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Research Article

Random Trigonometric Polynomials with Nonidentically Distributed Coefficients

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This paper provides asymptotic estimates for the expected number of real zeros of two different forms of random trigonometric polynomials, where the coefficients of polynomials are normally distributed random variables with different means and variances. For the polynomials in the form of $a_0+a_1\cos\theta+a_2\cos 2\theta+\cdots+a_n\cos n\theta$ and $a_0+a_1\cos\theta+b_1\sin\theta+a_2\cos 2\theta+b_2\sin 2\theta+\cdots+a_n\cos n\theta+b_n\sin n\theta$, we give a closed form for the above expected value. With some mild assumptions on the coefficients we allow the means and variances of the coefficients to differ from each others. A case of reciprocal random polynomials for both above cases is studied.

1. Introduction

There are mainly two different forms of random trigonometric polynomial previously studied. They are

$$T(\theta) = \sum_{j=0}^{n} a_j \cos j\theta,$$
$$D(\theta) = \sum_{j=0}^{n} \left(a_j \cos j\theta + b_j \sin j\theta\right). \tag{1.1}$$

Dunnage [1] first studied the classical random trigonometric polynomial $T(\theta)$. He showed that in the case of identically and normally distributed coefficients $\{a_j\}_{j=0}^n$ with $\mu_j \equiv 0$ and $\sigma_j^2 \equiv 1$, $j = 0,1,2,\ldots,n$, the number of real zeros in the interval $(0,2\pi)$, outside of an exceptional set of measure zero, is $2n/\sqrt{3} + O\{n^{11/13}(\log n)^{3/13}\}$, when n is large. Subsequent papers mostly assumed an identical distribution for the coefficients and obtained $2n/\sqrt{3}$ as the asymptotic formula for the expected number of real zeros. In [2–4] it is shown that this asymptotic formula remains valid when the expected number of real zeros of the equation $T(\theta) = K$, known as K-level crossing, is considered. The work of Sambandham and Renganathan [5] and Farahmand [6] among others obtained this result for different assumptions on the distribution of the coefficients. Earlier works on random polynomials have been reviewed in Bharucha-Reid and Sambandham [7], which includes a comprehensive reference.

Later Farahmand and Sambandham [8] study a case of coefficients with different means and variances, which shows an interesting result for the expected number of level crossings in the interval $(0, 2\pi)$. Based on this work, we study the following two cases in order to better understand how the behavior of random trigonometric polynomials is affected by the different assumptions of the distribution on the coefficients for both $T(\theta)$ and $D(\theta)$, defined above.

To this end we allow all the coefficients to have different means and variances. Also, motivated by the recent developments on random reciprocal polynomials, we assume the coefficients a_j and a_{n-j} have the same distribution. In [9] for the case of random algebraic polynomial $a_j \equiv a_{n-j}$ is assumed. Further in order to overcome the analysis we have to make the following assumptions on the means and variances. Let $\max\{\sigma_j^2\} = \sigma_n^{*2} = O(n^{2/3})$ and also $|\sigma_j^2 - \sigma_{j+1}^2| \le \sigma_n^{*2}/n$. For the means, we assume $\max\{|\mu_j|\} = \mu_n^* = O(\sqrt{n})$ and $|\mu_j - \mu_{j+2}| \le \mu_n^*/n$. We also need $\sigma_n^{*2}, \sigma_{*n}^2$, where ${\sigma_{*n}}^2 = \min\{\sigma_j^2\}$ and μ_n^* is chosen such that for any positive constant δ, $(\sigma_n^{*3} n^{\delta-1}/\sigma_{*n}^3) \to 0$ and $(\mu_n^* n^{\delta-1/2}/\sigma_{*n}) \to 0$ as $n \to \infty$. Then for σ_{*n}^2 finite, we have the following theorem.

Theorem 1.1. *If the coefficients a_j, $j = 1, \ldots, n$ of $T(\theta) = \sum_{j=0}^{n} a_j \cos j\theta$ are normally distributed with mean μ_j and variance σ_j^2, where $\sigma_j^2 = \sigma_{n-j}^2$, then the mathematical expectation of the number of real zeros of the $T(\theta)$ satisfies*

$$EN_T(0, 2\pi) = 2\sqrt{\frac{\sum_{j=0}^{(n-1)/2} \sigma_j^2 \left(n^2/2 - nj + j^2\right)}{\sum_{j=0}^{(n-1)/2} \sigma_j^2}}. \tag{1.2}$$

We study the case of $D(\theta)$ in Theorem 3.1 later. We first give some necessary identities.

