

ANALYSIS OF STABILITY PROBLEMS VIA MATRIX LYAPUNOV FUNCTIONS¹

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ABSTRACT

The stability of nonlinear systems is analyzed by the direct Lyapunov's method in terms of Lyapunov matrix functions. The given paper surveys the main theorems on stability, asymptotic stability and nonstability. They are applied to systems of nonlinear equations, singularly-perturbed systems and hybrid systems. The results are demonstrated by an example of a two-component system.

Key words: Lyapunov Matrix Function, Stability, Asymptotic Stability, Nonlinear Systems.

AMS Subject Classification: 34D20, 34A34.

1. INTRODUCTION

In 1979, A. A. Martynyuk (see [32]) gave a description of two-indexed system of functions called *matrix Lyapunov function* (MLF). In later works, [17, 20, 22], some main concepts of MLF's were developed. As it is well-known in the theory of Lyapunov stability, the notion of definiteness of the scalar Lyapunov function V is of importance, for V is a positive measure in one of the variables. The definiteness remains essential also for MLF's.

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2. THE DEFINITENESS OF MATRIX-FUNCTIONS

Let $(\mathbb{R}^n, \|\cdot\|)$ be a real Euclidian normed space. Denote $B(\rho) = B(0, \rho) = \{x: \|x\| < \rho\}$ an open ball centered at the origin with radius ρ ; let $D = \overline{\mathbb{R}} \times B(\rho)$ be the Cartesian product of $\overline{\mathbb{R}} = [-\infty, +\infty]$ and $B(\rho)$. Moreover, we shall consider an open connected time-invariant neighborhood $N \subseteq \mathbb{R}^n$ of point $x=0$ and intervals $T_0 = \{t: t_0 \leq t < +\infty\}$, $t_0 \in T$, $T \subseteq \mathbb{R}$ and $T_\tau = \{t: \tau \leq t < +\infty\}$, $\tau \in \mathbb{R}$, where T is the set of initial values.

Consider a perturbed motion equation

$$\frac{dx}{dt} = f(t, x), \quad f(t, 0) = 0, \quad (2.1)$$

where $x \in \mathbb{R}^n$ and $f \in C(D, \mathbb{R}^n)$.

Suppose that a solution $\chi(t; t_0, x_0)$ of (2.1) is continuous for all $t \in T_0$ and $\chi(t_0; t_0, x_0) = x_0$.

Since the definitions of stability of the trivial solution of (2.1) with respect to T , attraction with respect to T and asymptotic stability with respect to T are given in [5, 7], and the same concepts relative to a part of variables are introduced in [16], we omit this notions here.

Matrix-Function (MF). Together with (2.1) we consider a two-indexed system of functions

$$U(t, x) = (u_{ij}(t, x)), \quad i, j \in \{1, \dots, m\}, \quad m > 1 \quad (2.2)$$

with $u_{ij} \in C(T_\tau \times B(\rho), \mathbb{R})$.

The notion of the definiteness of an auxiliary function (that is used in the direct Lyapunov's method) is a main one [15], since this behaves as a scalar function having all norm properties. The definiteness of the MF can be defined in terms of the definiteness of a corresponding scalar function whose construction is based on the MF.

By means of the matrix-function U , we can introduce the following scalar functions

- 1⁰. $U_0(t, x) = \max\{u_{ij}(t, x): 1 \leq i, j \leq m\}$;
- 2⁰. $U_0(t, x) = \phi^T U(t, x) \phi$, where $\phi \in \mathbb{R}_+^m$ or $\phi \in C[\mathbb{R}^n, \mathbb{R}_+^m]$, $\phi(0) = 0$;
- 3⁰. $U_0(t, x) = \sum_{i,j=1}^m u_{ij}(t, x)$.

Definition 2.1. Matrix-function $U: T_\tau \times \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$ is called

- 1) positive definite on T_τ iff there exists a time-invariant connected neighborhood $N \subseteq \mathbb{R}^n$ of point $x=0$, a positive definite function, $w: N \rightarrow \mathbb{R}_+$, and a vector $\phi \in \mathbb{R}_+^m$ or a vector-function $\phi \in C[\mathbb{R}^n, \mathbb{R}_+^m]$, $\phi(0) = 0$ such that
 - a) $U = C(T_\tau \times N, \mathbb{R}^{m \times m})$;

- b) $U(t, 0) = 0 \quad \forall t \in T_\tau$;
 c) function $U_0(t, x)$, defined by 1⁰, 2⁰ and 3⁰, satisfies the condition

$$U_0(t, x) \geq w(x) \quad \forall (t, x \neq 0) \in T_\tau \times \mathbb{R}^n.$$

- 2) positive definite on $T_\tau \times S$ iff the above conditions a) - c) are satisfied for $N=S$ (where $S \subseteq \mathbb{R}^n$ is an arbitrary set);
 3) positive definite in the whole on T_τ iff the conditions a) - c) are satisfied for $N=\mathbb{R}^n$;
 4) negative definite (in the whole) on $T_\tau \times N$ (on T_τ) iff $(-U)$ is positive definite (in the whole) on $T_\tau \times N$ (on T_τ).

Remark 1. The expression “on T_τ ” in Definition 2.1 will be dropped if all above requirements are satisfied for every $\tau \in \mathbb{R}$.

