Hindawi Publishing Corporation
Journal of Applied Mathematics and Stochastic Analysis
Volume 2007, Article ID 37848, 23 pages
doi:10.1155/2007/37848

Research Article

Comparison of Inventory Systems with Service, Positive Lead-Time, Loss, and Retrial of Customers

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Received 22 November 2006; Accepted 21 June 2007

We analyze and compare three (s,S) inventory systems with positive service time and retrial of customers. In all of these systems, arrivals of customers form a Poisson process and service times are exponentially distributed. When the inventory level depletes to s due to services, an order of replenishment is placed. The lead-time follows an exponential distribution. In model I, an arriving customer, finding the inventory dry or server busy, proceeds to an orbit with probability γ and is lost forever with probability $(1-\gamma)$. A retrial customer in the orbit, finding the inventory dry or server busy, returns to the orbit with probability δ and is lost forever with probability $(1-\delta)$. In addition to the description in model I, we provide a buffer of varying (finite) capacity equal to the current inventory level for model II and another having capacity equal to the maximum inventory level S for model III. In models II and III, an arriving customer, finding the buffer full, proceeds to an orbit with probability γ and is lost forever with probability $(1-\gamma)$. A retrial customer in the orbit, finding the buffer full, returns to the orbit with probability δ and is lost forever with probability $(1-\delta)$. In all these models, the interretrial times are exponentially distributed with linear rate. Using matrix-analytic method, we study these inventory models. Some measures of the system performance in the steady state are derived. A suitable cost function is defined for all three cases and analyzed using graphical illustrations.

1. Introduction

In all works reported in inventory prior to 1993, it was assumed that the time required to serve the items to the customer is negligible. Berman et al. [1] were the first to attempt to introduce positive service time in inventory, where it was assumed that service time is a constant. Later, Berman and Kim [2] extended this result to random service time. Berman and Sapna [3] studied inventory control at a service facility, where exactly one item from the inventory was used for each service provided. Using Markov renewal theory, they analyzed a finite state space process. Arivarignan et al. [4] considered a perishable inventory system with service facility with the arrival of customers forming a Poison process. Each customer requires single item, which is delivered through a service of random duration having exponential distribution.

Queuing systems, in which customers who find all servers busy and waiting positions occupied may retry for service after a period of time, are called retrial queues or queues with repeated attempts. This has been extensively investigated (see Yang and Templeton [5], Falin [6], and Falin and Templeton [7]). Retrials of failed components for service were introduced into the reliability of k-out-of-n system by Krishnamoorthy and Ushakumari [8]. Artalejo et al. [9] were the first to study inventory policies with positive lead-time and retrial of customers who could not get service during their earlier attempts to access the service station; it may be noted that recently Ushakumari [10] obtained analytical solution to this model.

So far, very little investigation is done in retrial inventory with service time. Krishnamoorthy and Islam [11] analyzed a production inventory with retrial of customers. In that paper, they considered an (s, S) inventory system where arrivals of customers form a Poisson process and demands arising from the orbital customers are exponentially distributed with linear rate. Recently, Krishnamoorthy and Jose [12] studied an inventory system with service and retrial of customers. They calculated the expected number of departures after receiving service, the expected number of customers lost without getting service, and the expected total cost of the system using matrix-analytic method. For a detailed discussion of the matrix-analytic method, see [13].

In this paper, we compare three (s,S) inventory systems with service and retrial of customers. Arrival of customers forms a Poisson process rate λ and service times are exponentially distributed with parameter μ. When the inventory level depletes to s due to service, an order of replenishment is placed. The lead-time follows an exponential distribution with rate β. In model I, an arriving customer who finds the inventory level zero or server busy proceeds to an orbit with probability γ and is lost forever with probability $(1-\gamma)$. A retrial customer in the orbit who finds the inventory level zero or server busy returns to the orbit with probability δ and is lost forever with probability $(1-\delta)$. In models II and III, an arriving customer who finds buffer full proceeds to an orbit with probability γ and is lost forever with probability $(1-\gamma)$. A retrial customer from the orbit who finds buffer full returns to the orbit with probability δ and is lost forever with probability $(1-\delta)$. In all these cases, the interretrial times follow an exponential distribution with linear rate $i\theta$ when there are i customers in the system.

This paper is organized as follows. Sections 2, 3, and 4 provide the analysis of the models I, II, and III, respectively. Section 5 presents the cost analysis and numerical results.

Assumption 1.1. (i) Interarrival times of demands are exponentially distributed with parameter λ.

(ii) Service time follows exponential distribution with rate μ.

(iii) Lead-time follows exponential distribution with rate β.

(iv) Interretrial times are exponential with linear rate $i\theta$ when there are i customers in the orbit.

Notation 1.2.

$I(t)$: inventory level at time t.
$N(t)$: number of customers in the orbit at time t.
$M(t)$: number of customers in the buffer at time t.

$$C(t) = \begin{cases} 0, & \text{if the server is idle,} \\ 1, & \text{if the server is busy.} \end{cases}$$

e:$(1,1,\ldots,1)'$ column vector of 1's of appropriate order.

For easy representation of the infinitesimal generator of the process, we use the following notations:

$$\begin{aligned}
\circ &= -(\lambda + i\theta),\\
* &= -(\lambda + \beta + i\theta),\\
\square &= -(\lambda + \mu + i\theta),\\
\Theta &= -(\lambda + \beta + \mu + i\theta),\\
\Delta &= -(\lambda\gamma + \beta + i\theta(1-\delta)),\\
\nabla &= -(\lambda\gamma + \mu + i\theta(1-\delta)),\\
\Omega &= -(\lambda\gamma + \beta + \mu + i\theta(1-\delta)).
\end{aligned} \tag{1.1}$$

2. Analysis of model I

We consider an (s,S) inventory system with retrial of customers. Arrival of customers forms a Poisson process with rate λ. When the inventory level depletes to s due to demands, an order of replenishment is placed. The lead-time is exponentially distributed with rate β. An arriving customer who finds the inventory level zero proceeds to an orbit with probability γ and is lost forever with probability $(1-\gamma)$. A retrial customer who finds the inventory level zero returns to the orbit with probability δ and is lost forever with probability $(1-\delta)$. The interretrial times follow an exponential distribution with linear rate $i\theta$ when there are i customers in the orbit.

