Journal of Applied Mathematics and Stochastic Analysis 5, Number 3, Fall 1992, 283-290

MULTIPOINT FOCAL BOUNDARY VALUE PROBLEMS ON INFINITE INTERVALS[1]

S. UMAMAHESWARAM and M. VENKATA RAMA[2]
University of Hyderabad
School of Mathematics and Computer Information Sciences
Central University P.O.
Hyderabad-500 134, INDIA

ABSTRACT

For the differential equation $y^{(n)} = f(x,y)$, we state a set of necessary and sufficient conditions for the existence of a solution (i) on a semi-infinite interval for a k-point right focal boundary value problem and (ii) on $(-\infty,\infty)$ for a $(n-1)$-point right focal boundary value problem. The conditions are in terms of the existence of a pair of solutions $u(x)$, $v(x)$ satisfying some auxiliary boundary conditions and algebraic inequatilities.

Key words: Focal boundary value problems.

AMS (MOS) subject classifications: 34B15.

1. INTRODUCTION

Let n be a fixed positive integer greater than 1, k, $n(1),\ldots,n(k)$ be arbitrary but fixed integers satisfying $1 < k \leq n$, $n(1) \geq 2$, $1 \leq n(r) \leq n-1$, $r = 2,\ldots,k$, $n(1)+\ldots+n(k) = n$, and $x_1 < \ldots < x_k$ be arbitrary real numbers. Define $s(0) = 0$ and $s(r) = n(1)+\ldots+n(r)$ for $r = 1,\ldots,k$.

In this paper we obtain in Theorems 2.3 and 3.1 necessary and sufficient conditions for the existence of a solution (i) on the interval $(-\infty, x_1]$ of the k-point right focal boundary value problem (BVP) (1.1), (1.2) with $(r,i) \neq (1,0)$ and (ii) on the interval $(-\infty,\infty)$ of the BVP (1.1), (1.3) with $i \neq m$, the underlying equations being

$$y^{(n)} = f(x,y) \tag{1.1}$$

$$y^{(i)}(x_r) = y_{ri},\ r = 1,\ldots,k$$

$$i = s(r-1),\ldots,s(r)-1 \tag{1.2}$$

[1]Received: October, 1991. Revised: December, 1991.

[2]This author is supported by a Post-Doctoral Research Associateship of the University Grants Commission, New Delhi, India.

and

$$y^{(i)}(x_1) = y_i,\ i = 0,\dots,n-1. \tag{1.3}$$

These conditions are stated in terms of the existence of a pair of solutions $u(x), v(x)$ of (1.1) satisfying some auxiliary boundary conditions (BCs) and algebraic inequalities. We assume throughout this paper that the differential equation (1.1) satisfies some of the following hypotheses.

A. f is continuous on $\mathbb{R}^2$.

UR. Solutions of n-point right focal BVPs, if they exist, are unique; that is, if $y(x)$, $z(x)$ are solutions of the BVP (1.1), (1.2) with $x_1 < \dots < x_k$ and $k = n$ then $y(x) \equiv z(x)$ on $[x_1, x_k]$.

U. Solutions of initial value problems (IVPs) are unique.

E. All solutions of (1.1) exist on $(-\infty, \infty)$.

E'. All solutions of (1.1) exist on $(-\infty, c)$, where $-\infty < c \leq \infty$ is a constant depending on the solution.

Some existence theorems on infinite intervals for conjugate BVPs have been proved for the cases $n = 2$ and 3 in [5,6] and for arbitrary n in [7]. However, existence theorems on infinite intervals for focal BVPs do not seem to be included in the literature so far.

2. AN EXISTENCE THEOREM FOR SEMI-INFINITE INTERVALS

We first prove the following lemma which is useful in the proofs of our main theorems.

Lemma 2.1: *Assume the hypotheses A, UR, and E' hold. Let ℓ, m be arbitrary but fixed integers with $1 \leq \ell \leq k$, $s(\ell-1) \leq m \leq s(\ell)$, and $(\ell, m) \neq (1, 0)$. Suppose $u(x)$, $v(x)$ are distinct solutions of the BVP (1.1), (1.2) with $(r, i) \neq (1, 0)$, (ℓ, m) and satisfying $u(x_0) = v(x_0)$ for some $x_0 < x_1$ and $w(x) \equiv u(x) - v(x)$. Then*

(i) $w'(x) \neq 0$ *for* $x_0 \leq x < x_1$, $w(x) \neq 0$ *for* $x_0 < x \leq x_1$.

(ii) $w^{(s(r-1))}(x) \neq 0$ *for* $x_{r-1} \leq x < x_r$, $w^{(s(r-1)-1)}(x) \neq 0$ *for* $x_{r-1} < x \leq x_r$, $r = 2, \dots, \ell - 1$.

(iii) $w^{(s(\ell-1))}(x) \neq 0$ *for* $x_{\ell-1} \leq x < x_\ell$, $w^{(m)}(x) \neq 0$ *for* $x_{\ell-1} < x \leq x_\ell$.