Remark 2. If the (i, j) -th element $u_{ij}: \mathbb{R}^n \rightarrow \mathbb{R}$ does not depend on $t \in T_\tau$, the conditions of Definition 2.1 are simplified, in particular, condition 1 c) is reduced to

$$U_0(x) > 0 \quad \forall (x \neq 0) \in \mathbb{R}^n, (\phi \neq 0) \in \mathbb{R}^m(\mathbb{R}_+^m).$$

Remark 3. The definition of the semidefiniteness of the MF is formulated via Definition 2.1 and Definition 10 from [7, p. 23].

Remark 4. In the case when $m=1$ and $T_\tau=\mathbb{R}$, Definition 2.1 is reduced to the original definitions of positive semidefiniteness (in the whole) introduced earlier by Lyapunov [15] and Barbashin/Krassovsky [11].

Definition 2.2. Matrix-function $U: T_\tau \times \mathbb{R}^n \rightarrow \mathbb{R}^m \times m$ is called

- 1) decreasing on T_τ , $\tau \in \mathbb{R}$, iff there exist a time-invariant neighborhood $N \subseteq \mathbb{R}^n$ of point $x=0$, a positive definite function $u: N \rightarrow \mathbb{R}$, and a vector $\phi \in \mathbb{R}_+^m$ such that the conditions 1 a) and 1 b) of Definition 2.1 are satisfied and

$$U_0(t, x) \leq u(x) \quad \forall (x \neq 0) \in \mathbb{R}^n, t \in T_\tau;$$

- 2) decreasing on $T_\tau \times S$, iff the above conditions 1) hold for $N=S$ ($S \subseteq \mathbb{R}^n$);
 3) decreasing in the whole on T_τ if the conditions of 1) hold for $N=\mathbb{R}^n$.

3. MATRIX LYAPUNOV FUNCTION

First observe that the definiteness of MF's [17] was based on a generalization of the so-called “maximum function” introduced by LaSalle [14] for the case of vector functions. Later [18, 30], the concept of the definiteness has been modified, however no formal definition was introduced. We now introduce the definition of MLF's based on the definiteness of the MF from

Section 2 (whose special case agrees with the classical definiteness via Lyapunov [15] (cf., [11, 12, 38]).

We define the total derivatives of the matrix-function (2.2) by virtue of system (2.1) as

$$D^+ U(t, x) = (D^+ u_{ij}(t, x); i, j \in \{1, \dots, m\}), \quad (3.1)$$

where

$$D^+ u_{ij}(t, x) = \limsup_{\theta \rightarrow 0^+} \{ [u_{ij}(t+\theta, x+\theta f(t, x)) - u_{ij}(t, x)] \theta^{-1} \};$$

$$D_+ U(t, x) = (D_+ u_{ij}(t, x); i, j \in \{1, \dots, m\}), \quad (3.2)$$

where

$$D_+ u_{ij}(t, x) = \liminf_{\theta \rightarrow 0^+} \{ [u_{ij}(t+\theta, x+\theta f(t, x)) - u_{ij}(t, x)] \theta^{-1} \}.$$

Unless specified, $D^* U$ is either $D^+ U(t, x)$ or $D_+ U(t, x)$.

Definition 3.1. The two-indexed system of functions (2.2) is called a *matrix Lyapunov function* for system (2.1) if on $T_\tau \times N$

- a) the function U is definite;
- b) $D^* U$ is semidefinite and has either the same sign as U or the opposite sign of U .

It is clear that the function U (cf. (2.2)) and its total derivative in terms of (2.1) solves the stability problem of state $x=0$ of equation (2.1).

Definition 3.1 can be specified for various dynamic properties of state $x=0$. Denote $S(\phi)$ the set of MLF's that solve the stability problem.

Definition 3.2. A MF $U: T_\tau \times \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$ is called a MLF of $S(\phi)$ -type on $T_\tau \times N$ if

- 1) U is positive definite (decreasing),
- 2) $D^+ U$ is nonpositive definite and $D^+ U(t, 0) = 0 \forall t \in T_\tau$.

Let $AS(\phi)$ be the set of MLF solving the problem of asymptotic stability of state $x=0$.

Definition 3.3. A MF $U: T_\tau \times \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$ is called a MLF of $AS(\phi)$ -type on $T_\tau \times N$ if

- 1) U is positive definite (decreasing),
- 2) $D^+ U$ is negative definite and

$$D^+ U(t, 0) = 0 \quad \forall t \in T_\tau.$$

Let $NS(\phi)$ be the set of MF's solving the problem of instability of state $x=0$.

Definition 3.4. A MF $U: T_\tau \times \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$ is called a Lyapunov-Chetayev MF of $NS(\phi)$ -type if there exist a $t_0 \in T$, a $\Delta > 0$, such that $\bar{B}(\Delta) \subset N$ and an open set $W \subset B(\Delta)$ such that on $T_0 \times W$

- a) $0 < U(t, x) \leq C$, where C is an $m \times m$ real-valued matrix;
- b) $\phi^T D^+ U(t, x) \phi \geq a(v(t, x, \phi))$, a is an element of Hahn class K
- c) $0 \in \partial W$;
- d) $U=0$ on $T_0 \times (\partial W \cap B(\Delta))$.

4. THEOREMS OF DIRECT LYAPUNOV METHOD IN TERMS OF MATRIX-FUNCTIONS³

The meaning of the introduced definition of MLF's is revealed in view of the following classical results of stability theory of motion.

