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Research Article

Nonlinear Vector Variational Inequality Problems for η-Pseudomonotone Maps

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We consider a new class of complementarity problems for η-pseudomonotone maps and obtain an existence result for their solutions in real Hausdorff topological vector spaces. Our results extend the same previous results in this literature.

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1. Introduction

Variational inequalities were introduced and considered by Stampacchia [1] in early sixties. It has been shown that a wide class of linear and nonlinear problems arising in various branches of mathematical and engineering sciences can be studied in the unified and general framework of variational inequalities. Variational inequalities have been generalized and extended in several directions using new techniques. Giannessi [2] introduced a new class of variational inequalities, which is vector variational inequality. Vector variational inequalities have many applications in vector optimization, approximate vector optimization, and other areas (see, e.g., [3]). Noor [4] introduced a class of variational inequalities involving two operators, which are called general variational inequalities. It has been shown that nonsymmetric and odd-order obstacle, free, moving, and equilibrium problems can be studied via the general variational inequalities. For the applications, formulation, and numerical methods for solving variational inequalities, see [5–8] and the references therein.

Inspired and motivated by the recent research activities going on in this dynamic field, we introduce a new class of complementarity problems for η-pseudomonotone maps. Moreover, we obtain an existence result for their solutions in real Hausdorff topological vector spaces setting for a moving cone by relaxing continuity and compactness. This is done by using a new version of famous Ky Fan lemma which is due to Ben-El-Mechaiekh et al. [9]. Our results represent an improvement and refinement of the recent results obtained in [10].

In the rest of this section, we recall some definitions and preliminaries results which are used in the next section.

We will denote by 2^A the family of all subsets of A and by $\mathcal{F}(A)$ the family of all nonempty finite subsets of A. Let X be a real Hausdorff topological vector space (t.v.s.). A nonempty subset P of X is called convex cone if (i) $P+P=P$, (ii) $\lambda P\subseteq P$, for all $\lambda\geq 0$. Cone P is said to be pointed whenever $P\cap -P=\{0\}$. Let Y be a t.v.s. and let $P\subseteq Y$ be a cone. The cone P induces an ordering on Y (in this case the pair (Y,P) is called an ordered t.v.s.) which is defined as follows:

$$x\leq y\Longleftrightarrow y-x\in P. \tag{1.1}$$

This ordering is antisymmetric if P is pointed. Let K be a nonempty convex subset of a t.v.s. X and let K_0 be a subset of K. A multivalued map $\Gamma: K_0\to 2^K$ is said to be a KKM map if

$$\operatorname{co}A\subseteq\bigcup_{x\in A}\Gamma(x),\quad \forall A\in\mathcal{F}(K_0), \tag{1.2}$$

where co denotes the convex hull.

Definition 1.1 (see [9]). Consider a subset A of a topological vector space and a topological space Y. A family $\{C_i,K_i\}_{i\in I}$ of pairs of sets is said to be coercing for a map $G:A\to 2^Y$ if and only if

(i) for each $i\in K, C_i$ is contained in a compact convex subset of A, and K_i is a compact subset of Y;
(ii) for each i,j, there exists $k\in I$ such that $C_i\cup C_j\subseteq C_k$;
(iii) for each $i\in I$, there exists $k\in I$ with $\bigcap_{x\in C_k}G(x)\subseteq K_i$.

Theorem 1.2 (see [9]). *Let $F:K\to 2^Y$ be a KKM map with compactly closed (in K) values. If F admits a coercing family, then $\bigcap_{x\in K}F(x)\neq\emptyset$.*

2. Main results

Throughout this section we let X and Y be two topological vector spaces, K a nonempty convex subset of $X, C:K\to 2^Y$ with convex cone values, and let $\eta:K\times K\to L(X,Y)$ and $T:K\to L(X,Y)$ be two nonlinear mappings.

