}$ be independent and identically distributed (i.i.d.) with symmetric stable distributions, which belong to the domain of normal attraction and nongeneration. So, their characteristic functions are of the forms:

$$E\exp(itX_i) = \exp(-|t|^{\alpha}), \quad t \in R,\ i \geq 1. \tag{1.1}$$

Chover [4] has obtained that

$$\limsup_{n\to\infty}\left(\frac{|\sum_{i=1}^{n} X_i|}{n^{1/\alpha}}\right)^{1/\log\log n} = e^{1/\alpha} \quad \text{a.s.} \tag{1.2}$$

We call this a Chover-type LIL (laws of the iterated logarithm). This type LIL has been established by Vasudeva and Divanji [13], Zinchenko [14] for delayed sums, by Chen and Huang [3] for geometric weighted sums, and by Chen [2] for weighted sums. Qi and Cheng [11] extended the Chover-type law of the iterated logarithm for the partial sums to the case where the underlying distribution is in the domain of attraction of a nonsymmetric stable distribution (see below for details).

Let L_α denote a stable distribution with exponent $\alpha \in (0,2)$. Recall that the distribution of X is said to be *in the domain of attraction of* L_α if there exist some constants $A_n \in R$

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and $B_n > 0$ such that

$$\frac{S_n - A_n}{B_n} \xrightarrow{d} L_\alpha. \tag{1.3}$$

Under (1.3), Qi and Cheng [11] and Peng and Qi [10] showed that

$$\limsup_{n\to\infty}\left(\frac{\left|\sum_{i=1}^{n} X_i - A_n\right|}{B_n}\right)^{1/\log\log n} = e^{1/\alpha} \quad \text{a.s.} \tag{1.4}$$

It is well known that (1.3) holds if and only if

$$1 - F(x) = \frac{C_1(x)l(x)}{x^\alpha}, \quad F(-x) = \frac{C_2(x)l(x)}{x^\alpha}, \quad \text{for } x > 0, \tag{1.5}$$

where, for $x > 0$, $C_i(x) \geq 0$, $\lim_{x\to\infty} C_i(x) = C_i$, $i = 1,2$, $C_1 + C_2 > 0$, and $l(x) \geq 0$ is slowly varying in the sense of Karamata function, that is,

$$\lim_{t\to\infty}\frac{l(tx)}{l(t)} = 1, \quad \text{for } x > 0. \tag{1.6}$$

By Lin et al. [6, page 76, Exercise 21], we have $B_n = (nl(n))^{1/\alpha}$.

For nonempty sets $S, T \subset \mathcal{N}$, we define $\mathcal{F}_S = \sigma(X_k, k \in S)$. And we define the maximal correlation coefficient $\rho_n^* = \sup \operatorname{corr}(f, g)$ where the supremum is taken over all (S,T) with $\operatorname{dist}(S,T) \geq n$ and for all $f \in L_2(\mathcal{F}_S)$, $g \in L_2(\mathcal{F}_T)$, and $\operatorname{dist}(S,T) = \inf_{x\in S,\, y\in T} |x - y|$.

A sequence of random variables $\{X_n,\ n \geq 1\}$ on a probability space $\{\Omega, \mathcal{F}, P\}$ is called ρ^*-mixing if

$$\lim_{n\to\infty} \rho_n^* = 0. \tag{1.7}$$

As for ρ^*-mixing sequences of random variables, one can refer to Bryc and Smolenski [1], who established bounds for the moments of partial sums for a sequence of random variables satisfying

$$\lim_{n\to\infty} \rho_n^* < 1. \tag{1.8}$$

Peligrad [7] established a CLT. Peligrad [8] established an invariance principle. Peligrad and Gut [9] established Rosenthal-type maximal inequalities and Baum-Katz-type results. Utev and Peligrad [12] established an invariance principle of nonstationary sequences.

To derive a Baum-Katz-type result, the main purpose of this paper is to establish a Chover-type law of the iterated logarithm for the weighted sums of ρ^*-mixing and identically distributed random variables with a distribution in the domain of a stable law. Our result not only generalizes the main results of Peng and Qi [10] and Qi and Cheng [11] to ρ^*-mixing sequences of random variables, but also improves them.

Throughout this paper, let $h \in B[0,1]$ denote that the function h is bounded on [0,1]. C will represent a positive constant though its value may change from one appearance to the next, and $a_n = O(b_n)$ will mean $a_n \leq Cb_n$.