Let $I(t)$ be the inventory level and let $N(t)$ be the number of customers in the orbit at time t. Let $C(t)$ be the sever status which is equal to 0 if the server is idle and 1 if the sever is busy. Now, $\{X(t), t \geq 0\}$, where $X(t) = (N(t), C(t), I(t))$, is a level-dependent quasi-birth-death (LDQBD) process on the state space $\{(i,0,j),\ i \geq 0,\ 0 \leq j \leq S\} \cup \{(i,1,j),\ i \geq 0, 1 \leq j \leq S\}$. The infinitesimal generator Q of the process is a block tridiagonal matrix and it has the following form:

$$Q=\begin{pmatrix} A_{1,0} & A_0 & 0 & 0 & 0 & \cdots \\ A_{2,1} & A_{1,1} & A_0 & 0 & 0 & \cdots \\ 0 & A_{2,2} & A_{1,2} & A_0 & 0 & \cdots \\ 0 & 0 & A_{2,3} & A_{1,3} & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2.1}$$

where the blocks $A_0, A_{1,i}$ $(i \geq 0)$, and $A_{2,i} (i \geq 1)$ are square matrices, each of order $(2S+1)$; they are given by

$$A_0=\begin{pmatrix} B_0 & 0 \\ 0 & \lambda\gamma I_S \end{pmatrix}, \quad A_{1,i}=\begin{pmatrix} E_0 & E_1 \\ E_2 & E_3 \end{pmatrix}, \quad A_{2,i}=\begin{pmatrix} C_0 & C_1 \\ 0 & i\theta(1-\delta)I_S \end{pmatrix},$$

$$B_0=\begin{pmatrix} \lambda\gamma & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{(S+1)\times(S+1)},$$

$$E_0=\begin{matrix} 0 \\ 1 \\ \vdots \\ s \\ (s+1) \\ \vdots \\ (S-s) \\ (S-s+1) \\ \vdots \\ S \end{matrix}\begin{pmatrix} \Delta & & & & & & \beta & & & \\ & * & & & & & & \beta & & \\ & & \ddots & & & & & & \ddots & \\ & & & * & & & & & & \beta \\ & & & & \circ & & & & & \\ & & & & & \ddots & & & & \\ & & & & & & \circ & & & \\ & & & & & & & \circ & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & \circ \end{pmatrix}_{(S+1)\times(S+1)},$$

$$E_1=\begin{pmatrix} 0 & 0 & \cdots & 0 \\ \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{pmatrix}_{(S+1)\times S}, \quad E_2=\begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & \mu & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu & 0 \end{pmatrix}_{S\times(S+1)},$$

$$E_3 = \begin{array}{c} 1 \\ 2 \\ \vdots \\ s \\ (s+1) \\ \vdots \\ (S-s) \\ (S-s+1) \\ \vdots \\ S \end{array} \begin{pmatrix} \Omega & & & & & & \beta & & & \\ & \Omega & & & & & & \beta & & \\ & & \ddots & & & & & & \ddots & \\ & & & \Omega & & & & & & \beta \\ & & & & \nabla & & & & & \\ & & & & & \ddots & & & & \\ & & & & & & \nabla & & & \\ & & & & & & & \nabla & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & \nabla \end{pmatrix}_{S\times S},$$

$$C_0 = \begin{pmatrix} i\theta(1-\delta) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{(S+1)\times(S+1)}, \quad C_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ i\theta & 0 & \cdots & 0 & 0 \\ 0 & i\theta & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & i\theta & 0 \\ 0 & 0 & \cdots & 0 & i\theta \end{pmatrix}_{(S+1)\times S}. \tag{2.2}$$

2.1. System stability. For the model under consideration, we define the following Lyapunov test function (see Falin [6]):

$$\phi(s) = i, \quad \text{if } s \text{ is a state in the level } i. \tag{2.3}$$

The mean drift y_s for any s belonging to the level $i \geq 1$ is given by

$$\begin{aligned} y_s &= \sum_{p\neq s} q_{sp}\left(\phi(p) - \phi(s)\right) \\ &= \sum_u q_{su}\left(\phi(u) - \phi(s)\right) + \sum_v q_{sv}\left(\phi(v) - \phi(s)\right) + \sum_w q_{sw}\left(\phi(w) - \phi(s)\right), \end{aligned} \tag{2.4}$$

where u, v, w vary over the states belonging to the levels $(i-1), i$, and $(i+1)$, respectively. Then, by the definition of ϕ, $\phi(u) = i-1$, $\phi(v) = i$ and $\phi(w) = i+1$ so that

$$\begin{aligned} y_s &= -\sum_u q_{su} + \sum_w q_{sw} \\ &= \begin{cases} -i\theta, & \text{if the server is idle,} \\ -i\theta(1-\delta) + \lambda\gamma, & \text{otherwise.} \end{cases} \end{aligned} \tag{2.5}$$

Since $(1-\delta) > 0$, for any $\varepsilon > 0$, we can find N' large enough so that $y_s < -\varepsilon$ for any s belonging to the level $i \geq N'$. Hence, by Tweedie's [14] result, the system under consideration is stable.

2.2. Choice of N. To find the truncation level N, one can use Neuts–Rao method (see [15]). As outlined in Neuts [16], Elsner's algorithm is used to determine the spectral radius $\eta(N)$ of $R(N)$. To minimize the effect of the approximation on the probabilities, N must be chosen such that $|\eta(N)-\eta(N+1)|<\varepsilon$, where ε is an arbitrarily small value.

2.3. System performance measures. We partition the $(i+1)$th component of the steady state probability vector $x=(x_0,x_1,x_2,\dots,x_{N-1},x_N,\dots)$ as

$$x_i=\left(y_{i,0,0},y_{i,0,1},\dots,y_{i,0,S},y_{i,1,1},y_{i,1,2},\dots,y_{i,1,S}\right). \tag{2.6}$$

Then,

(i) the expected inventory level, EI, in the system is given by

$$\mathrm{EI}=\sum_{i=0}^{\infty}\sum_{j=0}^{S}j\,y_{i,0,j}+\sum_{i=0}^{\infty}\sum_{j=1}^{S}j\,y_{i,1,j}; \tag{2.7}$$

(ii) the expected number of customers, EO, in the orbit is given by

$$\mathrm{EO}=\left(\sum_{i=1}^{\infty}ix_i\right)e=\left(\left(\sum_{i=1}^{N-1}ix_i\right)+x_{N-1}R\left(N(I-R)^{-1}+R(I-R)^{-2}\right)\right)e; \tag{2.8}$$

(iii) the expected reorder rate, EROR, is given by

$$\mathrm{EROR}=\mu\sum_{i=0}^{\infty}y_{i,1,s+1}; \tag{2.9}$$

(iv) the expected number of departures, EDS, after completing service is given by

$$\mathrm{EDS}=\mu\sum_{i=0}^{\infty}\sum_{j=1}^{S}y_{i,1,j}; \tag{2.10}$$

(v) the expected number of customers lost, EL_1, before entering the orbit per unit time is given by

$$\mathrm{EL}_1=(1-\gamma)\lambda\sum_{i=0}^{\infty}\left(y_{i,0,0}+\sum_{j=1}^{S}y_{i,1,j}\right); \tag{2.11}$$

(vi) the expected number of customers lost, EL_2, after retrials per unit time is given by

$$\mathrm{EL}_2=\theta(1-\delta)\sum_{i=1}^{\infty}i\left(y_{i,0,0}+\sum_{j=1}^{S}y_{i,1,j}\right); \tag{2.12}$$

(vii) the overall retrial rate, ORR, is given by

$$\text{ORR} = \theta\left(\sum_{i=1}^{\infty} i x_i\right)e; \tag{2.13}$$

(viii) the successful retrial rate, SRR, is given by

$$\text{SRR} = \theta\sum_{i=0}^{\infty} i\left(\sum_{j=1}^{S} y_{i,0,j}\right). \tag{2.14}$$

