MEAN CONVERGENCE THEOREM FOR MULTIDIMENSIONAL ARRAYS OF RANDOM ELEMENTS IN BANACH SPACES

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For a d-dimensional array of random elements $\{V_n,\ n \in \mathbb{Z}_+^d\}$ in a real separable stable type p $(1 \leq p < 2)$ Banach space, a mean convergence theorem is established. Moreover, the conditions for the convergence in mean of order p are shown to completely characterize stable-type p Banach spaces.

1. Introduction

Let $\mathbb{Z}_+^d$, where d is a positive integer, denote the positive integer d-dimensional lattice points. The notation $m \prec n$, where $m = (m_1, m_2, \dots, m_d)$ and $n = (n_1, n_2, \dots, n_d) \in \mathbb{Z}_+^d$, means that $m_i \leqslant n_i$, $1 \leqslant i \leqslant d$, $|n|$ is used for $\prod_{i=1}^d n_i$.

Gut [5] proved that if $\{X, X_n, n \in \mathbb{Z}_+^d\}$ is a d-dimensional array of i.i.d. random variables with $E|X|^p < \infty$ $(0 < p < 2)$ and $EX = 0$ if $1 \leqslant p < 2$, then

$$\frac{\sum_{j \prec n} X_j}{|n|^{1/p}} \longrightarrow 0 \quad \text{in } L^p \text{ as } \min_{1 \leqslant i \leqslant d} n_i \longrightarrow \infty, \tag{1.1}$$

where $(n_1, n_2, \dots, n_d) = n \in \mathbb{Z}_+^d$.

Recently, Thanh [11] proved (1.1) under condition of uniform integrability of $\{|X_n|^p, n \in \mathbb{Z}_+^d\}$.

Mean convergence theorems for sums of random elements Banach-valued are studied by many authors. The reader may refer to Wei and Taylor [12], Adler et al. [2], Rosalsky and Sreehari [9], or more recently, Rosalsky et al. [10], Cabrera and Volodin [3]. However, we are unaware of any literature of investigation on the mean convergence for multidimensional arrays of random elements in Banach spaces.

Consider a d-dimensional array $\{V_n,\ n \in \mathbb{Z}_+^d\}$ of independent random elements defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in a real separable Banach space

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$\mathscr{X}$ with norm $\|\cdot\|$. In the current work, we establish the convergence in mean of order p $(1 \leqslant p < 2)$ of the sums $\sum_{j \prec n} V_j / |n|^{1/p}$, $n \in \mathbb{Z}_+^d$, under the condition that $\{\|V_n\|^p,\ n \in \mathbb{Z}_+^d\}$ is uniformly integrable. The main results of this paper are Theorems 2.1 and 2.2. Theorem 2.1 is a stable-type p Banach space version of the main result of Thanh [11]. While the proof of Theorem 2.1 and the proof of the main result in Thanh [11] are similar, we will show in Theorem 2.2 that the implication in Theorem 2.1 indeed completely characterizes stable-type p Banach spaces.

Let $0 < p \leqslant 2$ and let $\{\theta_n,\ n \geqslant 1\}$ be independent and identically distributed stable random variables each with characteristic function $\phi(t) = \exp\{-|t^p|\}$. The real separable Banach space $\mathscr{X}$ is said to be of *stable-type* p if $\sum_{n=1}^{\infty} \theta_n v_n$ converges a.s. whenever $v_n \in \mathscr{X}$, $n \geqslant 1$ with $\sum_{n=1}^{\infty} \|v_n\|^p < \infty$. Equivalent characterizations of a Banach space being of stable-type p, properties of stable-type p Banach spaces, as well as various relationships between the conditions Rademacher-type p, and stable-type p may be found by Woyczyński in [13], by Marcus and Woyczyński in [7], and by Pisier in [8], see also the discussion by Adler et al. in [1]. We now mention explicitly some characterizations of this concept. The first theorem was obtained by Mandrekar and Zinn [6] and by Marcus and Woyczyński [7].

Theorem 1.1. *Let $1 \leqslant p < 2$ and let $\mathscr{X}$ be a real separable Banach space. Then the following statements are equivalent.*

(i) *$\mathscr{X}$ is of stable-type p.*

(ii) *For every symmetric random elements V, the condition $n^p P(\|V\| > n) \to 0$ as $n \to \infty$ implies that*

$$\frac{\sum_{j=1}^{n} V_j}{n^{1/p}} \longrightarrow 0 \quad \textit{in probability}, \tag{1.2}$$

where $\{V_j,\ j \geqslant 1\}$ are independent copies of V.

