ON THE NONEXISTENCE OF A LAW OF THE ITERATED LOGARITHM FOR WEIGHTED SUMS OF IDENTICALLY DISTRIBUTED RANDOM VARIABLES[1]

André Adler

Illinois Institute of Technology
Chicago, IL 60616

ABSTRACT

For weighted sums of independent and identically distributed random variables, conditions are placed under which a generalized law of the iterated logarithm cannot hold, thereby extending the usual nonweighted situation.

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1. INTRODUCTION.

Heyde [1] established the fact that partial sums of independent and identically distributed (i.i.d.) random variables $\{X, X_n,\ n \geq 1\}$ whose common distribution is of the form $P\{|X| > x\} = L(x)x^{-\alpha}$ $(0 \leq \alpha < 2,\ \alpha \neq 1)$, where $L(x)$ is slowly varying at infinity and where $EX = 0$ if $E|X| < \infty$, cannot be normalized in the sense that there exist constants $0 < b_n \uparrow$ with $\sum_{k=1}^{n} X_k/b_n \to 1$ a.s. The purpose of this paper is to present similar results in the weighted case.

Herein, we define $S_n = \sum_{k=1}^{n} a_k X_k$ where $\{a_n,\ n \geq 1\}$ are constants and the random variables $\{X, X_n,\ n \geq 1\}$ are identically distributed with common distribution

$$P\{|X| > x\} = \begin{cases} L(x)x^{-\alpha} & x \geq 1, \\ 1 & x < 1, \end{cases}$$

where $L(cx)/L(x) \to 1$ as $x \to \infty$ for all $c > 0$, and $\alpha \geq 0$.

A remark about notation is needed. Throughout, the symbol C will denote a generic finite nonzero constant which is not necessarily the same in each appearance. Also, we let $c_n = b_n/|a_n|$, $n \geq 1$, where $\{b_n,\ n \geq 1\}$ is our norming sequence.

It should be noted that the techniques involved with the main results (Theorems 2 and 3) follow a similar pattern to those that can be found in Heyde [1]. As usual, via the Borel-Cantelli lemma, one need only consider a truncated version of the random variables $\{X_n,\ n \geq 1\}$. Instead of truncating X_n at b_n the trick, in the weighted case, is to cut off X_n at c_n. Then by classical arguments the remaining terms are shown to be almost surely negligible. Also of particular interest is the discussion (Section 3) of the $\alpha = 1$ situation.

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2. RESULTS.

Our first theorem examines what happens when $P\{|X_n| > c_n \ i.o.(n)\} = 1$.

Theorem 1. Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables. If $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ are constants satisfying $b_n = O(b_{n+1})$, $b_n \to \infty$, and $\sum_{n=1}^{\infty} P\{|X| > c_n\} = \infty$, then $\limsup_{n\to\infty} |S_n|/b_n = \infty$ a.s.

Proof. If $c_n \to \infty$, then for all large M

$$\begin{aligned}
\sum_{n=1}^{\infty} P\{|a_n X_n| > M b_n\} &= \sum_{n=1}^{\infty} L(Mc_n)(Mc_n)^{-\alpha} \\
&\geq C\sum_{n=1}^{\infty} L(c_n)c_n^{-\alpha} \\
&\geq C\sum_{n=n_0}^{\infty} P\{|X_n| > c_n\} \text{ (for a suitably chosen } n_0) \\
&= \infty.
\end{aligned}$$

Otherwise, if $\liminf_{n\to\infty} c_n < \infty$, then there exists a subsequence $\{n_k, k \geq 1\}$ and a finite constant B such that $c_{n_k} \leq B$. Hence for all $0 < M < \infty$

$$\begin{aligned}
\sum_{n=1}^{\infty} P\{|X| > Mc_n\} &\geq \sum_{k=1}^{\infty} P\{|X_n| > Mc_{n_k}\} \\
&\geq \sum_{k=1}^{\infty} P\{|X| > MB\} \\
&= \infty.
\end{aligned}$$

So in either case we conclude, via the Borel-Cantelli lemma, that

$$\limsup_{n\to\infty} \left|\frac{a_n X_n}{b_n}\right| = \infty \text{ a.s.}$$

Since

$$\left|\frac{a_n X_n}{b_n}\right| \leq \left|\frac{S_n}{b_n}\right| + \left|\frac{b_{n-1}}{b_n}\right| \cdot \left|\frac{S_{n-1}}{b_{n-1}}\right|$$

the conclusion follows. □

Note that in the next result independence is not necessary.