2. Preliminary Analysis

In order to be able to prove the theorem, we need to define some auxiliary results. Let

$$\begin{gathered} A^2 = \operatorname{var}\{T(\theta)\}, \qquad B^2 = \operatorname{var}\{T'(\theta)\}, \\ C = \operatorname{cov}\{T(\theta), T'(\theta)\}, \qquad \Delta = A^2B^2 - C^2, \\ \alpha = E\{T(\theta)\}, \qquad \beta = E\{T'(\theta)\}. \end{gathered} \tag{2.1}$$

Then from Farahmand [10, page 43], we have the extension of the Kac-Rice formula for our case as

$$EN(a, b) = I_1(a, b) + I_2(a, b), \tag{2.2}$$

where

$$I_1(a,b) = \int_a^b \frac{\Delta}{\pi A^2} \exp\left(-\frac{\alpha^2 B^2 + \beta^2 A^2 - 2\alpha\beta C}{2\Delta^2}\right) d\theta,$$
$$I_2(a,b) = \int_a^b \frac{\sqrt{2}\left|\beta A^2 - C\alpha\right|}{\pi A^3} \exp\left(-\frac{\alpha^2}{2A^2}\right) \operatorname{erf}\left(\frac{\left|\beta A^2 - C\alpha\right|}{\sqrt{2} A\Delta}\right) d\theta. \tag{2.3}$$

As usual, erf $(|\beta A^2 - C\alpha|/\sqrt{2}A\Delta)$ is the function defined as

$$\operatorname{erf}(x) = \int_0^x \exp\left(-t^2\right) dt = \sqrt{\pi}\Phi\left(x\sqrt{2}\right) - \frac{\sqrt{\pi}}{2}. \tag{2.4}$$

Now we are going to define the following functions to make the estimations. At first, we define $S(\theta) = \sin(2n+1)\theta / \sin\theta$ and to be continuous at $\theta = j\pi$ see also [10, page 74]. Let ϵ be any positive value arbitrary at this point, to be defined later. Since for $\theta \in (\varepsilon, \pi - \varepsilon), (\pi + \varepsilon, 2\pi - \varepsilon)$, we have $|S(\theta)| < 1/\sin\varepsilon$, we can obtain

$$S(\theta) = O(1/\varepsilon). \tag{2.5}$$

Furthermore,

$$S'(\theta) = \frac{(2n+1)\cos(2n+1)\theta}{\sin\theta} - \cot\theta S(\theta) = O\left(\frac{n}{\varepsilon}\right),$$
$$\begin{aligned} S''(\theta) = {} & \frac{-(2n+1)^2 \sin(2n+1)\theta}{\sin\theta} \\ & - \frac{(2n+1)\cos\theta\cos(2n+1)\theta}{\sin^2\theta} \\ & - \cot\theta S'(\theta) + \csc^2\theta S(\theta) \\ = {} & O\left(\frac{n^2}{\varepsilon}\right). \end{aligned} \tag{2.6}$$

Now using the above identities and by expanding $\sin\theta(1 + 2\sum_{j=1}^n \cos 2j\theta)$, we can show

$$\sum_{j=1}^n \cos 2j\theta = \frac{\sin(2n+1)\theta}{2\sin\theta} - \frac{1}{2} = \frac{S(\theta) - 1}{2} = O\left(\frac{1}{\varepsilon}\right),$$
$$\sum_{j=1}^n j\sin 2j\theta = -\frac{1}{2}\left\{\sum_{j=1}^n \cos 2j\theta\right\}' = -\frac{1}{4}S'(\theta) = O\left(\frac{n}{\varepsilon}\right), \tag{2.7}$$
$$\sum_{j=1}^n j^2 \cos 2j\theta = -\frac{1}{4}\left\{\sum_{j=1}^n \cos 2j\theta\right\}'' = -\frac{1}{8}S''(\theta) = O\left(\frac{n^2}{\varepsilon}\right).$$