2. The main results

In order to prove our results, we need the following lemma and definition.

LEMMA 2.1 (Utev and Peligrad [12]). *Let* $\{X_i,\ i \geq 1\}$ *be a* ρ^**-mixing sequence of random variables,* $EX_i = 0$, $E|X_i|^p < \infty$ *for some* $p \geq 2$ *and for every* $i \geq 1$*. Then there exists* $C = C(p)$*, such that*

$$E \max_{1\leq k\leq n} \left| \sum_{i=1}^{k} X_i \right|^p \leq C\left\{ \sum_{i=1}^{n} E\left|X_i\right|^p + \left(\sum_{i=1}^{n} EX_i^2 \right)^{p/2} \right\}. \tag{2.1}$$

DEFINITION 2.2 (Lin and Lu [5]). *A function* $f(x) > 0$ $(x > 0)$ *is said to be quasimonotone nondecreasing, if*

$$\limsup_{x\to\infty} \sup_{0\leq t\leq x} \frac{f(t)}{f(x)} < \infty. \tag{2.2}$$

Here are our main results.

THEOREM 2.3. *Let* $\{X, X_i,\ i \geq 1\}$ *be a* ρ^**-mixing sequence of identically distributed random variables. Let* h *be a bounded function on* $[0,1]$*, continuous at* $x_0 \in (0,1)$*. Let* $S_n = \sum_{i=1}^{n} h(i/n)X_i$, $EX = 0$*, when* $\alpha > 1$*. Let* $f(x) > 0$ *be quasimonotone nondecreasing and* $\int_1^\infty (1/xf(x))dx < \infty$*. Then under condition (1.3), for any* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leq j\leq n} \left|S_j\right| > \varepsilon \left(nf(n)l(n)\right)^{1/\alpha} \right) < \infty. \tag{2.3}$$

Proof of Theorem 2.3. For any $i \geq 1$, define $X_i^{(n)} = X_i I(|X_i| \leq a_n)$, $S_j^{(n)} = \sum_{i=1}^{j}(h(i/n)X_i^{(n)} - Eh(i/n)X_i^{(n)})$, where $a_n = (nf(n)l(n))^{1/\alpha}$. Then for any $\varepsilon > 0$,

$$\begin{aligned} &P\left(\max_{1\leq j\leq n} \left|S_j\right| > \varepsilon a_n \right) \\ &\quad\leq P\left(\max_{1\leq j\leq n} \left|X_j\right| > a_n \right) + P\left(\max_{1\leq j\leq n} \left|S_j^{(n)}\right| > \varepsilon a_n - \max_{1\leq j\leq n} \left| \sum_{i=1}^{j} Eh\left(\frac{i}{n}\right) X_i^{(n)} \right| \right). \end{aligned} \tag{2.4}$$

First we show that

$$\frac{1}{a_n} \max_{1\leq j\leq n} \left| \sum_{i=1}^{j} Eh\left(\frac{i}{n}\right) X_i^{(n)} \right| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{2.5}$$

In fact, (i) when $0 < \alpha \le 1$, $h \in B[0,1]$. For any positive integers n, N,

$$\begin{aligned}
&\frac{1}{a_n}\max_{1\le j\le n}\left|\sum_{i=1}^{j} Eh\left(\frac{i}{n}\right)X_i^{(n)}\right| \\
&\qquad \le \frac{1}{a_n}\sum_{i=1}^{n} E\left|h\left(\frac{i}{n}\right)X_i^{(n)}\right| \le \frac{Cn}{a_n}\int_{|x|\le a_n}|x|\,dF(x) \\
&\qquad \le \frac{Cn}{a_n}a_N + \frac{Cn}{a_n}\int_{a_N<|x|\le a_n}|x|\,dF(x) =: C(A+B).
\end{aligned} \tag{2.6}$$