Theorem 4.1. *In order that the trivial solution of the equation (2.1) to be (uniformly) stable, it is necessary and sufficient that for (2.1) there exists a MLF $U: T_\tau \times \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$ of $S(\phi)$ -type for any positive integer m .*

Theorem 4.2. *In order that the trivial solution of the equation (2.1) to be (uniformly) asymptotically stable, it is necessary and sufficient that for (2.1) there exists a MLF $U: T_\tau \times \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$ of $AS(\phi)$ -type for any positive integer m .*

Theorem 4.3. *In order that the trivial solution of the equation (2.1) be unstable it is necessary and sufficient that for (2.1) there exists a Lyapunov-Chetayev MF $U: T_\tau \times \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$ of $NS(\phi)$ -type for any positive integer m .*

Remark 5. For $m=1$ Theorems 4.1 - 4.3 are reduced to the classical results on stability, asymptotic stability and non-stability obtained in ([11, 12, 38]). For autonomous systems results similar to Theorems 4.1 and 4.2 are given in [6], in terms of the MF (2.2) with a given vector $\phi \triangleq (b_i \neq 0, i=1, \dots, m)$ (see Remark 2).

5. METHODS OF CONSTRUCTION OF MLFs

We consider time-invariant nonlinear system

³The proofs to Theorem 4.1-4.3 are not yet published and are contained in a monograph by the author that will appear later.

$$\frac{dx}{dt} = f(x), \tag{5.1}$$

where $x \in \mathbb{R}^n$ and $f \in C(\mathbb{R}^n, \mathbb{R}^n)$ is subject to physical or mathematical decomposition into m interconnected subsystems

$$\frac{dx_i}{dt} = g_i(x_i) + h_i(x) \triangleq f_i(x), \quad i=1, 2, \dots, m. \tag{5.2}$$

Here $x_i \in \mathbb{R}^{n_i}$, $x = (x_1^T, x_2^T, \dots, x_m^T)^T$, $f = (f_1^T, f_2^T, \dots, f_m^T)^T$, $t \in \mathbb{R}$, $t_0 = 0$. The free subsystems corresponding to the system (5.2), contain the main information on dynamic properties of the system (5.2):

$$\frac{dx_i}{dt} = g_i(x_i), \quad i \in \{1, \dots, m\} \tag{5.3}$$

Assumption 5.1. There exist open connected neighborhoods N_i of states $x_i = 0$ $\forall i \in \{1, \dots, m\}$ such that

a) motions $x_i(t, x_{i0})$ of subsystems (5.3) are continuous on

$$\mathbb{R}_+ \times N_i \text{ and } x_i(0, x_{i0}) = 0 \quad \forall i \in \{1, \dots, m\};$$

b) motion $x(t, x_0)$ of system (5.1) is continuous on $\mathbb{R}_+ \times N$, where $N = N_1 \times \dots \times N_m$ and $x(0, x_0) = 0$.

The decomposition of system (5.1) into interconnected subsystems (5.2) is called the *first level decomposition*, which corresponds to the following approach of the construction of the MLF.

In matrix function (2.2) the diagonal elements $v_{ii}(x)$, are subject to the dynamical properties of the free subsystems (5.3) and the non-diagonal elements $v_{ij}(x_i, x_j)$ are constructed in response with interconnections $h_i(x)$, $\forall i, j \in \{1, \dots, m\}$. Therefore, the MLF for the system (5.2) has the form

$$U(x) = \begin{bmatrix} v_{11}(x_1) & v_{12}(x_1, x_2), \dots, v_{1m}(x_1, x_m) \\ \cdot & \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ v_{m1}(x_m, x_1) & v_{m2}(x_m, x_2), \dots, v_{mm}(x_m) \end{bmatrix} \tag{5.4}$$

The definiteness of this MLF is defined according to Definitions 2.1 and 2.2 and Remark 2.

It is possible by the construction of a vector-function to implement the interconnections h_i , $i=1, \dots, m$. However, even in this case, the use of a MF improves the estimate of the region of

asymptotic stability [cf., 36].

Example 1. (See [3]). Let $x \in \mathbb{R}^n$ and let system (5.1) be linear. Then, setting $x = (x_1^T, x_2^T)^T$, where $x_i \in \mathbb{R}^{n_i}$, $n_1 + n_2 = n$, we have

$$\frac{dx}{dt} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x.$$

The elements of the matrix U are taken in form

$$v_{ii}(x_i) = x_i^T C_{ii} x_i; \quad v_{ij}(x_i, x_j) = v_{ji}(x_j, x_i) = x_i^T C_{ij} x_j,$$

for $i, j = 1, 2$. For the function

$$v(x, \eta) = \eta^T U(x) \eta, \quad \eta \in \mathbb{R}^2 \quad (5.5)$$

we obtain the estimate

$$v(x, \eta) \geq u^T B^T C B u, \quad (5.6)$$

where $B = \text{diag}(\eta_1, \eta_2)$, $u = (\|x_1\|, \|x_2\|)^T$, $C = (C_{ij})$,

$$C_{ii} = \lambda_{\min}(D_{ii}),$$

$$C_{ij} = -\text{sign}(\eta_i, \eta_j) \lambda^{1/2} \max(D_{ij} D_{ij}^T), \quad i, j = 1, 2.$$

Obtaining the total derivative of the function v in (5.5) by virtue of system

$$\frac{dx_i}{dt} = A_{ii} x_i + \sum_{\substack{i=1 \\ j \neq i}}^2 A_{ij} x_j, \quad (5.7)$$

it is easy to get the upper bound

$$Dv(x, \eta) \leq u^T Q u, \quad (5.8)$$

where Q is a 2×2 -matrix constructed in terms of certain estimates of the derivatives of the components of the MF U [cf. 2, 28]. The stability conditions for the state $(x_1^T, x_2^T)^T = 0$ of system (5.7) are formulated by means of the definiteness of the matrices A and Q .