In a similar way to [10], we define $Q(\theta) = \cos\theta - \cos(2n+1)\theta/2\sin\theta$, then for $\theta \in (\varepsilon, \pi - \varepsilon)$, $(\pi+\varepsilon, 2\pi-\varepsilon)$, since $\cos\theta - \cos(2n+1)\theta = 2\sin(n+1)\theta\sin n\theta$, we have $|Q(\theta)| < 1/\sin\varepsilon$. Hence, we can obtain

$$Q(\theta) = O\left(\frac{1}{\varepsilon}\right). \tag{2.8}$$

Furthermore, we have

$$\begin{aligned} Q'(\theta) &= \frac{-\sin\theta + (2n+1)\sin(2n+1)\theta}{2\sin\theta} - \cot\theta Q(\theta) \\ &= O\left(\frac{n}{\varepsilon}\right), \\ Q''(\theta) &= \frac{-\cos\theta + (2n+1)^2\cos(2n+1)\theta}{2\sin\theta} \\ &\quad + \frac{\cos\theta[\sin\theta - (2n+1)\sin(2n+1)\theta]}{2\sin^2\theta} \\ &\quad - \cot\theta Q'(\theta) + \csc^2\theta Q(\theta) \\ &= O\left(\frac{n^2}{\varepsilon}\right). \end{aligned} \tag{2.9}$$

Now using these identities for $Q(\theta)$, $Q'(\theta)$, and $Q''(\theta)$ and by expanding $2\sin\theta\sum_{j=1}^{n}\sin 2j\theta$, we can get a series of the following results:

$$\begin{aligned} \sum_{j=1}^{n}\sin 2j\theta &= \frac{\cos\theta - \cos(2n+1)\theta}{2\sin\theta} = Q(\theta) = O\left(\frac{1}{\varepsilon}\right), \\ \sum_{j=1}^{n} j\cos 2j\theta &= \frac{1}{2}\left\{\sum_{j=1}^{n}\sin 2j\theta\right\}' = \frac{1}{2}Q'(\theta) = O\left(\frac{n}{\varepsilon}\right), \\ \sum_{j=1}^{n} j^2\sin 2j\theta &= -\frac{1}{4}\left\{\sum_{j=1}^{n}\sin 2j\theta\right\}'' = -\frac{1}{4}Q''(\theta) = O\left(\frac{n^2}{\varepsilon}\right). \end{aligned} \tag{2.10}$$

Now we are in position to give a proof for Theorem 1.1 for $T(\theta)$ in the intervals $(\epsilon, \pi-\epsilon)$ and $(\pi+\epsilon, 2\pi-\epsilon)$. In order to avoid duplication the remaining intervals for both cases of $T(\theta)$ and $D(\theta)$ are discussed together later.

3. The Proof

Case 1. Here we study the random trigonometric polynomial in the classical form of $T(\theta) = \sum_{j=0}^{n} a_j \cos j\theta$ as assumed in Theorem 1.1 and prove the theorem in this section. To this end, we have to get all the terms in the Kac-Rice formula, such as $A^2, B^2, C, \alpha,$ and β. Since the property $\sigma_j^2 = \sigma_{n-j}^2$, using the results obtained in Section 2 of (2.7) and (2.10), we can have all the terms needed to calculate formula (2.2).At first, we get the variance of the polynomial, that is,

$$\begin{aligned} A_T^2 &= \sum_{j=0}^{(n-1)/2} \operatorname{var}\{a_j\}\left\{\cos^2 j\theta + \cos^2(n-j)\theta\right\} \\ &= \sum_{j=0}^{(n-1)/2} \sigma_j^2 + \cos n\theta \sum_{j=0}^{(n-1)/2} \sigma_j^2\{\cos(n-2j)\theta\} \\ &= \sum_{j=0}^{(n-1)/2} \sigma_j^2 + O\left(\frac{\sigma_n^{*2}}{\varepsilon}\right). \end{aligned} \tag{3.1}$$