3. Analysis of model II

Here, in addition to the description in model I, we assume that there is a buffer of varying (finite) capacity, equal to the current inventory level. Customers, finding the buffer full, are directed to an orbit. Let $M(t)$ be the number of customers in the buffer at time t. Now, $\{X(t), t \geq 0\}$, where $X(t) = (N(t), I(t), M(t))$, is an LDQBD on the state space $\{(i,j,k), i \geq 0, 0 \leq j \leq S, 0 \leq k \leq j\}$. Then, the generator has the form (2.1), where the blocks $A_0, A_{1,i}$ $(i \geq 0)$, and $A_{2,i} (i \geq 1)$ are square matrices of the same order $(1/2)(S+1)(S+2)$ and they are given by

$$A_0 = \begin{array}{c} \underline{0} \\ \underline{1} \\ \underline{2} \\ \vdots \\ \underline{S} \end{array} \begin{pmatrix} B_0 & & & & \\ & B_1 & & & \\ & & B_2 & & \\ & & & \ddots & \\ & & & & B_S \end{pmatrix}, \quad A_{2,i} = \begin{array}{c} \underline{0} \\ \underline{1} \\ \underline{2} \\ \vdots \\ \underline{S} \end{array} \begin{pmatrix} C_0 & & & & \\ & C_1 & & & \\ & & C_2 & & \\ & & & \ddots & \\ & & & & C_S \end{pmatrix},$$

$$A_{1,i} = \begin{array}{c} \underline{0} \\ \underline{1} \\ \underline{2} \\ \vdots \\ \underline{s} \\ \underline{s+1} \\ \underline{S-s} \\ \underline{S-s+1} \\ \underline{S-s+2} \\ \vdots \\ \underline{S} \end{array} \begin{pmatrix} H_0 & & & & & & & U_0 & & & \\ P_1 & H_1 & & & & & & & U_1 & & \\ & P_2 & H_2 & & & & & & & U_2 & \\ & & \ddots & \ddots & & & & & & & \ddots \\ & & & P_s & H_s & & & & & & & U_s \\ & & & & P_{s+1} & G_{s+1} & & & & & \\ & & & & & \ddots & \ddots & & & & \\ & & & & & & P_{S-s} & G_{S-s} & & & \\ & & & & & & & P_{S-s+1} & G_{S-s+1} & & \\ & & & & & & & & P_{S-s+2} & G_{S-s+2} & \\ & & & & & & & & & \ddots & \ddots \\ & & & & & & & & & & P_S & G_S \end{pmatrix}, \tag{3.1}$$

where

$$B_0=(\lambda\gamma)_{1\times 1},\quad B_n=\begin{pmatrix}0&\cdots&0&0\\ \vdots&\ddots&\vdots&\vdots\\ 0&\cdots&0&0\\ 0&\cdots&0&\lambda\gamma\end{pmatrix}_{(n+1)\times(n+1)}\quad (n=1,2,\ldots,S),\qquad H_0=(\Delta)_{1\times 1},$$

$$C_n=\begin{pmatrix}0&i\theta&&&&\\ &0&i\theta&&&\\ &&\ddots&\ddots&&\\ &&&0&i\theta&\\ &&&&0&i\theta\\ &&&&&i\theta(1-\delta)\end{pmatrix}_{(n+1)\times(n+1)}\quad (n=1,2,\ldots,S),\qquad C_0=\nu\big(i\theta(1-\delta)\big)_{1\times 1},$$

$$H_n=\begin{pmatrix}*&\lambda&&&&\\ &\Theta&\lambda&&&\\ &&\Theta&\lambda&&\\ &&&\ddots&\ddots&\\ &&&&\Theta&\lambda\\ &&&&&\Omega\end{pmatrix}_{(n+1)\times(n+1)}\quad (n=1,2,\ldots,S),$$

$$G_n=\begin{pmatrix}\circ&\lambda&&&&\\ &\square&\lambda&&&\\ &&\square&\lambda&&\\ &&&\ddots&\ddots&\\ &&&&\square&\lambda\\ &&&&&\nabla\end{pmatrix}_{(n+1)\times(n+1)}\quad (n=s+1,s+2,\ldots,S),$$

$$U_n=\begin{matrix}0\\1\\2\\ \vdots\\ s\end{matrix}\begin{pmatrix}\beta&0&0&\cdots&0&0&\cdots&0\\ 0&\beta&0&\cdots&0&0&\cdots&0\\ 0&0&\beta&\cdots&0&0&\cdots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\ddots&\vdots\\ 0&0&0&\cdots&\beta&0&\cdots&0\end{pmatrix}_{(n+1)\times((S-s)+n+1)}\quad (n=0,1,2,\ldots,s),$$

$$P_n=\begin{pmatrix}0&0&\cdots&0\\ \mu&0&\cdots&0\\ 0&\mu&\cdots&0\\ \vdots&\vdots&\ddots&\vdots\\ 0&0&\cdots&\mu\end{pmatrix}_{(n+1)\times n}\quad (n=1,2,\ldots,S). \tag{3.2}$$

3.1. System stability. Here, mean drift y_s is given by

$$y_s = \begin{cases} -i\theta(1-\delta)+\lambda\gamma, & \text{if the buffer is full,} \\ -i\theta, & \text{otherwise.} \end{cases} \tag{3.3}$$

Since $(1-\delta)>0$, for any $\varepsilon>0$, we can find N' large enough so that $y_s<-\varepsilon$ for any s belonging to the level $i\geq N'$. Hence, by Tweedie's [14] result, the system under consideration is stable.

3.2. System performance measures. For computing various measures of performance, we judiciously obtain a truncation level N. To find N, we adopt the procedure in Section 2.2. Here, again we partition the steady state probability vector $x=(x_0,x_1,x_2,\dots,x_{N-1},x_N,\dots)$ such that its $(i+1)$th component is

$$x_i = (y_{i,0,0},y_{i,1,0},y_{i,1,1},y_{i,2,0},y_{i,2,1},y_{i,2,2},\dots,y_{i,S,0},y_{i,S,1},y_{i,S,2},\dots,y_{i,S,S}). \tag{3.4}$$

(i) The expected inventory level, EI, in the system is given by

$$\text{EI} = \sum_{i=0}^{\infty}\sum_{j=0}^{S}\sum_{k=0}^{j} j\,y_{i,j,k}. \tag{3.5}$$

(ii) The expected number of customers, EO, in the orbit is given by

$$\begin{aligned}\text{EO} &= \left(\sum_{i=1}^{\infty} i x_i\right)e \\ &= \left(\left(\sum_{i=1}^{N-1} i x_i\right)+x_{N-1}R\left(N(I-R)^{-1}+R(I-R)^{-2}\right)\right)e.\end{aligned} \tag{3.6}$$

(iii) The expected number of customers, EB, in the buffer is given by

$$\text{EB} = \sum_{i=0}^{\infty}\sum_{j=0}^{S}\sum_{k=0}^{j} k\,y_{i,j,k}. \tag{3.7}$$

(iv) The expected reorder rate, EROR, is given by

$$\text{EROR} = \mu\sum_{i=0}^{\infty}\sum_{k=1}^{s+1} y_{i,s+1,k}. \tag{3.8}$$

(v) The expected number of departures, EDS, after completing service is given by

$$\mathrm{EDS} = \mu \sum_{i=0}^{\infty} \sum_{j=1}^{S} \sum_{k=1}^{j} y_{i,j,k}. \tag{3.9}$$