We consider two following problems; the first is called nonlinear vector variational inequality (NVVI) problem with respect to η that consists in finding $x\in K$ such that

$$\langle T(x),\eta(y,x)\rangle\in C(x),\quad \forall y\in K. \tag{2.1}$$

The second problem is called dual nonlinear vector variational inequality (DNVVI) problem with respect to η that consists in finding $x\in K$ such that

$$\langle T(y),\eta(x,y)\rangle\in -C(y),\quad \forall y\in K. \tag{2.2}$$

We denote the solution set of (2.1) and (2.2) with NVVIS and DNVVIS, respectively.

Definition 2.1. T is C-pseudomonotone with respect to η if, for all $x, y \in K$, the following implication holds:

$$\langle T(x), \eta(y,x)\rangle \in C(x) \Longrightarrow \langle T(y), \eta(x,y)\rangle \in -C(y). \tag{2.3}$$

Remark that the definition of monotonicity of T with respect to η given in [10] implies C-pseudomonotonicity of T with respect to η, for a constant cone C, that is $C(x) = C$, for all $x \in K$.

Definition 2.2. T is said to be C-upper sign continuous with respect to η if, for all $x, y \in K$, the following holds:

$$\langle T(u), \eta(y,u)\rangle \in C(u), \quad \forall u \in]x,y[\Longrightarrow \langle T(x), \eta(x,y)\rangle \in C(x). \tag{2.4}$$

Let us recall that the above definition is a very weak kind of continuity. This notion is introduced by Hadjisavvas [11] in the framework of variational inequalities and later by Bianchi and Pini [12] for real bifunctions.

The following proposition improves Theorem 3.1 in [10].

Proposition 2.3. *If η is antisymmetric, that is, $\eta(x,y) = -\eta(y,x)$, the set $\{y \in K : \langle T(x), \eta(y,x)\rangle = 0\} = \{x\}$, and T is C-pseudomonotone with respect to η, then the solution set of (NVVI) is empty or singleton.*

Proof. Let x_1, x_2 be two solutions of (NVVI). Hence

$$\langle T(x_1), \eta(x_2,x_1)\rangle \in C(x_1), \qquad \langle T(x_2), \eta(x_1,x_2)\rangle \in C(x_2). \tag{2.5}$$

From C-pseudomonotone with respect to η of T, η is antisymmetric, and from (2.1), we get

$$\langle T(x_2), \eta(x_1,x_2)\rangle \in C(x_2) \cap -C(x_2) = \{0\}. \tag{2.6}$$

Thus

$$x_1 \in \{y \in K : \langle T(x_2), \eta(y,x_2)\rangle = 0\} = \{x_2\}. \tag{2.7}$$

This completes the proof. □

The following theorem generalizes Theorem 3.2 in [10].

Theorem 2.4. *Let $T : K \to L(X,Y)$ and $\eta : K \times X \to L(X,Y)$ be two mappings satisfying the following conditions:*

(i) *T is C-pseudomonotone with respect to η;*
(ii) *η is convex in the first variable with $\eta(x,x) = 0$, for all $x \in K$;*
(iii) *T is C-upper sign continuous with respect to η.*

Then, NVVIS = DNVVIS.

Proof. By the definition of C-pseudomonotone with respect to η, we have

$$\text{NVVIS} \subseteq \text{DNVVIS}. \tag{2.8}$$

Conversely, let $x_0 \in$ DNVVIS and $x \in K$. By letting $x_t = x_0 + t(x - x_0)$, for $t \in]0,1[$, from (2.2), we have

$$\langle Tx_t, \eta(x_0, x_t)\rangle \in -C(x_t). \tag{2.9}$$

If $\langle T(x_s), \eta(x, x_s)\rangle \notin C(x_s)$, for some $s \in]0,1[$, then it is obvious from (2.9) and (ii) that

$$0 = \langle T(x_s), (1-s)\eta(x_0, x_s) + s\eta(x, x_s) - \eta(x_s, x_s)) \notin C(x_s), \tag{2.10}$$

which is a contradiction, since $C(x_s)$ is a pointed convex cone and $0 \in C(x_s)$. Hence we have

$$\langle T(x_s), \eta(x, x_s)\rangle \in C(x_s), \quad \forall s \in]0,1[. \tag{2.11}$$