Next, we calculate the variance of its derivative $T'(\theta)$ with respect to θ:

$$\begin{aligned} B_T^2 &= \sum_{j=0}^{(n-1)/2} \operatorname{var}\{a_j\}\left\{j^2\sin^2 j\theta + (n-j)^2\sin^2(n-j)\theta\right\} \\ &= \sum_{j=0}^{(n-1)/2} \sigma_j^2\left(\frac{n^2}{2} - nj + j^2\right) - \sum_{j=0}^{(n-1)/2} \sigma_j^2 \frac{j^2}{2}\cos 2j\theta \\ &\quad - \sum_{j=0}^{(n-1)/2} \sigma_j^2\left(\frac{n^2}{2} - nj + j^2\right)\cos 2(n-j)\theta \\ &= \sum_{j=0}^{(n-1)/2} \sigma_j^2\left(\frac{n^2}{2} - nj + j^2\right) + O\left(\frac{n^2\sigma_n^{*2}}{\varepsilon}\right). \end{aligned} \tag{3.2}$$

At last, it turns to the covariance between the polynomial and its derivative

$$\begin{aligned} C_T &= \sum_{j=0}^{(n-1)/2} - j\sigma_j^2 \sin j\theta \cos j\theta \\ &= -\sum_{j=0}^{(n-1)/2} \frac{j}{2}\sigma_j^2\{\sin 2j\theta - \sin 2(n-j)\theta\} - \frac{n}{2}\sum_{j=0}^{(n-1)/2} \sigma_j^2 \sin 2(n-j)\theta \\ &= O\left(\frac{n\sigma_n^{*2}}{\varepsilon}\right). \end{aligned} \tag{3.3}$$

Then, from (3.1), (3.2), and (3.3), we can get

$$\Delta_T^2 = A_T^2 B_T^2 - C_T^2 = \sum_{j=0}^{(n-1)/2} \sigma_j^2 \sum_{j=0}^{(n-1)/2} \sigma_j^2\left(\frac{n^2}{2} - nj + j^2\right) + O\left(\frac{n^2\sigma_n^{*2}}{\varepsilon}\right). \tag{3.4}$$

It is also easy to obtain the means of $T(\theta)$ and its derivative as

$$\begin{aligned}\alpha_T &= \sum_{j=0}^{n} \mu_j \cos j\theta = O\left(\frac{\mu_n^*}{\varepsilon}\right),\\ \beta_T &= -\sum_{j=0}^{n} j\mu_j \sin j\theta = O\left(\frac{n\mu_n^*}{\varepsilon}\right).\end{aligned} \tag{3.5}$$

From (2.3) and the results of (3.1)–(3.5), we therefore have

$$\begin{aligned}EN_T(\varepsilon, \pi-\varepsilon) &\sim \int_{\varepsilon}^{\pi-\varepsilon} \frac{\Delta}{\pi A^2}\, d\theta\\ &\sim \int_{\varepsilon}^{\pi-\varepsilon} \sqrt{\frac{\sum_{j=0}^{(n-1)/2} \sigma_j^2 \left(n^2/2 - nj + j^2\right)}{\pi^2 \sum_{j=0}^{(n-1)/2} \sigma_j^2}}\, d\theta\\ &= \sqrt{\frac{\sum_{j=0}^{(n-1)/2} \sigma_j^2 \left(n^2/2 - nj + j^2\right)}{\sum_{j=0}^{(n-1)/2} \sigma_j^2}}.\end{aligned} \tag{3.6}$$

Now before considering the zeros in the small interval of length ε we consider the polynomial $D(\theta)$.

Case 2. We have to make the assumptions a little different. In this case let $\max|\{\sigma_{aj}^2\} - \{\sigma_{bj}^2\}| = \sigma_n^{*2}$ and $|\sigma_{aj}^2 + \sigma_{bj}^2 - \sigma_{aj+1}^2 - \sigma_{bj+1}^2| \le \sigma_n^{*2}/n$. For the means, we assume $\max\{\mu_{aj}\} = \mu_{an}^*$ and $|\mu_{aj} - \mu_{aj+2}| \le \mu_{an}^*/n$, $\max\{\mu_{bj}\} = \mu_{bn}^*$ and $|\mu_{bj} - \mu_{bj+2}| \le \mu_{bn}^*/n$, and $\max|\{\mu_{aj}\} - \{\mu_{bj}\}| = \mu_n^*$.