Theorem 2. Let $\{X, X_n, n \geq 1\}$ be identically distributed random variables. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be constants satisfying $0 < b_n \uparrow \infty$ and $\sum_{n=1}^{\infty} P\{|X| > c_n\} < \infty$. If $0 \leq \alpha < 1$, then $S_n/b_n \to 0$ a.s.

Proof. Notice, via the Borel-Cantelli lemma, that

$$\sum_{k=1}^{n} a_k X_k I(|X_k| > c_k) = o(b_n) \text{ a.s.}$$

Hence it remains to show that

$$\sum_{k=1}^{n} a_k X_k I(|X_k| \leq c_k) = o(b_n) \text{ a.s.} \tag{1}$$

Since, for all large k

$$\begin{aligned} E|X|I(|X| \le c_k) &\le \int_0^{c_k} P\{|X| > t\}dt \\ &= \int_0^1 dt + \int_1^{c_k} L(t)t^{-\alpha}dt \\ &\le CL(c_k)c_k^{-\alpha+1} \end{aligned}$$

(by Theorem 1b of Feller [2, p. 281]), it follows that

$$\begin{aligned} \sum_{k=1}^{\infty} c_k^{-1} E|X|I(|X| \le c_k) &\le C\sum_{k=1}^{\infty} L(c_k)c_k^{-\alpha} \\ &\le C\sum_{k=1}^{\infty} P\{|X| > c_k\} \\ &< \infty, \end{aligned}$$

whence

$$\sum_{k=1}^{\infty} c_k^{-1}|X_k|I(|X_k| \le c_k) < \infty \text{ a.s.}$$

This, via Kronecker's lemma, implies (1). □

Next, we examine the mean zero situation.

Theorem 3. Let $\{X, X_n, n \ge 1\}$ be i.i.d. mean zero random variables. Let $\{a_n, n \ge 1\}$ and $\{b_n, n \ge 1\}$ be constants satisfying $0 < b_n \uparrow \infty$ and $\sum_{n=1}^{\infty} P\{|X| > c_n\} < \infty$. If $1 < \alpha < 2$, then $S_n/b_n \to 0$ a.s.

Proof. Again, note that

$$\sum_{k=1}^{n} a_k X_k I(|X_k| > c_k) = o(b_n) \text{ a.s.}$$

Since

$$\begin{aligned} \sum_{k=1}^{n} a_k X_k &= \sum_{k=1}^{n} a_k[X_k I(|X_k| \le c_k) - EXI(|X| \le c_k)] \\ &\quad + \sum_{k=1}^{n} a_k EXI(|X| \le c_k) + \sum_{k=1}^{n} a_k X_k I(|X_k| > c_k) \end{aligned}$$

we need only show that the first two terms are $o(b_n)$. In view of the Khintchine-Kolmogorov convergence theorem and Kronecker's lemma, all that one needs to show, in order to prove that the first term is $o(b_n)$ a.s., is that

$$\sum_{k=1}^{\infty} c_k^{-2} EX^2 I(|X| \le c_k) < \infty. \tag{2}$$

By integration by parts and Theorem 1b of Feller [2, p. 281] we observe that

$$\sum_{k=1}^{\infty} c_k^{-2} EX^2 I(|X| \le c_k) \le 2\sum_{k=1}^{\infty} c_k^{-2} \int_0^{c_k} tP\{|X| > t\}dt$$

$$\begin{aligned} &\leq C\sum_{k=1}^{\infty} L(c_k)c_k^{-\alpha} \\ &\leq C\sum_{k=1}^{\infty} P\{|X|>c_k\} \\ &< \infty. \end{aligned}$$