Now, (iii) entails the result. □

Theorem 2.5. *Assume that*

(i) *for each $x \in K$, $\eta(x,x) = 0$, and any compact subset W of K, the set $\{y \in W : \langle Ty, \eta(x,y)\rangle \in C(y)\}$ is closed in W;*

(ii) *for each finite subset A of K and any $y \in coA\backslash A$, there exists $x \in A$ such that $\langle Ty, \eta(x,y)\rangle \in C(y)$;*

(iii) *there exist compact subset B and compact convex subset D of K such that for all $x \in K\backslash B$, $\exists y \in D; \langle Tx, \eta(y,x)\rangle \notin C(x)$.*

Then the NVVIS is nonempty and compact.

Proof. We define $\Gamma : K \to 2^K$ as follows:

$$\Gamma(y) = \{x \in K : \langle Tx, \eta(y,x)\rangle \in C(x)\}. \tag{2.12}$$

By (i), Γ has compactly closed values. We claim that Γ is a KKM mapping. Indeed, if it is false, then there exist elements $y_1, y_2, \dots, y_n$ of K and $z \in \text{co}(\{y_1, y_2, \dots, y_n\})$ such that $z \notin \bigcup_{i=1}^n \Gamma(y_i)$. Thus by the definition of Γ, we have $\langle Tz, \eta(y_i, z)\rangle \notin C(z)$, for $i = 1, 2, \dots, n$, which is a contradiction (by (ii)). It is clear that $\{(D,B)\}$ is a coercing family for Γ. Now, by Theorem 1.2, NVVIS $= \bigcap_{x\in K}\Gamma(x) \neq \varnothing$. Using (iii), we obtain

$$\text{NVVIS} = \bigcap_{x\in K}\Gamma(x) \subseteq B, \tag{2.13}$$

and hence

$$\text{NVVIS} = \bigcap_{x\in K}\Gamma(x) = \bigcap_{x\in K}(\Gamma(x) \cap B), \tag{2.14}$$

which is closed in B (by (i)), and so a compact subset of B. □

Theorem 2.6. *Assume that*

(i) *for each $x \in K$, $\eta(x,x) = 0$, and any compact subset W of K, the set $\{y \in W : \langle Ty, \eta(y,x)\rangle \in -C(y)\}$ is closed in W;*

(ii) *for each finite subset A of K and any $y \in coA\backslash A$, there exists $x \in A$ such that $\langle Ty, \eta(y,x)\rangle \in -C(x)$;*

(iii) *there exist compact subset B and compact convex subset D of K such that for all* $x \in K \backslash B, \exists y \in D; \langle Ty, \eta(x,y)\rangle \notin -C(y)$.

Then the DNVVIS is nonempty and compact.

Proof. We define $\Gamma : K \to 2^K$ as follows:

$$\Gamma(y) = \{x \in K : \langle Ty, \eta(x,y)\rangle \in -C(y)\}. \tag{2.15}$$

By (i), Γ has compactly closed values. By (ii), Γ is a KKM mapping. It is obvious that $\{(D,B)\}$ is a coercing family for Γ. Now, by Theorem 1.2, DNVVIS $= \bigcap_{x\in K}\Gamma(x) \neq \emptyset$. Moreover, using (iii),

$$\text{DNVVIS} = \bigcap\nolimits_{x\in K}\Gamma(x) \subseteq B, \tag{2.16}$$

and hence

$$\text{DNVVIS} = \bigcap\nolimits_{x\in K}\Gamma(x) = \bigcap\nolimits_{x\in K}(\Gamma(x) \cap B), \tag{2.17}$$

which is closed in B (by (i)), and so a compact subset of B. □

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