Theorem 3.1. *Consider the polynomial $D(\theta) = \sum_{j=0}^{n} a_j \cos j\theta + b_j \sin j\theta$, where a_j, b_j are independent, normally distributed random variables, divided into n groups each with its own mean μ_{aj}, μ_{bj} and variance $\sigma_{aj}^2, \sigma_{bj}^2$, $j = 1, 2, \ldots, n$,. The expected number of real zeros of $D(\theta)$ satisfies*

$$EN_D(0, 2\pi) = 2\sqrt{\frac{\sum_{j=0}^{n} j^2 \left\{\sigma_{aj}^2 + \sigma_{bj}^2\right\}}{\sum_{j=0}^{n} \left\{\sigma_{aj}^2 + \sigma_{bj}^2\right\}}}. \tag{3.7}$$

Similarly, using the same results obtained from Section 2, we can get the following terms. At first, we get the means of the polynomial and its derivative separately

$$\begin{aligned}\alpha_D &= \sum_{j=0}^{n} \mu_{aj} \cos j\theta + \sum_{j=0}^{n} \mu_{bj} \sin j\theta = O\left(\frac{\mu_{an}^* + \mu_{bn}^*}{\varepsilon}\right),\\ \beta_D &= -\sum_{j=0}^{n} j\mu_{aj} \sin j\theta + \sum_{j=0}^{n} j\mu_{bj} \cos j\theta = O\left(\frac{n\mu_n^*}{\varepsilon}\right).\end{aligned} \tag{3.8}$$

Then we obtain the variance of the polynomial

$$A_D^2 = \operatorname{var}\left\{\sum_{j=0}^{n} a_j \cos j\theta + b_j \sin j\theta\right\}$$
$$= \frac{1}{2}\sum_{j=0}^{n}\left\{\sigma_{aj}^2 + \sigma_{bj}^2\right\} + \frac{1}{2}\sum_{j=0}^{n}\left\{\sigma_{aj}^2 - \sigma_{bj}^2\right\}\cos 2j\theta \tag{3.9}$$
$$= \frac{1}{2}\sum_{j=0}^{n}\left\{\sigma_{aj}^2 + \sigma_{bj}^2\right\} + O\left(\frac{\sigma_n^{*2}}{\varepsilon}\right).$$

Next, we calculate the variance of its derivative with respect to θ:

$$B_D^2 = \operatorname{var}\left\{\sum_{j=0}^{n} - ja_j \sin j\theta + b_j \cos j\theta\right\}$$
$$= \frac{1}{2}\sum_{j=0}^{n} j^2\left\{\sigma_{aj}^2 + \sigma_{bj}^2\right\} + \frac{1}{2}\sum_{j=0}^{n} j^2\left\{\sigma_{bj}^2 - \sigma_{aj}^2\right\}\cos 2j\theta \tag{3.10}$$
$$= \frac{1}{2}\sum_{j=0}^{n} j^2\left\{\sigma_{aj}^2 + \sigma_{bj}^2\right\} + O\left(\frac{n^2\sigma_n^{*2}}{\varepsilon}\right).$$

At last, it turns to the covariance between the polynomial and its derivative:

$$C_D = -E\left\{a_j^2\right\} j \sin j\theta \cos j\theta + E\left\{b_j^2\right\} j \sin j\theta \cos j\theta$$
$$+ E^2\{a_j\} j \sin j\theta \cos j\theta - E^2\{b_j\} j \sin j\theta \cos j\theta$$
$$= \sum_{j=0}^{n} - j\left\{\sigma_{bj}^2 - \sigma_{aj}^2\right\} \sin j\theta \cos j\theta \tag{3.11}$$
$$= O\left(\frac{n\sigma_n^{*2}}{\varepsilon}\right).$$

Then, from (3.9), (3.10), and (3.11), we can get

$$\Delta_D^2 = \frac{1}{4}\sum_{j=0}^{n}\left\{\sigma_{aj}^2 + \sigma_{bj}^2\right\}\sum_{j=0}^{n} j^2\left\{\sigma_{aj}^2 + \sigma_{bj}^2\right\} + O\left(\frac{n}{\varepsilon}\right). \tag{3.12}$$