Since $f(x) > 0$ is a quasimonotone nondecreasing and by (1.5), we have, for $n \ge N$, N large enough,

$$\begin{aligned}
B &= \frac{n}{a_n}\sum_{k=N+1}^{n}\int_{a_{k-1}<|x|\le a_k}|x|\,dF(x) \le \frac{n}{a_n}\sum_{k=N+1}^{n} a_k P\left(a_{k-1} < |X| \le a_k\right) \\
&\le C\sum_{k=N+1}^{n} kP\left(a_{k-1} < |X| \le a_k\right) \le CNP\left(|X| \ge a_N\right) + C\sum_{k=N}^{\infty} P\left(|X| \ge a_k\right) \\
&\le C\frac{1}{f(N)} + C\sum_{k=N}^{\infty}\frac{1}{kf(k)} \le C\frac{1}{f(N)} + C\int_N^{\infty}\frac{dx}{kf(k)} < \frac{\varepsilon}{4}.
\end{aligned} \tag{2.7}$$

It is obvious that for each given N,

$$A \le C\frac{a_N}{\left(f(n)\right)^{1/\alpha}} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{2.8}$$

So, for $0 < \alpha \le 1$, we have (2.5).

(ii) When $1 < \alpha < 2$, using $EX_i = 0$, $h \in B[0,1]$, and (1.5), when $n \to \infty$, we have

$$\begin{aligned}
&\frac{1}{a_n}\max_{1\le j\le n}\left|\sum_{i=1}^{j} Eh\left(\frac{i}{n}\right)X_i^{(n)}\right| \\
&\quad = \frac{1}{a_n}\max_{1\le j\le n}\left|\sum_{i=1}^{j} Eh\left(\frac{i}{n}\right)X_i I\left(|X_i| > a_n\right)\right| \le \frac{1}{a_n}\sum_{i=1}^{n} E\left|h\left(\frac{i}{n}\right)X_i\right| I\left(|X_i| > a_n\right) \\
&\quad \le \frac{Cn}{a_n}E|X|I\left(|X| > a_n\right) = \frac{Cn}{a_n}\int_{a_n}^{\infty} P\left(|X| \ge x\right)dx = \frac{Cn}{a_n}\int_{a_n}^{\infty}\frac{Cl(n)}{x^{\alpha}}dx \\
&\quad = \frac{n}{a_n}Ca_n^{1-\alpha} = \frac{C}{f(n)} < \frac{\varepsilon}{2}.
\end{aligned} \tag{2.9}$$

So, for $1 < \alpha < 2$, we also have (2.5). Hence (2.5) holds for $0 < \alpha < 2$.

By (2.4) and (2.5), we have that

$$P\left(\max_{1\le j\le n}|S_j| > \varepsilon a_n\right) \le \sum_{j=1}^{n} P\left(|X_j| > a_n\right) + P\left(\max_{1\le j\le n}|S_j^{(n)}| > \frac{\varepsilon}{2}a_n\right), \tag{2.10}$$

for n large enough. Hence we need only to prove

$$I =: \sum_{n=1}^{\infty} n^{-1} \sum_{j=1}^{n} P(|X_j| > a_n) < \infty,$$
$$II =: \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le j\le n} |S_j^{(n)}| > \frac{\varepsilon}{2} a_n\right) < \infty. \tag{2.11}$$

From (1.5), it is easily seen that

$$I = \sum_{n=1}^{\infty} P(|X| > a_n) \le \sum_{n=1}^{\infty} \frac{C}{nf(n)} \le C\int_1^{\infty} \frac{dx}{xf(x)} < \infty. \tag{2.12}$$

By Lemma 2.1 and the fact that $h \in B[0,1]$, it follows that

$$\begin{aligned}
II &\le C\sum_{n=1}^{\infty} n^{-1} E\max_{1\le j\le n} |S_j^{(n)}|^2 \frac{1}{a_n^2} \le C\sum_{n=1}^{\infty} n^{-1} \frac{1}{a_n^2} \left(\sum_{i=1}^{n} E\left|h\left(\frac{i}{n}\right)X_i^{(n)}\right|^2\right) \\
&\le C\sum_{n=1}^{\infty} \frac{1}{a_n^2} E|X|^2 I(|X| \le a_n) = C\sum_{n=1}^{\infty} \frac{1}{a_n^2} \int_{|x|\le a_n} x^2 dF(x) \\
&= C\sum_{n=1}^{\infty} \frac{1}{a_n^2} \sum_{k=1}^{n} \int_{a_{k-1}<|x|\le a_k} x^2 dF(x) \le C\sum_{k=1}^{\infty} a_k^2 P(a_{k-1} < |X| \le a_k) \sum_{n=k}^{\infty} \frac{1}{a_n^2} \\
&\le C\sum_{k=1}^{\infty} kP(a_{k-1} < |X| \le a_k) \le C\int_1^{\infty} \frac{dx}{xf(x)} < \infty,
\end{aligned} \tag{2.13}$$

which completes the proof of Theorem 2.3. □

Corollary 2.4. *Under the conditions of Theorem 2.3,*

$$\limsup_{n\to\infty} \left(\frac{|S_n|}{B_n}\right)^{1/\log\log n} \le e^{1/\alpha} \quad a.s. \tag{2.14}$$