Hence (2) holds. Finally, we need to show that

$$\sum_{k=1}^{n} a_k EXI(|X|\leq c_k) = o(b_n).$$

Due to the fact that $|EXI(|X|\leq c_k)| \leq E|X|I(|X|>c_k)$ it is sufficient to show that

$$\sum_{k=1}^{n} |a_k| E|X| I(|X|>c_k) = o(b_n). \tag{3}$$

However, since

$$\begin{aligned} \sum_{k=1}^{\infty} c_k^{-1}E|X|I(|X|>c_k) &= \sum_{k=1}^{\infty} P\{|X|>c_k\} + \sum_{k=1}^{\infty} c_k^{-1}\int_{c_k}^{\infty} P\{|X|>t\}dt \\ &\leq O(1) + C\sum_{k=1}^{\infty} c_k^{-1}\int_{c_k}^{\infty} L(t)t^{-\alpha}dt \\ &\leq O(1) + C\sum_{k=1}^{\infty} L(c_k)c_k^{-\alpha} \quad \text{(see Feller, [2, p.281])} \\ &\leq O(1) + C\sum_{k=1}^{\infty} P\{|X|>c_k\} \\ &= O(1), \end{aligned}$$

it is clear that (3) obtains. □

3. DISCUSSION.

In this section we combine the previous theorems. The conclusion is that for all $\alpha \in [0,1)\cup(1,2)$ a law of the iterated logarithm cannot hold.

Theorem 4. Let $\{X, X_n,\ n\geq 1\}$ be i.i.d. random variables with

$$P\{|X|>x\} = \begin{cases} L(x)x^{-\alpha} & x\geq 1, \\ 1 & x<1, \end{cases}$$

with $EX=0$ if $\alpha>1$. If $\{a_n,\ n\geq 1\}$ and $\{b_n,\ n\geq 1\}$ are constants with $0<b_n\uparrow\infty$, then for all $\alpha\in[0,1)\cup(1,2)$

$$\limsup_{n\to\infty}\left|\frac{\sum_{k=1}^{n} a_k X_k}{b_n}\right| = 0 \text{ or } \infty \text{ a.s.}$$

depending on whether $\sum_{n=1}^{\infty} P\{|X| > c_n\}$ converges or diverges.

Proof. In view of Theorems 1, 2, and 3 the conclusion is immediate. □

Now, clearly if a law of the iterated logarithm does not exist, then a strong law of large numbers (with limit one) is also not feasible.

Corollary. If the hypotheses of Theorem 4 hold, then

$$P\{\lim_{n\to\infty} \sum_{k=1}^{n} a_k X_k / b_n = 1\} = 0.$$

It is well known that if $\alpha > 2$, then a classical law of the iterated logarithm can be obtained provided suitable conditions are imposed on the constants $\{a_n, \ n \geq 1\}$. An interesting question is what happens when $\alpha = 1$. If we allow $\alpha = 1$, then not only can a law of the iterated logarithm obtain, but a strong law of large numbers can also occur where the limit is one. The following example is of the flavor of those that can be found in Adler [3].

Example. If $\{X_n, \ n \geq 1\}$ are i.i.d. random variables with common density $f(x) = x^{-2} I_{(1,\infty)}(x)$, $-\infty < x < \infty$, then

$$\frac{\sum_{k=1}^{n} \frac{2}{k} X_k}{(\log n)^2} \to 1 \ a.s.$$

Proof. Since

$$\sum_{n=1}^{\infty} P\{|X| > \frac{n(\log n)^2}{2}\} = 2 + \sum_{n=3}^{\infty} \frac{2}{n(\log n)^2} < \infty$$

and

$$\left[\frac{n(\log n)^2}{2}\right]^2 \sum_{j=n}^{\infty} \left[\frac{2}{j(\log j)^2}\right]^2 = O(n)$$

we have, by Theorem 1 of Adler and Rosalsky [4],

$$\frac{\sum_{k=1}^{n} \frac{2}{k}(X_k - \mu_k)}{(\log n)^2} \to 0 \ a.s.$$

where

$$\begin{aligned} \mu_n &= EXI(|X| \leq \frac{n(\log n)^2}{2}) \\ &= \int_1^{n(\log n)^2/2} x^{-1} dx \\ &\sim \log n. \end{aligned}$$

Noting that

$$\frac{\sum_{k=1}^{n} \frac{2}{k} \log k}{(\log n)^2} \to 1$$

the proof is complete. □

Here we exhibited a strong law in the nonintegrable case. One can obtain similar strong laws for mean zero random variables when $P\{|X| > x\} = L(x)/x$ (see, e.g., Adler and Rosalsky [5]).

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