For an autonomous nonlinear large scale system (LSS) an algorithm for construction of estimate of asymptotic stability (ASD) is developed by means of a MLF (see [9]). The algorithm is as follows. Consider a class of MF's, whose $m \times m$ elements are positive definite forms. A class of scalar functions (SF) is also, constructed based on the class of MLF's as a subclass of MF's, i.e. for $U(x, B_{ij}) \in MF$, $B_{ij} = B_{ij}^T > 0$ the function

$$v(x, \eta, B_{ij}) = \eta^T U(x, B_{ij}) \eta, \quad \eta \in \overset{0}{\mathbb{R}}_+^s,$$

is constructed, where $(\eta, B_{ij}) \in \overset{\circ}{\mathbb{R}}_+^s \times \mathcal{P} = \mathcal{P}S$ is a space of admissible parameters. Given a matrix-function $U(x, B_{ij}) \in MF$, the estimate of the ASD of LSS

$$\mathfrak{D} = \{x \in \mathbb{R}^n: v(x, \eta, B_{ij}) < \mu(\eta, B_{ij})\}$$

is made such that

$$Dv(x, \eta, B_{ij}) = \eta^T DU(x, B_{ij})\eta < 0 \quad \forall (x \neq 0) \in \mathfrak{D}$$

and

$$\mu(\eta, B_{ij}) = \min_{x \in \mathfrak{Z}} v(x, \eta, B_{ij}),$$

$$\mathfrak{Z} = \{x \in \mathbb{R}^n: Dv(x, \eta, B_{ij}) = 0\}.$$

The hypervolume of the set \mathfrak{D} in \mathbb{R}^n is taken as its measure, and the problem of ASD extension is reduced to the problem of the conditional maximization.

Remark 6. In the MF (5.4) the diagonal elements $v_{ii}(x_i)$ are constructed in order to satisfy the inequalities

$$\underline{\alpha}_{ii}\phi_i(\|x_i\|) \leq v_{ii}(x_i) \leq \bar{\alpha}_{ii}\psi_i(\|x_i\|), \quad i=1, 2, \dots, m, \quad (5.9)$$

where $\underline{\alpha}_{ii}, \bar{\alpha}_{ii} > 0$, $\phi_i, \psi_i \in K$ are of Hahn class $K(K\mathbb{R})$, and nondiagonal elements satisfy the inequalities

$$\underline{\alpha}_{ij}\phi_i(\|x_i\|)\phi_j(\|x_j\|) \leq v_{ij}(x_i, x_j) \leq \bar{\alpha}_{ij}\psi_i(\|x_i\|)\psi_j(\|x_j\|), \quad (5.10)$$

where $\underline{\alpha}_{ij}, \bar{\alpha}_{ij} = \text{const}$, $(i \neq j) \in \{1, \dots, m\}$.

Another way of a MF construction for system (5.1) appeared in [21] where there were considered pairs (i, j) , $(i \neq j) \in \{1, \dots, m\}$ of the interconnected subsystems

$$\begin{aligned} \frac{dx_i}{dt} &= q_i(x_i, x_j); \\ \frac{dx_j}{dt} &= q_j(x_j, x_i), \end{aligned} \quad (5.11)$$

where $q_i: \mathbb{R}^{n_i} \times \mathbb{R}^{n_j} \rightarrow \mathbb{R}^{n_i}$; $q_j: \mathbb{R}^{n_j} \times \mathbb{R}^{n_i} \rightarrow \mathbb{R}^{n_j}$, $q_i(0, 0) = q_j(0, 0) = 0$ for all $i, j \in \{1, \dots, m\}$.

Such a decomposition is naturally referred as the second level decomposition, and the MF here is regarded as a hierarchical one [27, 9]. The MF elements

$$U(x) = (v_{ij}(\cdot)), \quad i, j \in \{1, \dots, m\} \quad (5.12)$$

are defined as

$$v_{i,j} = \begin{cases} v_{i,i}(x_i) & \text{for } i=j; \\ v_{i,j}(x_{i,j}) & \text{for } i < j; \\ v_{j,i}(x_{j,i}) & \text{for } i > j, (i \neq j) \in \{1, \dots, m\}. \end{cases}$$

The functions $v_{i,j} \in C[\mathbb{R}^{n_i} \times \mathbb{R}^{n_j}, \mathbb{R}]$ correspond to the (i, j) - pairs of (5.11). Under some assumptions on the functions $v_{i,j}$ and their total Dini derivatives by virtue of systems (5.11) and interconnected system (5.1), sufficient conditions for stability of equilibrium state $x=0$ are obtained (see [9, 21]).