Theorem 1.2 (see [13, Theorem V.9.3]). *Let $1 \leqslant p < 2$ and let $\mathscr{X}$ be a real separable Banach space. Then the following statements are equivalent.*

(i) *$\mathscr{X}$ is of stable-type p.*

(ii) *For each bounded sequence $\{x_n,\ n \geqslant 1\}$ of elements of $\mathscr{X}$,*

$$\frac{\sum_{j=1}^{n} x_k \epsilon_k}{n^{1/p}} \longrightarrow 0 \quad \textit{a.s.}, \tag{1.3}$$

where $\{\epsilon_n,\ n \geqslant 1\}$ is a Rademacher sequence.

The symbol C denotes throughout a generic constant $(0 < C < \infty)$ which is not necessarily the same one in each appearance.

2. Main results

We can now present the main results. Theorem 2.1 is a stable-type p Banach space version of the main result of Thanh [11].

THEOREM 2.1. *Let $\{V_n,\ n \in \mathbb{Z}_+^d\}$ be a d-dimensional array of independent mean-zero random elements in a real separable stable-type p $(1 \leqslant p < 2)$ Banach space $\mathscr{X}$. If*

$$\{\|V_n\|^p,\ n \in \mathbb{Z}_+^d\} \text{ is uniformly integrable,} \tag{2.1}$$

then

$$\frac{\sum_{j \prec n} V_j}{|n|^{1/p}} \longrightarrow 0 \quad \text{in } L^p \text{ as } |n| \longrightarrow \infty. \tag{2.2}$$

Proof. For arbitrary $\epsilon > 0$, there exists $M > 0$ such that

$$E(\|V_n\|^p I(\|V_n\| > M)) < \epsilon, \quad \forall n \in \mathbb{Z}_+^d. \tag{2.3}$$

Set

$$V_n' = V_n I(\|V_n\| \leqslant M), \quad n \in \mathbb{Z}_+^d, \qquad V_n'' = V_n I(\|V_n\| > M), \quad n \in \mathbb{Z}_+^d. \tag{2.4}$$

Since $\mathscr{X}$ is of stable-type p and $p < 2$, it is of Rademacher-type q for some $p < q < 2$. Thus

$$\begin{aligned}
E\left\|\sum_{j \prec n} V_j\right\|^p &\leqslant 2^{p-1}\left[E\left\|\sum_{j \prec n}(V_j' - EV_j')\right\|^p + E\left\|\sum_{j \prec n}(V_j'' - EV_j'')\right\|^p\right] \\
&\leqslant 2^{p-1}E\left\|\sum_{j \prec n}(V_j' - EV_j')\right\|^p + C\sum_{j \prec n} E\|V_j'' - EV_j''\|^p \\
&\leqslant 2^{p-1}\left(E\left\|\sum_{j \prec n}(V_j' - EV_j')\right\|^q\right)^{p/q} + C\sum_{j \prec n} E\|V_j'' - EV_j''\|^p \\
&\qquad \text{(by the Jensen inequality)} \\
&\leqslant C\left(\sum_{j \prec n} E\|V_j' - EV_j'\|^q\right)^{p/q} + C\sum_{j \prec n} E\|V_j''\|^p \\
&\leqslant C(|n|M^q)^{p/q} + C|n|\epsilon \\
&= o(|n|), \quad \text{as } |n| \longrightarrow \infty.
\end{aligned} \tag{2.5}$$

□

While the proof of Theorem 2.1 and the proof of the main result in Thanh [11] are similar, we now show in Theorem 2.2 that the implication ((2.1)⇒(2.2)) in Theorem 2.1 indeed completely characterizes stable-type p Banach spaces.

Theorem 2.2. *Let $1 \leqslant p < 2$ and let $\mathscr{X}$ be a real separable Banach space. Then the following statements are equivalent.*

(i) *$\mathscr{X}$ is of stable-type p.*

(ii) *For every d-dimensional array $\{V_n,\ n \in \mathbb{Z}_+^d\}$ of independent mean-zero random elements in $\mathscr{X}$, the condition (2.1) implies (2.2).*

(iii) *For every d-dimensional array $\{V, V_n,\ n \in \mathbb{Z}_+^d\}$ of independent mean-zero random elements in $\mathscr{X}$, the conditions*

$$E\|V\|^p < \infty, \qquad \sup_{n\in\mathbb{Z}_+^d} P\{\|V_n\| > t\} \leqslant CP\{\|V\| > t\}, \quad \forall t > 0, \tag{2.6}$$

imply (2.2).