Proof: If $w'(x') = 0$ for some $x', x_0 \leq x' \leq x_1$ then using the BCs (1.2), successive applications of Rolle's theorem to $w', \dots, w^{(m-1)}$ on appropriate subintervals of $[x', x_\ell]$ and the theorem in [3] result in the contradiction $w \equiv 0$. Thus the first inequality in (i) holds.

Now $w'(x) \neq 0$ for $x_0 < x < x_1$ implies $w(x) \neq 0$ for $x_0 < x \leq x_1$.

The proofs for (ii) and (iii) are similar.

We also need the following lemma due to Kolmogorov [4] which is stated here for the sake of convenience.

Lemma 2.2: *Let $M > 0$, $[a,b] \subset \mathbb{R}$ and $y(x) \in C^n[a,b]$ be an arbitrary function with the property that $|y(x)| \leq M$ and $|y^{(n)}(x)| \leq M$ on $[a,b]$. Then there exists a constant $K > 0$ depending on M and the interval $[a,b]$ such that $|y^{(r)}(x)| \leq K$ on $[a,b]$ for $1 \leq r \leq n-1$.*

Theorem 2.3: *Assume the hypotheses A, UR, U and E' hold. Let ℓ, m be arbitrary but fixed integers with $1 \leq \ell \leq k$, $s(\ell-1) \leq m \leq s(\ell)-1$ and $(\ell,m) \neq (1,0)$. Then a necessary and sufficient condition that the BVP (1.1), (1.2) with $(r,i) \neq (1,0)$ has a solution $y(x)$ on $(-\infty, x_k]$ is that there exist solutions $u(x)$, $v(x)$ of (1.1) on $(-\infty, x_k]$ satisfying the condition (1.2) with $(r,i) \neq (1,0)$, (ℓ,m);*

$$u(x) \geq v(x) \text{ on } (-\infty, x_1]$$

and

$$(-1)^m u^{(m)}(x_\ell) \leq (-1)^m y_{\ell m} \leq (-1)^m v^{(m)}(x_\ell).$$

In the sufficiency part the solution $y(x)$ satisfies $u(x) \geq y(x) \geq v(x)$ on $(-\infty, x_1]$.

Proof:

Necessity: This is obvious since we can choose $u(x) = v(x) = y(x)$ where $y(x)$ is the assumed solution of (1.1), (1.2) with $(r,i) \neq (1,0)$.

Sufficiency: If $(-1)^m y_{\ell m} = (-1)^m u^{(m)}(x_\ell)$ (or $(-1)^m v^{(m)}(x_\ell)$) we can choose $y(x) = u(x)$ (or $v(x)$) and there is nothing to prove. Suppose

$$(-1)^m u^{(m)}(x_\ell) < (-1)^m y_{\ell m} < (-1)^m v^{(m)}(x_\ell). \tag{2.1}$$

If $u(x') = v(x')$ for some $x' < x_1$, then since $u(x) \geq v(x)$ on $(-\infty, x_1]$ we must have $u'(x') = v'(x')$ and this contradicts Lemma 2.1 (i). Hence $u(x) > v(x)$ on $(-\infty, x_1)$. For $j = 1, 2, \ldots$, let $v_j(x)$ be the solution of the BVP (1.1), (1.2) with $(r,i) \neq (1,0)$ and $y(x_1 - j) = v(x_1 - j)$ which exists by theorem 3 of [2].

Now we claim that $v_j'(x_1 - j) > v'(x_1 - j)$. Clearly $v_j'(x_1 - j) \neq v'(x_1 - j)$ by Lemma 2.1 (i). Also, due to the same reason, if $v_j'(x_1 - j) < v'(x_1 - j)$ then $v_j'(x) < v'(x)$ for all x, $x_1 - j < x < x_1$. Let $g(x) \equiv v_j(x) - v(x)$ so that $g'(x) < 0$ on $[x_1 - j, x_1)$, $g^{(i)}(x_1) = 0$, $i = 1, \ldots, s(1)-1$ and by Lemma 2.1 (ii), $g^{(s(1))}(x_1) \neq 0$. Hence for $x_1 - j \leq x < x_1$ we have by Taylor's theorem

$$\begin{aligned}-1 &= Sgn\ g'(x)\\ &= Sgn(g'(x)-g'(x_1))\\ &= Sgn\left\{\frac{(x-x_1)^{s(1)-1}}{(s(1)-1)!}g^{(s(1))}(x_1)\right\}.\end{aligned}$$

This implies $Sgn\ g^{(s(1))}(x_1) = (-1)^{s(1)}$ and by Lemma 2.1 (*ii*), $Sgn\ g^{(s(1))}(x) = (-1)^{s(1)}$ for $x_1 < x < x_2$. Further $g^{(i)}(x_2) = 0$ for $i = s(1),\ldots,s(2)-1$ and by Lemma 2.1 (*ii*) $g^{(s(2))}(x_2) \neq 0$. Hence for $x_1 < x < x_2$, we again by Taylor's theorem

$$\begin{aligned}(-1)^{s(1)} &= Sgn\ g^{(s(1))}(x)\\ &= Sgn(g^{(s(1))}(x)-g^{(s(1))}(x_2))\\ &= Sgn\left\{\frac{(x-x_2)^{s(2)-s(1)}}{(s(2)-s(1))!}g^{(s(2))}(x_2)\right\}.\end{aligned}$$