6. MATRIX-FUNCTIONS FOR SINGULARLY-PERTURBED SYSTEMS

It is known that systems of the type

$$dx/dt=f(t, x, y), \tag{6.1}$$

$$\mu dy/dt=g(t, x, y, \mu), \tag{6.2}$$

where $(x^T, y^T)^T \in \mathbb{R}^{m+n}$, $\mu \in [0, 1]$ and t and g are vector-functions continuous in all their arguments, are of a considerable importance for applications. This is the most general example of systems admitting the motion separations for $\mu \rightarrow 0$. In [19, 33], it was proposed to use the matrix-function

$$U(t, x, y, \mu) = \begin{bmatrix} v_{11}(t, x), & v_{12}(t, x, y, \mu) \\ v_{21}(t, x, y, \mu), & v_{22}(t, y, \mu) \end{bmatrix} \tag{6.3}$$

for solving the problem on stability of equilibrium of state $x=0$ in the equations (6.1), (6.2).

We consider two concrete classes of systems (6.1), (6.2) and matrix-functions corresponding to them.

CASE A. Linear singularly-perturbed system has the form

$$dx/dt=A_{11}x+A_{12}y, \tag{6.4}$$

$$\mu dy/dt=A_{21}x+A_{22}y, \tag{6.5}$$

where $A_{11} \dots A_{12}$ are constant matrices of the corresponding dimensions. For convenience, the elements of matrix-function (6.3) are taken in the form

$$\begin{aligned} v_{11}(x) &= x^T B_1 x, \quad v_{22}(y, \mu) = \mu y^T B_2 y, \\ v_{12}(x, y, \mu) &= v_{21}(x, y, \mu) = \mu x^T B_3 y, \end{aligned} \tag{6.6}$$

where matrices B_1, B_2 are symmetric positive-definite and B_3 is a constant matrix, similarly to the conditions from Example 1, for various types of stability of state $x=0$ of (6.4), (6.5) were obtained by means of functions of the type of (5.5) for $\mu \in (0, \mu^0]$ and for $\mu \rightarrow 0$, (see [34]).

7. STABILITY OF HYBRID SYSTEMS

The problems of hybrid system stability in sense of Lyapunov are of essential interest both in theoretical and application aspects. Recently a problem of hybrid system practical stability has also been considered (see [20, 25]). The hybrid interconnected system (HIS) is a model of a really functioning system, a part of which is usually described by a (not necessarily linear) operator L and the rest of which is described by a system of ordinary differential equations (ODE's). In many cases, the operator L describes the object and the system of ODE's describes the regulator. In general, the operator L may represent either finite-dimensional system ODE's, infinite-dimensional system, delay equations, partial differential equations, Volterra integral equations or integro-differential equations. For the systems constructed by the operator L , are introduced the Banach space $E = \prod_{i=1} E_i$ is introduced, where E_i are Banach spaces corresponding to the subsystems $(\sigma_k, g_k), k \in \{1, \dots, m\}$ with norms $\|\cdot\|_k$. Here σ_k is a free k -th subsystem, g_k is the k -th interconnection function, the pair (σ_k, g_k) determines the k -th interconnected subsystem $\tilde{\sigma}_k$.

To be specific, we suppose that the HIS consists of isolated subsystems

$$\sigma_i: \frac{dx_i}{dt} = X_i(t, x_i), \quad X_i(t, 0) = 0, \quad i = 1, 2, \dots, m, \quad (7.1)$$

where $X_i: T_\tau \times E_i \rightarrow E_i$ are connected by interconnection functions

$$g_i: g_i = g_i(t, x_1, \dots, x_m), \quad i \in \{1, \dots, m\} \quad (7.2)$$

and such that $g_i: T_\tau \times E_1 \times \dots \times E_m \rightarrow E_i$.

The fixed interacting k -th subsystem is described by the pair

$$(\sigma_k, g_k): \frac{dx_k}{dt} = X_k(t, x_k) + g_k(t, x_1, \dots, x_m), \quad (7.3)$$

where $x_k \in E_k$. We introduce some assumptions.

Assumption 7.1. There exist connected neighborhoods $N_i \subseteq E_i$ of states $x_i = 0$ $\forall i = 1, 2, \dots, m$ such that

- a) the motions $x_i(t, t_0, x_i^0)$ of subsystems (7.1) are continuous on $(t, x_{i0}) \in T_\tau \times N_i$ where $x(0; t_0, x_{i0}) = 0, i = 1, 2, \dots, m$;

- b) the motion $x(t; t_0, x_0)$ of HIS is continuous on $(t, x_0) \in \tau_\tau \times N$, where $N = N_1 \times N_2 \times \dots \times N_m$ and $x(0; t_0, x_0) = 0$.

Let $U \in C[T_\tau \times E, \mathbb{R}^{m \times m}]$ be a MF for system (7.3). Like the functions $U_0(t, x)$ for the system of ordinary differential equations by (2.3), Theorems 4.1 - 4.3 are formulated for hybrid system (7.1) - (7.2).