Proof. The implication ((i)⇒(ii)) is precisely Theorem 2.1, whereas the implication ((ii)⇒(iii)) is immediate. It remains to verify the implication ((iii)⇒(i)). For reasons of clarity, we collect some of the steps in the following lemmas. The first lemma is a slight modification of de Acosta [4, Theorem 3.1] which holds for sequences of independent identically distributed random elements. The proof of the following modification can be obtained from de Acosta [4, Theorem 3.1] line by line, and so will be omitted. □

Lemma 2.3. *Let $\mathscr{X}$ be a real separable Banach space, $1 \leqslant p < 2$. Let $\{V, W_k,\ k \geq 1\}$ be sequence of independent random elements such that $E\|V\|^p < \infty$ and $\sup_{k\geqslant 1} P\{\|W_k\| > t\} \leqslant CP\{\|V\| > t\}$ for all $t > 0$. Then*

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n W_k}{n^{1/p}} = 0 \quad \textit{in probability} \tag{2.7}$$

if and only if

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n W_k}{n^{1/p}} = 0 \quad \textit{a.s.} \tag{2.8}$$

Lemma 2.4. *Let $1 \leqslant p < 2$ and let $\mathscr{X}$ be a real separable Banach space. Suppose that for every sequence $\{V, W_k,\ k \geqslant 1\}$ of independent mean-zero random elements in $\mathscr{X}$, the conditions*

$$E\|V\|^p < \infty, \qquad \sup_{k\geqslant 1} P\{\|W_k\| > t\} \leqslant CP\{\|V\| > t\}, \quad \forall t > 0, \tag{2.9}$$

imply that

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n W_k}{n^{1/p}} = 0 \quad \textit{in probability.} \tag{2.10}$$

Then $\mathscr{X}$ is of stable-type p.

Proof of Lemma 2.4. Let $\{\varepsilon_k,\ k \geqslant 1\}$ be a Rademacher sequence and let $\{x_k,\ k \geqslant 1\}$ be a sequence of elements in $\mathscr{X}$ such that

$$\sup_{k\geqslant 1} \|x_k\| < \infty. \tag{2.11}$$

Then by the hypothesis of the lemma,

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \varepsilon_k x_k}{n^{1/p}} = 0 \quad \text{in probability.} \tag{2.12}$$

By Lemma 2.3,

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \varepsilon_k x_k}{n^{1/p}} = 0 \quad \text{a.s.} \tag{2.13}$$

Hence, by Theorem 1.2, $\mathscr{X}$ is of stable-type p. The proof of Lemma 2.4 is completed. □

We now prove the implication ((iii)⇒(i)). If $d = 1$, then the conclusion follows directly from Lemma 2.4. So, we can assume that $d \geqslant 2$. Let $\{V, W_k, k \geqslant 1\}$ be a sequence of independent mean-zero random elements in $\mathscr{X}$ such that $E\|V\|^p < \infty$ and $\sup_{k\geqslant 1} P\{\|W_k\| > t\} \leqslant CP\{\|V\| > t\}$ for all $t > 0$. For $n = (n_1,\ldots,n_d) \in \mathbb{Z}_+^d$, set

$$\begin{aligned} V_{(n_1,\ldots,n_d)} &= W_{n_1}, \quad \text{if } n_2 = \cdots = n_d = 1, \\ V_{(n_1,\ldots,n_d)} &= 0, \quad \text{if } \max\{n_2,\ldots,n_d\} \geqslant 2. \end{aligned} \tag{2.14}$$

Then $\{V_n, n \in \mathbb{Z}_+^d\}$ is an array of independent mean-zero random elements, and

$$\sup_{n\in\mathbb{Z}_+^d} P\{\|V_n\| > t\} \leqslant CP\{\|V\| > t\}, \quad \forall t > 0. \tag{2.15}$$

By (iii),

$$\frac{1}{|n|^{1/p}} \sum_{j\prec n} V_j \longrightarrow 0 \quad \text{in } L^p \text{ as } |n| \longrightarrow \infty. \tag{2.16}$$

This implies by taking $n_2 = \cdots = n_d = 1$ and letting $n_1 \to \infty$ that

$$\frac{1}{n_1^{1/p}} \sum_{k=1}^{n_1} W_k \longrightarrow 0 \quad \text{in } L^p\text{, so in probability as } n_1 \longrightarrow \infty. \tag{2.17}$$

By Lemma 2.4, $\mathscr{X}$ is of stable-type p.

Remark 2.5. In Theorem 2.1, if $0 < p < 1$, then the independence hypothesis and the hypothesis that the $\{V_n, n \in \mathbb{Z}_+^d\}$ have mean-zero are not needed for the theorem to hold.

Indeed, for arbitrary $\epsilon > 0$, define V_n' and V_n'', $n \in \mathbb{Z}_+^d$ as in the proof of Theorem 2.1. If $0 < p < 1$, then

$$\begin{aligned} E\left\|\sum_{j\prec n} V_j\right\|^p &\leqslant E\left\|\sum_{j\prec n} V_j'\right\|^p + E\left\|\sum_{j\prec n} V_j''\right\|^p \\ &\leqslant E\left\|\sum_{j\prec n} V_j'\right\|^p + \sum_{j\prec n} E\|V_j''\|^p \\ &\leqslant (|n|M)^p + |n|\epsilon \\ &= o(|n|), \quad \text{as } |n| \longrightarrow \infty. \end{aligned} \tag{2.18}$$

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