From (2.3) and (3.8)–(3.12), we therefore have

$$EN_D(\epsilon, \pi-\epsilon) \sim \int_{\varepsilon}^{\pi-\varepsilon} \frac{\Delta}{\pi A^2}\,d\theta \sim \int_{\varepsilon}^{\pi-\varepsilon} \sqrt{\frac{\sum_{j=0}^{n} j^2\left\{\sigma_{aj}^2+\sigma_{bj}^2\right\}}{\pi^2\sum_{j=0}^{n}\left\{\sigma_{aj}^2+\sigma_{bj}^2\right\}}}\,d\theta = \sqrt{\frac{\sum_{j=0}^{n} j^2\left\{\sigma_{aj}^2+\sigma_{bj}^2\right\}}{\sum_{j=0}^{n}\left\{\sigma_{aj}^2+\sigma_{bj}^2\right\}}}. \tag{3.13}$$

This is the main contribution to the number of real zeros. In the following we show there is a negligible number of zeros in the remaining intervals of length ϵ. For the number of real roots in the interval $(0,\varepsilon)$, $(2\pi-\epsilon, 2\pi)$ or $(\pi-\varepsilon, \pi+\varepsilon)$, we use Jensen's theorem [11, page 300]. The method we used here is applicable to both of the cases we discussed above. Here we take the first case as the example to prove that the roots of these intervals are negligible. Let $m=\sum_{j=0}^{n}\mu_j$ and $s^2=\sum_{j=0}^{n}\sigma_j^2$. As $T(0)$ is normally distributed with mean m and variance s^2, for any constant k

$$\Pr\left(-n^{-k}\le D(0)\le n^{-k}\right) = \left(2\pi s^2\right)^{-1/2}\int_{-n^{-k}}^{n^{-k}} \exp\left\{-\frac{(t-m)^2}{2s^2}\right\}dt < \frac{2}{n^k\sqrt{\pi s^2}}. \tag{3.14}$$

Also since $|\cos(2\varepsilon e^{i\theta})|\le 2e^{2n\varepsilon}$ we have

$$D\left(2\varepsilon e^{i\theta}\right) \le 2e^{2n\varepsilon}\sum_{j=1}^{n}|a_j| \le 2(n+1)e^{2n\varepsilon}\max_{j=1,\dots,n}|a_j|. \tag{3.15}$$

Now by Chebyshev's inequality, for any $k>2$, we can find a positive constant d such that for $0\le j\le n$,

$$\Pr\left(|a_j|\ge n^{-k}\right) \le \frac{E\left(a_j^2\right)}{n^{2k}} \le \frac{d}{n^k}. \tag{3.16}$$

since $\mu_n^{*2}+\sigma_n^{*2}\le n^k$. Therefore, except for sample functions in an ω-set of measures not exceeding dn^-k,

$$T\left(2\varepsilon e^{i\theta}\right) \le 3n^{1+2k}\exp(2n\varepsilon). \tag{3.17}$$

So we obtain

$$\left|\frac{T(2\varepsilon e^{i\theta},\omega)}{T(0,\omega)}\right| \le 3n^{1+2k}\exp(2n\varepsilon), \tag{3.18}$$

except for the sample functions in an ω-set of measure not exceeding $dn^{-k}+2/n^k\sqrt{\pi s^2}$.

This implies that we can find an absolute constant d' such that

$$\Pr\{N(\varepsilon) > 2n\varepsilon + (1+2k)\log n + \log 3\} \le n^{-k}\left(d+\sqrt{\frac{2}{s^2\sqrt{\pi}}}\right) \le d'n^{-k}. \tag{3.19}$$

Let $[3n\varepsilon]$ be the greatest integer less than or equal to $3n\varepsilon$. Then since the number of real zeros of $D(\theta)$ is at most $2n$ we have

$$\begin{aligned} EN(\varepsilon) &= \sum_{j>0}^{2n}\Pr\{N(\varepsilon)\ge j\} \\ &\le 3n\varepsilon + \sum_{j=[3n\varepsilon]+1}^{2n}\Pr\{N(\varepsilon) > j\} \\ &\le 3n\varepsilon + d'n^{1-k} = O(n\varepsilon), \end{aligned} \tag{3.20}$$

where we choose

$$\epsilon = \max\left\{\frac{\mu_n^*}{\sigma_{*n}n^{1/2-\delta}}, \frac{\sigma_n^{*3}}{\sigma_{*n}^3 n^{1-\delta}}\right\}, \tag{3.21}$$

and δ is any positive number, so the error terms become small and we can prove the theorem.

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