Further we consider a vector approach in terms of matrix-function. Define a vector-function as

$$L(t, x) = AU(t, x)\phi, \quad (7.4)$$

where A is a constant $m \times m$ matrix, $\phi \in \mathbb{R}_+^m$ or $\phi \in C[\mathbb{R}^n, \mathbb{R}_+]$, $(\phi(0) = 0)$. For $(t, x) \in T_\tau \times E$ we introduce the Dini derivative of vector-function $L(t, x)$ by the formula

$$D^+ L(t, x) = \lim_{h \rightarrow 0^+} \sup \frac{1}{h} [L(t+h, x+hX(t, x)) - L(t, x)]. \quad (7.5)$$

It should be noted that if $\phi \in \mathbb{R}_+^m$, then

$$D^+ L(t, x) = AD^+ U(t, x)\phi, \quad (7.6)$$

where $D^+ U(t, x)$ is defined similarly to (7.5).

Lemma 7.1. *Let there exist*

- 1) a matrix-function $U(t, x) \in C[\mathbb{R}_+ \times E, \mathbb{R}^{m \times m}]$, such that $U(t, x)$ is locally Lipschitz in x for $(t, x) \in \mathbb{R}_+ \times E$;
- 2) a constant $m \times m$ matrix A , a vector $\phi \in \mathbb{R}_+^m$ and a vector $\eta \in \overset{0}{\mathbb{R}}_+$ such that

$$\eta^T L(t, x) \geq a(\|x\|), \quad (7.7)$$

where a is from Hahn class K .

- 3) a vector-function $G \in C[\mathbb{R}_+ \times \mathbb{R}_+^m, \mathbb{R}_+^m]$, such that $G(t, u)$ is quasimonotone nondecreasing on u for every $t \in \mathbb{R}_+$ and such that the estimate

$$D^+ L(t, x) \leq G(t, L(t, x)) \quad (7.8)$$

holds.

- 4) a maximal solution $r(t, t_0, u_0)$ of the system

$$du/dt = G(t, u), u(t_0) = u_0 \geq 0 \quad (7.9)$$

on interval $[t_0, \infty)$.

Then the inequality

$$L(t, x(t, t_0, x_0)) \leq r(t, t_0, u_0), \quad (7.10)$$

holds with respect to the solutions $x(t, t_0, x_0) = (x_1^T(t, t_0, x_{10}), \dots, x_m^T(t, t_0, x_{m0}))^T$ of hybrid system (7.3) as long as

$$L(t_0, x_0) \leq u_0. \quad (7.11)$$

The lemma is proved in a standard way by the comparison method (see [12] and [13]).

Lyapunov stability of hybrid system (7.3) will be investigated with respect to a measure $\rho = \|x\|$, where $\|x\| = \|x_1\|_1 + \dots + \|x_m\|_m$. For such a choice of measure for system (7.3) defined in spaces L_p , norms are defined by the formula

$$\|x_s\|_{L_p} = \int_a^b \|x_s(\tau)\|^p d\tau)^{1/p}, p \in [1, \infty),$$

$1 \leq s < m$. To formulate stability definitions for (7.3) we suppose that

$$\begin{aligned} 1) \quad & X_k(t, x_k) \equiv 0 \text{ for } x_k \equiv 0 \text{ and } g_k(t, 0, \dots, 0) \equiv 0, k = 1, 2, \dots, m; \\ 2) \quad & G(t, u) \equiv 0 \text{ for } u \equiv 0 \text{ for all } t \in \mathbb{R}_+. \end{aligned} \quad (7.13)$$

Definition 7.1. The equilibrium state $x = 0$ of hybrid system (7.3) is called *conditionally equi-stable*, if for every $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$ there exists $\delta = \delta(t_0, \epsilon) > 0$ such that $\|x_0\| < \delta$ implies $\|x(t, t_0, x_0)\| < \epsilon$ for all $t \geq t_0$.

Definition 7.2. The trivial solution $u = 0$ of comparison system (7.9) is called *conditionally equi-stable*, if for every $\epsilon^* > 0$ and $t_0 \in \mathbb{R}_+$ there exists $\delta^* = \delta^*(t_0, \epsilon^*)$ such that $u_0 \in H^+ \subset \mathbb{R}_+^m$ and $\sum_{i=1}^m u_{i0} < \delta$ imply the inequality $\sum_{i=1}^m u_i(t, t_0, u_0) < \epsilon^*$ for all $t \geq t_0$.

Theorem 7.1. Let

- 1) conditions 1 - 3 of Lemma 7.1 be satisfied
- 2) zero solution of (7.8) be
 - a) conditionally equi-stable;
 - b) uniformly asymptotically stable;

Then the equilibrium of state $x = 0$ of (7.3) is

- a) conditionally equi-stable;
- b) uniformly asymptotically stable, if for some function b of class K

$$\eta^T L(t, x) \leq b(\|x\|).$$

The theorem can be proved in a standard way applied for the proofs of stability theorems in the comparison method for equations in Banach spaces.