Proof of Corollary 2.4. Notice that for any positive integer n, there exists an nonnegative integer k, such that $2^k \le n < 2^{k+1}$. And there exists a $t \in [0,1)$, such that $n = 2^{k+t}$. By (2.3), we have

$$\sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} (2^{k+1}-1)^{-1} P\left(\max_{1\le j\le 2^{k+t}} |S_j| > \varepsilon\left(2^{k+1} f(2^{k+t}) l(2^{k+t})\right)^{1/\alpha}\right) < \infty. \tag{2.15}$$

Then

$$\sum_{k=0}^{\infty} P\left(\max_{1\le j\le 2^{k+t}} |S_j| > \varepsilon\left(2^{k+1} f(2^{k+t}) l(2^{k+t})\right)^{1/\alpha}\right) < \infty. \tag{2.16}$$

Then

$$\frac{\max_{1\le j\le 2^{k+t}}|S_j|}{\left(2^{k+1}f(2^{k+t})l(2^{k+t})\right)^{1/\alpha}}\longrightarrow 0 \quad \text{a.s.} \tag{2.17}$$

So

$$\begin{aligned}\frac{|S_n|}{\left(nf(n)l(n)\right)^{1/\alpha}}&\le\frac{\max_{1\le j\le 2^{k+t}}|S_j|}{\left(2^{k+1}f(2^{k+t})l(2^{k+t})\right)^{1/\alpha}}\frac{\left(2^{k+1}f(2^{k+t})l(2^{k+t})\right)^{1/\alpha}}{\left(nf(n)\right)^{1/\alpha}}\\&\le 2^{1/\alpha}\frac{\max_{1\le j\le 2^{k+t}}|S_j|}{\left(2^{k+1}f(2^{k+t})\right)^{1/\alpha}}\longrightarrow 0 \quad \text{a.s.}\end{aligned} \tag{2.18}$$

Then

$$\limsup_{n\to\infty}\frac{|S_n|}{\left(nf(n)l(n)\right)^{1/\alpha}}=0 \quad \text{a.s.} \tag{2.19}$$

Given $\varepsilon>0$, let $f(x)=\log^{1+\varepsilon}x$. It is obvious that $\int_1^\infty(1/xf(x))dx<\infty$. By (2.19), we have

$$\limsup_{n\to\infty}\frac{|S_n|}{\left(nl(n)\log^{1+\varepsilon}n\right)^{1/\alpha}}=0 \quad \text{a.s.} \tag{2.20}$$

Then

$$\limsup_{n\to\infty}\left(\frac{|S_n|}{B(n)}\right)^{1/\log\log n}\le e^{(1+\varepsilon)/\alpha} \quad \text{a.s.} \tag{2.21}$$

Therefore

$$\limsup_{n\to\infty}\left(\frac{|S_n|}{B(n)}\right)^{1/\log\log n}\le e^{1/\alpha} \quad \text{a.s.,} \tag{2.22}$$

which completes the proof of (2.14). □

Remark 2.5. Corollary 2.4 generalizes the estimate

$$\limsup_{n\to\infty}\left(\frac{|S_n|}{B_n}\right)^{1/\log\log n}\le e^{1/\alpha} \quad \text{a.s.} \tag{2.23}$$

obtained in Peng and Qi [10, Theorem 2.1] to ρ^*-mixing sequences of random variables.

Corollary 2.6. *Under the conditions of Corollary 2.4, letting $h(x)\equiv 1$, yields*

$$\limsup_{n\to\infty}\left(\frac{|\sum_{i=1}^n X_i|}{B_n}\right)^{1/\log\log n}\le e^{1/\alpha} \quad \textit{a.s.} \tag{2.24}$$

Remark 2.7. Corollary 2.6 generalizes in Qi and Cheng [11, Theorem 1.1] to ρ^*-mixing sequences of random variables.

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