(vi) The expected number of customers lost, EL_1, before entering the orbit per unit time is given by

$$\mathrm{EL}_2 = (1-\gamma)\lambda \sum_{i=0}^{\infty} \sum_{j=0}^{S} y_{i,j,j}. \tag{3.10}$$

(vii) The expected number of customers lost, EL_2, after retrials per unit time is given by

$$\mathrm{EL}_2 = \theta(1-\delta) \sum_{i=1}^{\infty} i \left(\sum_{j=1}^{S} y_{i,j,j} \right). \tag{3.11}$$

(viii) The overall rate of retrials, ORR, is given by

$$\mathrm{ORR} = \theta \left(\sum_{i=1}^{\infty} i x_i \right) e. \tag{3.12}$$

(ix) The successful rate of retrials, SRR, is given by

$$\mathrm{SRR} = \theta \sum_{i=0}^{\infty} i \left(\sum_{j=1}^{S} \sum_{k=0}^{j-1} y_{i,j,k} \right). \tag{3.13}$$

4. Analysis of model III

The distinguishing factor of this model from model II is that the capacity of the buffer is equal to S, the maximum inventory level, irrespective of the inventory held at any given instant of time. Here, $\{X(t), t \geq 0\}$, where $X(t) = (N(t), I(t), M(t))$, is an LDQBD on the state space $\{(i,j,k), i \geq 0, 0 \leq j \leq S, 0 \leq k \leq S\}$. Then, the generator has the form (2.1), where the blocks $A_0, A_{1,i}$ $(i \geq 0)$, and $A_{2,i}$ $(i \geq 1)$ are square matrices of the same order

$(S+1)^2$ and they are given by

$$A_0 = \begin{matrix} \underline{0} \\ \underline{1} \\ \underline{2} \\ \vdots \\ \underline{S} \end{matrix} \begin{pmatrix} B_0 & & & & \\ & B_1 & & & \\ & & B_2 & & \\ & & & \ddots & \\ & & & & B_S \end{pmatrix}, \quad A_{2,i} = \begin{matrix} \underline{0} \\ \underline{1} \\ \underline{2} \\ \vdots \\ \underline{S} \end{matrix} \begin{pmatrix} C_0 & & & & \\ & C_1 & & & \\ & & C_2 & & \\ & & & \ddots & \\ & & & & C_S \end{pmatrix},$$

$$\mathrm{A}_{1,i} = \begin{matrix} \underline{0} \\ \underline{1} \\ \underline{2} \\ \vdots \\ \underline{s} \\ \underline{s+1} \\ \underline{S-s} \\ \underline{S-s+1} \\ \underline{S-s+2} \\ \vdots \\ \underline{S} \end{matrix} \begin{pmatrix} H_0 & & & & & & & U_0 & & & \\ P_1 & H_1 & & & & & & & U_1 & & \\ & P_2 & H_2 & & & & & & & U_2 & \\ & & \ddots & \ddots & & & & & & & \ddots \\ & & & P_s & H_s & & & & & & & U_s \\ & & & & P_{s+1} & G_{s+1} & & & & & \\ & & & & & \ddots & \ddots & & & & \\ & & & & & & P_{S-s} & G_{S-s} & & & \\ & & & & & & & P_{S-s+1} & G_{S-s+1} & & \\ & & & & & & & & P_{S-s+2} & G_{S-s+2} & \\ & & & & & & & & & \ddots & \ddots \\ & & & & & & & & & & P_S & G_S \end{pmatrix}, \tag{4.1}$$

where

$$B_n = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \lambda\gamma \end{pmatrix}_{(S+1)\times(S+1)} \qquad (n = 0, 1, 2, \ldots, S),$$

$$C_n = \begin{pmatrix} 0 & i\theta & & & & \\ & 0 & i\theta & & & \\ & & \ddots & \ddots & & \\ & & & 0 & i\theta & \\ & & & & 0 & i\theta \\ & & & & & i\theta(1-\delta) \end{pmatrix}_{(S+1)\times(S+1)} \qquad (n = 0, 1, 2, \ldots, S),$$

$$H_0 = \begin{pmatrix} * & \lambda & & & \\ & * & \lambda & & \\ & & \ddots & \ddots & \\ & & & * & \lambda \\ & & & & \Delta \end{pmatrix}_{(S+1)\times(S+1)},$$

$$H_n = \begin{pmatrix} * & \lambda & & & & \\ & \Theta & \lambda & & & \\ & & \Theta & \lambda & & \\ & & & \ddots & \ddots & \\ & & & & \Theta & \lambda \\ & & & & & \Omega \end{pmatrix}_{(S+1)\times(S+1)} \quad (n = 1,2,\ldots,S),$$

$$G_n = \begin{pmatrix} \circ & \lambda & & & & \\ & \square & \lambda & & & \\ & & \square & \lambda & & \\ & & & \ddots & \ddots & \\ & & & & \square & \lambda \\ & & & & & \nabla \end{pmatrix}_{(S+1)\times(S+1)} \quad (n = s+1,s+2,\ldots,S),$$

$$U_n = \begin{pmatrix} \beta & & & & \\ & \beta & & & \\ & & \beta & & \\ & & & \ddots & \\ & & & & \beta \end{pmatrix}_{(S+1)\times(S+1)} \quad (n = 0,1,2,\ldots,s),$$

$$P_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \mu & 0 & \cdots & 0 & 0 \\ 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mu & 0 \end{pmatrix}_{(S+1)\times(S+1)} \quad (n = 1,2,\ldots,S). \tag{4.2}$$

(4.3)

4.1. System stability. Here, mean drift y_s is given by

$$y_s = \begin{cases} -i\theta(1-\delta) + \lambda\gamma, & \text{if the butter is full,} \\ -i\theta, & \text{otherwise.} \end{cases} \tag{4.4}$$

Since $(1-\delta) > 0$, for any $\varepsilon > 0$, we can find N' large enough so that $y_s < -\varepsilon$ for any s belonging to the level $i \geq N'$. Hence, by Tweedie's [14] result, the system under consideration is stable.

4.2. System performance measures. To find the truncation level N, we adopt the procedure as in Section 2.2. The $(i+1)$th component of the steady state probability vector $x = (x_0, x_1, x_2, \ldots, x_{N-1}, x_N, \ldots)$ can be partitioned as

$$x_i = (y_{i,0,0}, y_{i,0,1}, y_{i,0,2}, \ldots, y_{i,0,S}, y_{i,1,0}, y_{i,1,1}, y_{i,1,2}, \ldots, y_{i,1,S}, \ldots, y_{i,S,0}, y_{i,S,1}, y_{i,S,2}, \ldots, y_{i,S,S}). \tag{4.5}$$

Then,

(i) the expected inventory level, EI, in the system is given by

$$\mathrm{EI} = \sum_{i=0}^{\infty} \sum_{j=0}^{S} \sum_{k=0}^{S} j y_{i,j,k}; \tag{4.6}$$

(ii) the expected number of customers, EO, in the orbit is given by

$$\begin{aligned} \mathrm{EO} &= \left(\sum_{i=1}^{\infty} i x_i \right) e \\ &= \left(\left(\sum_{i=1}^{N-1} i x_i \right) + x_{N-1} R \left(N(I-R)^{-1} + R(I-R)^{-2} \right) \right) e; \end{aligned} \tag{4.7}$$