8. STABILITY ANALYSIS OF TWO-COMPONENT SYSTEM

We consider a system consisting of two subsystems, one of which is modeled by an ordinary

differential equation and the other one by a partial differential equation (see [1]). Let the interval $T_0 = [t_0, +\infty)$ and a bounded domain Ω be given in space \mathbb{R}^k . Consider the problem

$$dx/dt = X(t, x(t)) + \mu g_1(t, z, x(t), w(t, z)), x(t_0) = x_0, \quad (8.1)$$

$$dw/dt = L(t, z, \frac{\partial}{\partial z})w + \mu g_2(t, z, x(t), w(t, z)), \quad (8.2)$$

$$w(t_0, z) = w^0(z), M(t, z, \frac{\partial}{\partial z})\phi \Big|_{\partial\Omega} = w^1(t, x), s \in \partial\Omega.$$

Here $\mu > 0$ is a small positive parameter, $x: \tau_0 \rightarrow H, H \subseteq \mathbb{R}^n$;

$$w: T_0 \times \Omega \rightarrow Q, Q \subseteq \mathbb{R}^m; X: T_0 \times H \rightarrow \mathbb{R}^n; g_1: T_0 \times \Omega \times H \times Q \rightarrow \mathbb{R}^n;$$

$$L: B_1 \rightarrow B_2, M: B_1 \rightarrow B_2, M: B_1 \rightarrow B_3, w^0 \in B_4; g_2: T_0 \times \Omega \times H \times Q \rightarrow \mathbb{R}^m.$$

Because of the presence of the small parameter μ in the interconnection functions of subsystems

$$dx/dt = X(t, x(t)), x(t_0) = x_0; \quad (8.3)$$

$$dw/dt = L(t, z, \frac{\partial}{\partial z})w, w(t_0, z) = w^0(z); \quad (8.4)$$

$$M(t, z, \frac{\partial}{\partial z})\phi \Big|_{\partial\Omega} = w^1(t, s), s \in \partial\Omega,$$

the system (8.1) - (8.2) is a weakly connected hybrid systems (see [30]).

Assumption 1. There exist

- 1) functions $v_{11} \in C[T_0 \times N_x, \mathbb{R}_+]$, such that $v_{11}(t, x)$ is locally Lipschitz in x , $v_{22} \in C[T_0 \times N_w, \mathbb{R}_+]$, such that $v_{22}(t, w)$ is locally Lipschitz on w , and $N_x \subseteq H, N_w \subseteq B_1$ are open connected neighborhoods of points $x = 0$ and $w = 0$ respectively;
- 2) functions $\phi_i(\|x\|)$ and $\psi_i(\|w\|)$ of Hahn class $K, i = 1, 2$;
- 3) functions $v_{12}(t, x, w) = v_{21}(t, x, w), v_{12} \in C[T_0 \times N_x \times N_w, \mathbb{R}]$, such that $v_{12}(t, x, w)$ and $v_{21}(t, x, w)$ are locally Lipschitz in x and w ;
- 4) positive constants $\underline{\alpha}_{ii}$ and $\bar{\alpha}_{ii}, i = 1, 2$, and arbitrary constants $\underline{\alpha}_{12}$ and $\bar{\alpha}_{12}$ for which the estimates

$$a) \quad \underline{\alpha}_{11}\phi_1^2(\|x\|) \leq v_{11}(t, x) \leq \bar{\alpha}_{11}\phi_2^2(\|x\|);$$

$$b) \quad \underline{\alpha}_{22}\psi_1^2(\|w\|) \leq v_{22}(t, w) \leq \bar{\alpha}_{22}\psi_2^2(\|w\|);$$

$$c) \quad \underline{\alpha}_{12}\phi_1(\|x\|)\psi_1(\|w\|) \leq v_{12}(t, x, w) \leq \bar{\alpha}_{12}\phi_2(\|x\|)\psi_2(\|w\|).$$

hold.

Assumption 1 implies the following lemma.

Lemma 8.1. *If all conditions of Assumption 1 are satisfied and matrices*

$$A_1 = \begin{bmatrix} \underline{\alpha}_{11} & \underline{\alpha}_{12} \\ \underline{\alpha}_{21} & \underline{\alpha}_{22} \end{bmatrix} \quad A_2 = \begin{bmatrix} \bar{\alpha}_{11} & \bar{\alpha}_{12} \\ \bar{\alpha}_{21} & \bar{\alpha}_{22} \end{bmatrix}$$

$$\underline{\alpha}_{12} = \underline{\alpha}_{21}; \quad \bar{\alpha}_{21} = \bar{\alpha}_{12}$$

are positive-definite, the function

$$v(t, x, w) = \phi^T U(t, x, w) \phi, \quad (8.5)$$

(where $\phi^T = (\eta_1, \eta_2) - \eta_1, \eta_2 > 0$) is positive-definite and decreasing.

Assumption 2. There exist

- 1) functions $v_{ij}(t, \cdot)$ mentioned in Assumption 1;
- 2) constants β_{ik} , $i = 1, 2$; $k = 1, 2, \dots, 8$, functions $\zeta_1 = \zeta_1(\|x\|)$ and $\zeta_2(\|w\|)$, ζ_1, ζ_2 are of Hahn class K and estimates

- a) $D_t^+ v_{11} + (D_x^+ v_{11})^T X \leq 0$;
- b) $(D_x^+ v_{11})^T g_1 \leq \beta_{22} \zeta_1^2 + \beta_{13} \zeta_1 \zeta_2$;
- c) $D_t^+ v_{22} + (D_w^+ v_{22})^T L \leq 0$;
- d) $(D_w^+ v_{22})^T g_2 \leq \beta_{22} \zeta_2^2 + \beta_{23} \zeta_1 \zeta_2$;
- e) $D_t^+ v_{12} + (D_x^+ v_{12})^T X \leq \beta_{14} \zeta_1^2 + \beta_{15} \zeta_1 \zeta_2$;
- f) $(D_w^+ v_{12})^T L \leq \beta_{24} \zeta_1^2 + \beta_{25} \zeta_1 \zeta_2$;
- g) $(D_x^+ v_{12})^T g_1 \leq \beta_{16} \zeta_1^2 + \beta_{17} \zeta_1 \zeta_2 + \beta_{18} \zeta_2^2$;
- h) $(D_w^+ v_{12})^T g_2 \leq \beta_{26} \zeta_1^2 + \beta_{23} \zeta_1 \zeta_2 + \beta_{28} \zeta_2^2$

hold.