(iii) the expected number of customers, EB, in the buffer is given by

$$\mathrm{EB} = \sum_{i=0}^{\infty} \sum_{j=0}^{S} \sum_{k=0}^{S} k y_{i,j,k}; \tag{4.8}$$

(iv) the expected reorder rate, EROR, is given by

$$\mathrm{EROR} = \mu \sum_{i=0}^{\infty} \sum_{k=1}^{S} y_{i,s+1,k}; \tag{4.9}$$

(v) the expected number of departures, EDS, after completing service is given by

$$\mathrm{EDS} = \mu \sum_{i=0}^{\infty} \sum_{j=1}^{S} \sum_{k=1}^{S} y_{i,j,k}; \tag{4.10}$$

(vi) the expected number of customers lost, EL_1, before entering the orbit per unit time is given by

$$\mathrm{EL}_1 = (1-\gamma)\lambda \sum_{i=0}^{\infty} \sum_{j=0}^{S} y_{i,j,S}; \tag{4.11}$$

(vii) the expected number of customers lost, EL_2, after retrials per unit time is given by

$$\mathrm{EL}_2 = \theta(1-\delta) \sum_{i=1}^{\infty} i \left(\sum_{j=1}^{S} y_{i,j,S} \right); \tag{4.12}$$

(viii) the overall rate of retrials, ORR, is given by

$$\mathrm{ORR} = \theta \left(\sum_{i=1}^{\infty} i x_i \right) e; \tag{4.13}$$

(ix) the successful rate of retrials, SRR, is given by

$$\text{SRR} = \theta \sum_{i=0}^{\infty} i \left(\sum_{j=1}^{S} \sum_{k=0}^{S-1} y_{i,j,k} \right). \tag{4.14}$$

5. Cost analysis and numerical results

We define different costs as

K = fixed cost,
c_1 = procurement cost/unit,
c_2 = holding cost of inventory/unit /unit time,
c_3 = holding cost of customers/unit /unit time,
c_4 = cost due to loss of customers/unit /unit time,
c_5 = cost due to service/unit /unit time,
c_6 = revenue from service/unit/unit time.

We introduce a cost function, defined as the expected total cost (ETC) of the system, which is given by

$$\text{ETC} = (K + (S - s)c_1)\text{EROR} + c_2\text{EI} + c_3(\text{EO} + \text{EB}) + c_4(\text{EL}_1 + \text{EL}_2) + (c_5 - c_6)\text{EDS}. \tag{5.1}$$

In the following tables, we provide a comparison among the overall and successful rates of retrials for models I–III.

5.1. Interpretations of the numerical results in the tables. As the arrival rate λ increases, the number of customers in the orbit becomes larger so that the overall and successful rates of retrials from the orbit will increase (see Table 5.1). As the replenishment rate β or service rate μ increases, the arriving customers will get the inventory more rapidly, and thereby the number of customers in the orbit gets decreased. In that case, the overall and successful rates of retrials will decrease (see Tables 5.2 and 5.3). With the increase in probability γ of primary arrivals joining the orbit or in return probability δ of retrial customers, the orbit size increases. Here, again overall and successful rates of retrials increase (see Tables 5.4 and 5.5). Table 5.6 indicates that as the retrial rate θ of customers in the orbit increases, the overall and successful rates of retrials from the orbit will increase.

Next, we provide graphical illustrations of the performance measures of the above described models.

5.2. Interpretations of the graphs. The objective is to compare the three models and identify the one which is more profitable. For this, we compute the minimum value of the expected total cost per unit time by varying the parameters one at a time keeping others fixed. Since the objective function is known only implicitly, the analytical properties such as convexity of the analytic function cannot be studied in general. By fixing all the parameters except the arrival rate λ, it is clear from Figure 5.1 that the cost function is convex

TABLE 5.1. Variations in arrival rate λ. $\mu = 1.5$; $\beta = 1$; $\theta = 0.75$; $\gamma = 0.2$; $\delta = 0.4$; $N = 62$; $s = 1$; $S = 4$; $c_1 = 1$; $c_2 = 1$; $c_3 = 0.6$; $c_4 = 1$; $c_5 = 2$; $c_6 = 1$.

λ	Model I		Model II		Model III	
	ORR	SRR	ORR	SRR	ORR	SRR
2.6	0.288076	0.081170	1.250731	0.660523	0.561902	0.145595
2.7	0.307286	0.087625	1.274185	0.662086	0.605659	0.149230
2.8	0.326964	0.094214	1.297996	0.663758	0.649895	0.153096
2.9	0.347096	0.100919	1.322157	0.665546	0.694298	0.157210
3.0	0.367663	0.107724	1.346658	0.667455	0.738639	0.161582
3.1	0.388647	0.114610	1.371489	0.669494	0.782765	0.166204
3.2	0.410028	0.121562	1.396640	0.671670	0.826577	0.171040
3.3	0.431785	0.128564	1.422100	0.673990	0.870013	0.176015
3.4	0.453897	0.135600	1.447860	0.676462	0.913040	0.180994
3.5	0.476343	0.142656	1.473909	0.679093	0.955641	0.185780

TABLE 5.2. Variations in replenishment rate β. $\lambda = 2$; $\mu = 1.5$; $\theta = 0.75$; $\gamma = 0.2$; $\delta = 0.4$; $N= 48$; $s= 1$; $S= 4$; $c_1 = 1$; $c_2 = 5$; $c_3 = 0.5$; $c_4 = 1$; $c_5 = 2$; $c_6 = 1$.

β	Model I		Model II		Model III	
	ORR	SRR	ORR	SRR	ORR	SRR
1.1	0.169276	0.041360	1.106037	0.695448	0.311534	0.199893
1.2	0.156979	0.037028	1.095219	0.692633	0.293366	0.201550
1.3	0.146188	0.033169	1.085242	0.690077	0.277385	0.201175
1.4	0.136660	0.029745	1.076010	0.687745	0.263261	0.199345
1.5	0.128201	0.026713	1.067440	0.685611	0.250709	0.196505
1.6	0.120650	0.024032	1.059460	0.683649	0.239489	0.192984
1.7	0.113876	0.021662	1.052012	0.681840	0.229406	0.189028
1.8	0.107770	0.019564	1.045041	0.680167	0.220297	0.184812
1.9	0.102244	0.017706	1.038504	0.678616	0.212029	0.180464
2.0	0.097222	0.016058	1.032359	0.677173	0.204490	0.176075

TABLE 5.3. Variations in service rate μ. $\lambda = 2$; $\beta = 1$; $\theta = 0.75$; $\gamma = 0.2$; $\delta = 0.4$; $N= 37$; $s= 1$; $S= 4$; $c_1 = 1$; $c_2 = 1$; $c_3 = 1$; $c_4 = 1$; $c_5 = 2$; $c_6 = 1$.

μ	Model I		Model II		Model III	
	ORR	SRR	ORR	SRR	ORR	SRR
2.1	0.174671	0.045655	1.920751	1.497005	0.366320	0.290650
2.2	0.173506	0.045573	1.866814	1.442761	0.362866	0.286314
2.3	0.172406	0.045494	1.812846	1.388586	0.359369	0.281763
2.4	0.171364	0.045416	1.758860	1.334492	0.355847	0.276986
2.5	0.170376	0.045341	1.704867	1.280491	0.352321	0.271972
2.6	0.169438	0.045268	1.650881	1.226597	0.348820	0.266708
2.7	0.168546	0.045196	1.596919	1.172826	0.345380	0.261181
2.8	0.167697	0.045127	1.543002	1.119197	0.342050	0.255375
2.9	0.166888	0.045059	1.489156	1.065732	0.338890	0.249274
3.0	0.166116	0.044994	1.435410	1.012452	0.335977	0.242856

Table 5.4. Variations in probability γ of primary arrivals joining the orbit. $\lambda = 2$; $\beta = 1$; $\mu = 1.5$; $\theta = 0.75$; $\delta = 0.4$; $N = 56$; $s = 1$; $S = 4$; $c_1 = 1$; $c_2 = 5$; $c_3 = 1$; $c_4 = 1$; $c_5 = 2$; $c_6 = 1$.

γ	Model I		Model II		Model III	
	ORR	SRR	ORR	SRR	ORR	SRR
0.24	0.235962	0.058532	1.166675	0.716265	0.373307	0.197246
0.28	0.294115	0.071712	1.217041	0.734260	0.422046	0.201751
0.32	0.357502	0.085542	1.268831	0.752496	0.477661	0.209205
0.36	0.425575	0.099809	1.321993	0.770936	0.538306	0.219249
0.40	0.497650	0.114299	1.376494	0.789548	0.601978	0.231259
0.44	0.573019	0.128825	1.432317	0.808309	0.667075	0.244629
0.48	0.651051	0.143245	1.489458	0.827201	0.732477	0.258886
0.52	0.731269	0.157471	1.547918	0.846208	0.797423	0.273698
0.56	0.813361	0.171460	1.607709	0.865318	0.861393	0.288838
0.60	0.897137	0.185199	1.668843	0.884520	0.924018	0.304154

Table 5.5. Variations in return probability δ of retrial customers. $\lambda = 2$; $\beta = 1$; $\mu = 1.5$; $\theta = 0.75$; $\gamma = 0.2$; $N = 51$; $s = 1$; $S = 4$; $c_1 = 1$; $c_2 = 1.6$; $c_3 = 0.32$; $c_4 = 1$; $c_5 = 2$; $c_6 = 1$.

δ	Model I		Model II		Model III	
	ORR	SRR	ORR	SRR	ORR	SRR
0.35	0.179867	0.044661	1.077717	0.684408	0.325449	0.190152
0.40	0.183381	0.045382	1.117815	0.698563	0.332249	0.195492
0.45	0.187421	0.046189	1.162668	0.714154	0.338966	0.201308
0.50	0.192090	0.047095	1.213261	0.731433	0.345442	0.207632
0.55	0.197515	0.048111	1.270888	0.750721	0.351484	0.214490
0.60	0.203860	0.049248	1.337303	0.772436	0.356886	0.221897
0.65	0.211348	0.050516	1.414941	0.797138	0.361457	0.229859
0.70	0.220371	0.051921	1.507329	0.825602	0.365094	0.238379
0.75	0.232606	0.053478	1.619805	0.858943	0.367846	0.247469
0.80	0.266273	0.055392	1.760960	0.898851	0.369974	0.257184

Table 5.6. Variations in retrial rate θ. $\lambda = 2$; $\beta = 1$; $\mu = 1.5$; $\gamma = 0.2$; $\delta = 0.4$; $N = 28$; $s = 1$; $S = 4$; $c_1 = 1$; $c_2 = 1$; $c_3 = 1$; $c_4 = 60$; $c_5 = 2$; $c_6 = 1$.

θ	Model I		Model II		Model III	
	ORR	SRR	ORR	SRR	ORR	SRR
1.1	0.200807	0.035823	1.137674	0.284382	0.309113	0.049809
1.2	0.204819	0.036609	1.142996	0.297763	0.310083	0.053750
1.3	0.208480	0.037415	1.148167	0.312566	0.311154	0.058208
1.4	0.211829	0.038242	1.153192	0.329041	0.312339	0.063285
1.5	0.214901	0.039085	1.158074	0.347505	0.313649	0.069106
1.6	0.217728	0.039944	1.162817	0.368356	0.315098	0.075831
1.7	0.220334	0.040814	1.167424	0.392112	0.316703	0.083670
1.8	0.222745	0.041689	1.171901	0.419448	0.318480	0.092899
1.9	0.224979	0.042563	1.176251	0.451269	0.320448	0.103889
2.0	0.227055	0.043427	1.180478	0.488808	0.322629	0.117151

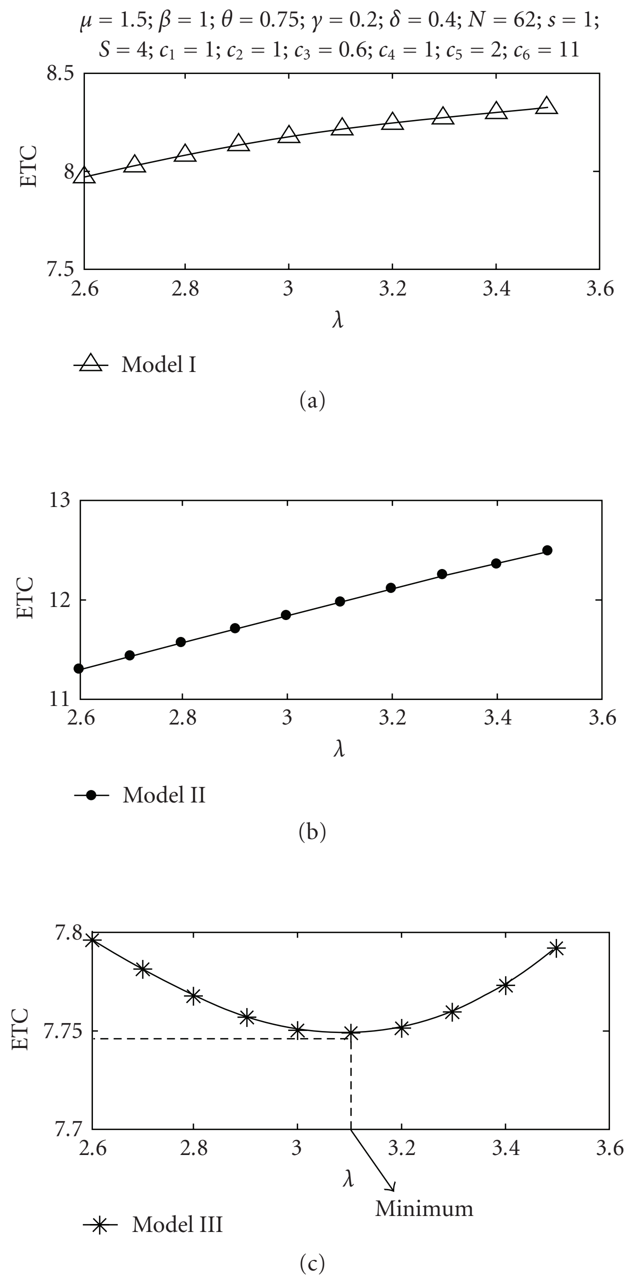

FIGURE 5.1. Arrival rate versus ETC.