Consider matrix $C(\mu)$ in form

$$C(\mu) = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \quad C_{12} = C_{21} \quad (8.6)$$

with the elements

$$C_{11} = \eta_1^2 \mu \beta_{12} + 2\eta_1 \eta_2 (\beta_{14} + \mu \beta_{16} + \mu \beta_{26});$$

$$C_{22} = \eta_2^2 \mu \beta_{22} + 2\eta_1 \eta_2 (\beta_{24} + \mu \beta_{18} + \mu \beta_{28});$$

$$C_{12} = \frac{1}{2}(\eta_1^2 \mu \beta_{13} + \eta_2^2 \mu \beta_{23}) + \eta_1 \eta_2 (\beta_{15} + \beta_{25} + \mu \beta_{17} + \mu \beta_{27}).$$

Introduce designations $a, p, q; \mu_1, \mu_2, \mu_3, \mu_4$ by formulas

$$a = \eta_1 \eta_2 [\eta_1 \beta_{12} + 2\eta_1 (\beta_{16} + \beta_{26})] [\eta_2 \beta_{22} + 2\eta_1 (\beta_{13} + \beta_{28})] - \\ - [\frac{1}{2}(\eta_1^2 \beta_{13} + \eta_2^2 \beta_{23}) + \eta_1 \eta_2 (\beta_{17} + \beta_{27})]^2;$$

$$p = \eta_1 \eta_2 \{ [\eta_1 \beta_{24} (\eta_1 \beta_{12} + 2\eta_1 \beta_{16} + 2\eta_2 \beta_{26}) + \eta_2 \beta_{14} (\eta_2 \beta_{22} + 2\eta_1 \beta_{18} + \\ + 2\eta_1 \beta_{28})] - (\beta_{15} + \beta_{25}) [\frac{1}{2}(\eta_1^2 \beta_{13} + \eta_2^2 \beta_{23}) + \eta_1 \eta_2 (\beta_{17} + \beta_{27})] \};$$

$$q = \eta_1^2 \eta_2^2 [4\beta_{14} \beta_{24} - (\beta_{15} + \beta_{25})^2];$$

$$\mu_1 = -2\eta_1 \beta_{14} / \eta_1 (\beta_{12} + 2\eta_2 (\beta_{16} + \beta_{26}));$$

$$\mu_2 = -2\eta_1 \beta_{24} / \eta_2 \beta_{22} + 2\eta_1 (\beta_{18} + \beta_{28});$$

$$\mu_3 = (p + \sqrt{p^2 - 4aq}) (-2a)^{-1};$$

$$\mu_4 = (-p - \sqrt{p^2 - 4aq}) (2a)^{-1}.$$

Further consider a value of parameter μ , whose boundary value is determined by the conditions

B₀. If $a > 0, p > 0$, then $\mu_0 = \min(\mu_1, \mu_2)$.

B₁. If $a > 0$ and p is arbitrary, then $\mu_0 = \min(\mu_1, \mu_2, \mu_3)$.

B₂. If $a > 0$ and $p > 0$, then $\mu_0 = \min(\mu_1, \mu_2, \mu_4)$.

The validity of the following lemma is easily verified.

Lemma 8.2. *If all conditions in Assumption 2 are satisfied and the inequalities*

a) $\eta_2 \beta_{14} < 0;$

b) $\eta_1 \beta_{12} + 2\eta_2 (\beta_{16} + \beta_{26}) > 0;$

c) $\eta_1 \beta_{24} < 0;$

d) $\eta_2 \beta_{22} + 2\eta_1 (\beta_{18} + \beta_{28}) > 0;$

e) $4\beta_{14} - (\beta_{15} + \beta_{25})^2 > 0,$

hold true, then the matrix $C(\mu)$ is negative-definite for $\mu \in]0, \mu_0[$ and $\mu \rightarrow 0$, where μ_0 is defined by one of the conditions B₀ - B₂.

In view of condition 2, Assumption 2, we have for function (8.5) the following estimate of Dini derivative

$$D^+ v(t, x, w) \leq u^T C(\mu) u, \quad (8.7)$$

where $u^T = (\zeta_1(\|x\|), \zeta_2(\|w\|)), \mu \in]0, \mu_0[$.

Theorem 8.1. *If two-component system (8.1) - (8.2) is such that all conditions of*

Lemmas 8.1 and 8.2 are satisfied, then its equilibrium state $x = 0$, $w = 0$ is uniformly asymptotically stable for $\mu \in]0, \mu_0[$ and for $\mu \rightarrow 0$.

The proof of the theorem follows from the conditions satisfied by the function (8.5) and its Dini derivative (8.7).

Remark 8.7. In consequence of conditions 2 a) and 2 c) of Assumption 2, the hybrid system (8.1) - (8.2) consists of (nonasymptotically) stable subsystems and uniform asymptotic stability of equilibrium state $x = 0$, $w = 0$ is achieved due to the interconnection functions stabilizing effect.

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