in λ for model III. For given parameter values, this function attains the following minimum values: (a) 7.966 at $\lambda = 2.6$ for model I, (b) 11.287 at $\lambda = 2.6$ for model II, and (c) 7.747 at $\lambda = 3.1$ for model III. Therefore, model III is the best for minimum cost per unit

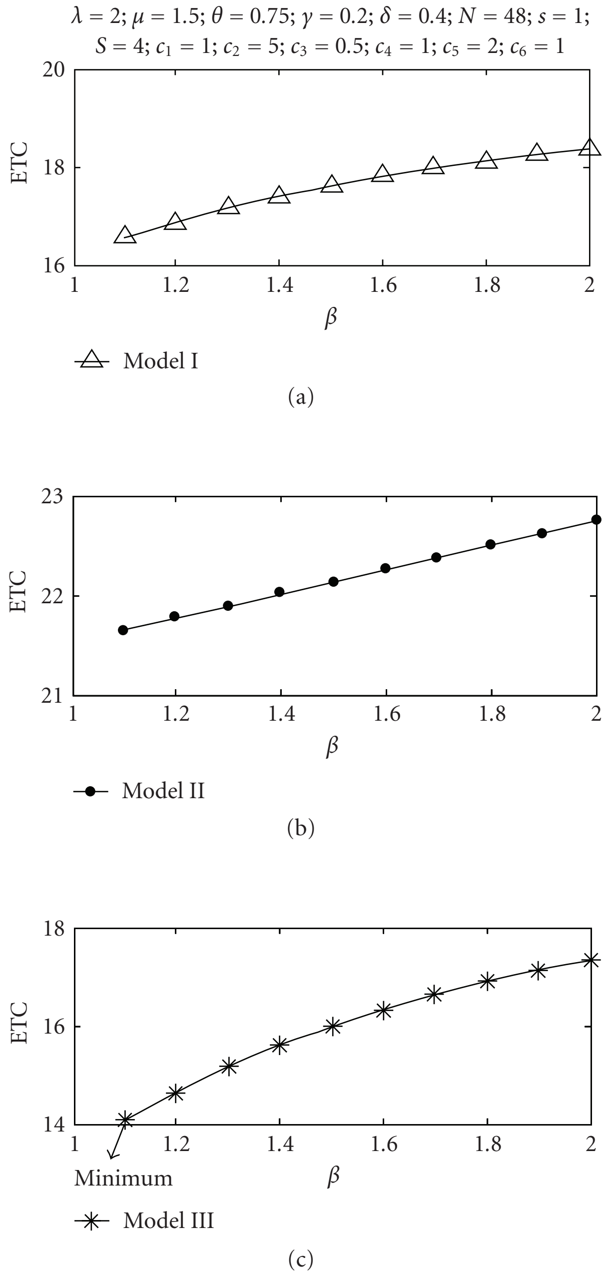

FIGURE 5.2. Replenishment rate versus ETC.

time. As replenishment rate β increases (keeping other parameters fixed), one can observe that cost function attains the minimum values 16.543, 21.647, and 14.057 at $\lambda = 1.1$ for models I, II, and III, respectively. Here, also model III is the best for minimum cost per

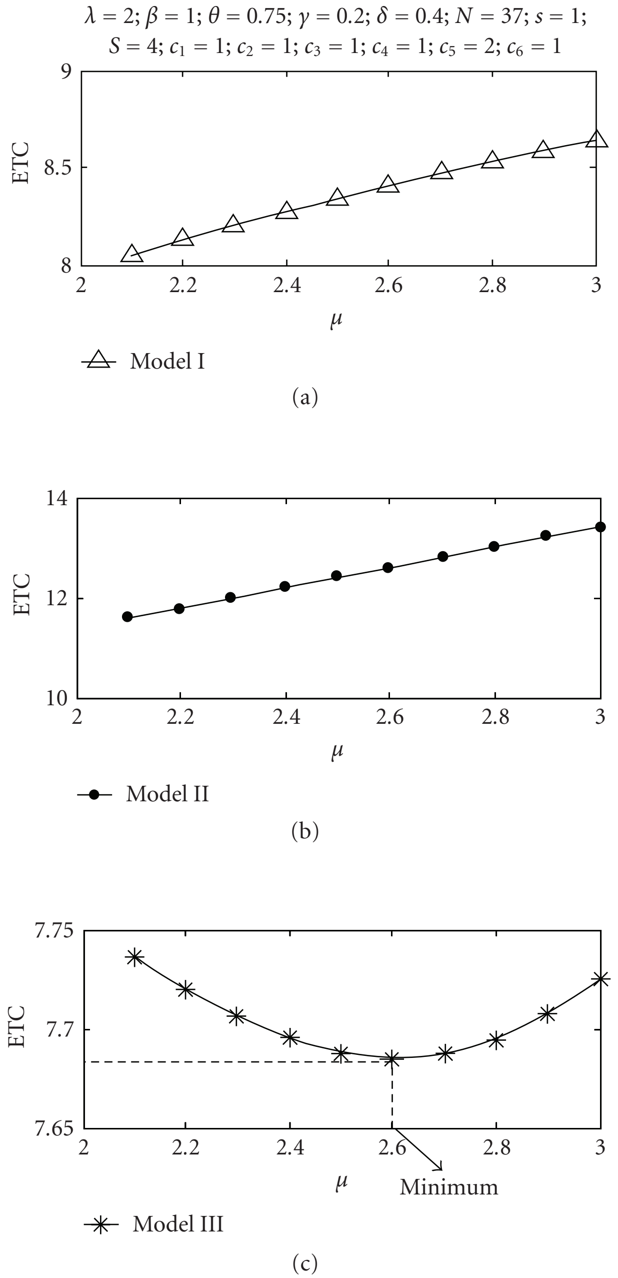

Figure 5.3. Service rate versus ETC.

unit time. One can observe the minimum value of the objective function by changing the other parameters μ, γ, δ, and θ (see Figures 5.3, 5.4, 5.5, and 5.6). In all examples considered here, the cost function has the minimum value for model III. Therefore, model III

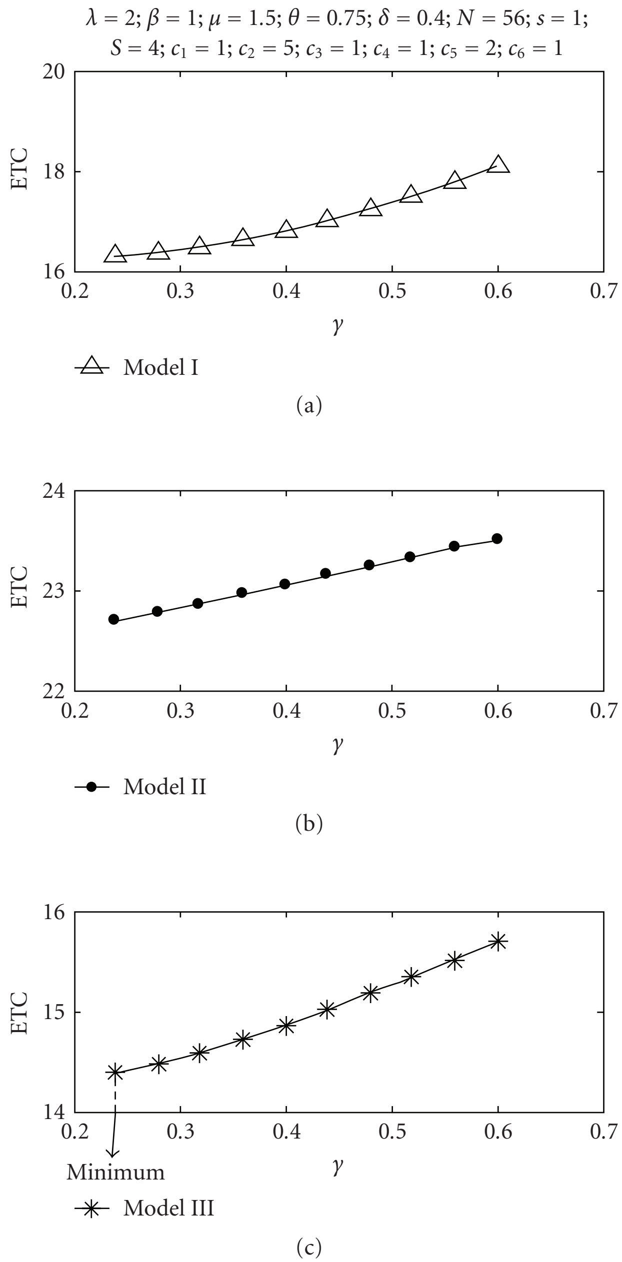

FIGURE 5.4. Gamma versus ETC.

(model with buffer size equal to the maximum inventory level S) can be considered as the best model for practical applications.

As indicated in the introduction, Ushakumari [10] obtained analytical solution to inventory system with retrial of unsatisfied customers. However, she considered a system

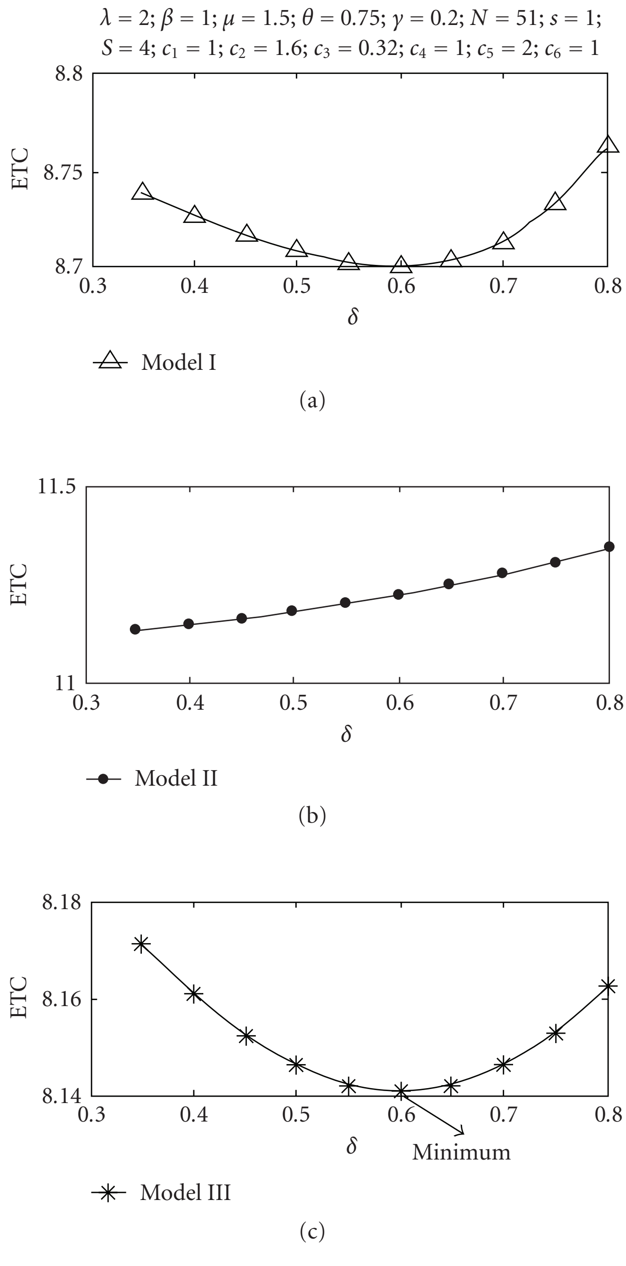

FIGURE 5.5. Delta versus ETC.

with negligible service time. Thus, that problem turns out to be just two-dimentional. This is not the case with the problem discussed in this paper since the authors considered positive lead time. This fact prevents the problem from getting analytical solution. Thus, we are forced to take recourse to numerical study.

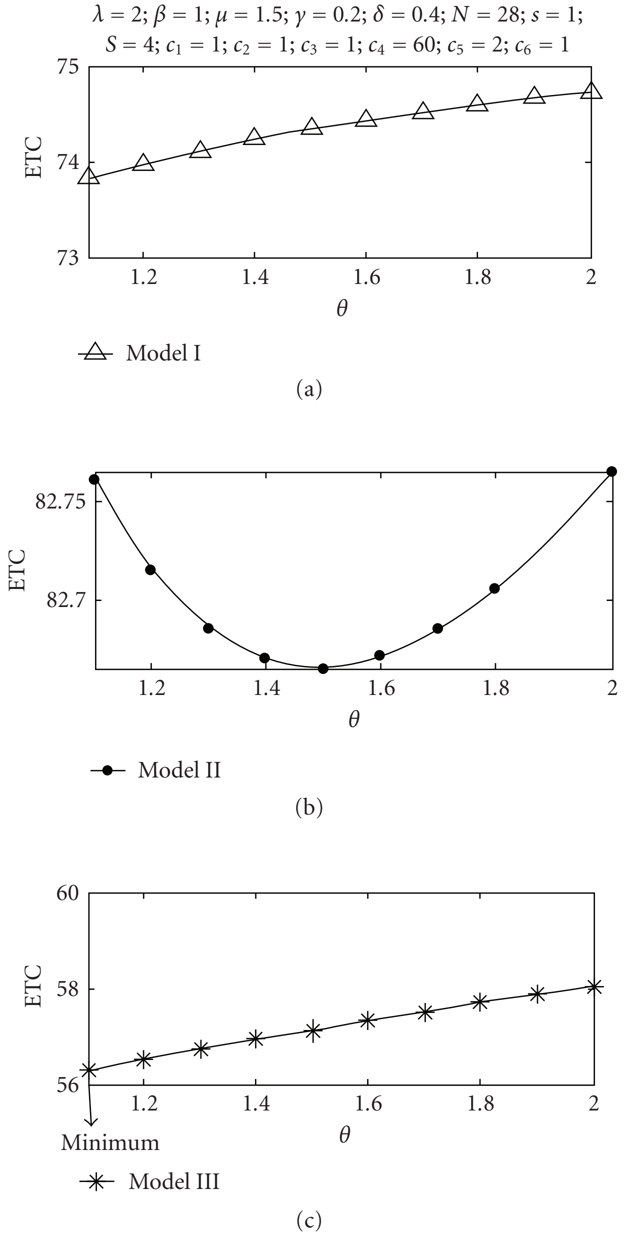

Figure 5.6. Theta versus ETC.

Acknowledgment

Krishnamoorthy acknowledges the financial support through NBHM (DAE) Research Project no. 48/5/2003/R&D/II/3269.

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