Thus $Sgn\ g^{(s(2))}(x_2) = (-1)^{s(2)}$ and consequently by Lemma 2.1 (*ii*) $Sgn\ g^{(s(2))}(x) = (-1)^{s(2)}$ for $x_2 < x < x_3$. Continuing this argument through the intervals $[x_2, x_3], \ldots, [x_{\ell-1}, x_\ell]$ we obtain $Sgn\ g^{(s(r))}(x_r) = (-1)^{s(r)}$, $r = 1,\ldots,\ell-1$ and, by Lemma 2.1 (*iii*), $Sgn\ g^{(s(\ell-1))}(x) = (-1)^{s(\ell-1)}$, $x_{\ell-1} < x < x_\ell$ whereas $g^{(i)}(x_\ell) = 0$, $i = s(\ell-1),\ldots,m-1$. Again an application of Taylor's theorem yields that for $x_{\ell-1} < x < x_\ell$

$$\begin{aligned}(-1)^{s(\ell-1)} &= Sgn\ g^{(s(\ell-1))}(x)\\ &= Sgn(g^{(s(\ell-1))}(x)-g^{(s(\ell-1))}(x_\ell))\\ &= Sgn\frac{(x-x_\ell)^{m-s(\ell-1)}}{(m-s(\ell-1))!}g^{(m)}(x_\ell)\\ &= (-1)^{m-s(\ell-1)}Sgn\ g^{(m)}(x_\ell).\end{aligned}$$

Thus $Sgn\ g^{(m)}(x_\ell) = (-1)^m$ or $(-1)^m(v_j - v)^{(m)}(x_\ell) > 0$, a contradiction to the inequality (2.1). Hence the claim $v'_j(x_1 - j) > v'(x_1 - j)$ is true. This implies by Lemma 2.1 (*i*) $v_j(x) > v(x)$ on $[x_1 - j, x_1]$ for all j.

Next we claim that $v_j(x) < u(x)$ for $x_1 - j \leq x \leq x_1$. If $v_j(x') = u(x')$ holds for some x', $x_1 - j < x' < x_1$ then $v'_j(x') \geq u'(x')$. However $v'_j(x') \neq u'(x')$ by Lemma 2.1 (*i*). On the other hand, if $v'_j(x') > u'(x')$ holds for some x', $x_1 - j < x' < x_1$, then by Lemma 2.1 (*i*) we should have $v'_j(x) > u'(x)$ for $x' \leq x \leq x_1$. However if $h(x) \equiv v_j(x) - u(x)$, $x' \leq x \leq x_1$ then $h'(x) > 0$

for $x' \le x < x_1$, $h^{(i)}(x_1) = 0$, $i = 1, \ldots, s(1) - 1$ and $h^{(s(1))}(x_1) \neq 0$. Hence for $x' < x < x_1$, we have by Taylor's theorem

$$\begin{aligned} 1 &= Sgn\ h'(x) \\ &= Sgn(h'(x) - h'(x_1)) \\ &= (-1)^{s(1)-1} Sgn\ h^{(s(1))}(x_1). \end{aligned}$$

Thus, $Sgn\ h^{(s(1))}(x_1) = (-1)^{s(1)-1}$. Continuing the arguments as in the case of $(v_j - v)$ in the earlier part of the proof we obtain, for $x_{\ell-1} < x < x_\ell$,

$$\begin{aligned} Sgn(v_j - u)^{(m)}(x_\ell) &= Sgn\ h^{(m)}(x_\ell) \\ &= (-1)^{m-1}. \end{aligned}$$

Thus, $(-1)^{m-1} v_j^{(m)}(x_\ell) > (-1)^{m-1} u^{(m)}(x_\ell)$, a contradiction to the inequality (2.1) and hence the claim is true.

Furthermore, since $v_j(x)$ are solutions of equation (1.1), it follows by hypothesis UR and the theorem in [3] that for each $j = 1, 2, \ldots$, $v(x) < v_j(x) < v_{j+1}(x) < u(x)$ on $[x_1 - j, x_1]$. By Lemma 2.2 and Kamke's convergence theorem [p.14, 1] there exists a subsequence called again $\{v_j(x)\}$ and a solution $v_0(x)$ of (1.1) such that $v_j^{(i)}(x) \to v_0^{(i)}(x)$, $i = 0, \ldots, j-1$ uniformly on compact subintervals of $(-\infty, x_1]$. Now the solution $y(x) = v_0(x)$ has the desired properties.

3. AN EXISTENCE THEOREM FOR $(-\infty, \infty)$

In this section we assume the additional hypothesis UL,

UL: Solutions of n-point left focal BVPs, if they exist, are unique; that is, if $y(x)$, $z(x)$ are solutions of the BVP (1.1), (1.2) with $x_k < \ldots < x_1$ and $k = n$ then $y(x) \equiv z(x)$ on $[x_k, x_1]$.

Theorem 3.1: *Assume the hypotheses A, UR, UL, U and E hold. Let m be a fixed but arbitrary integer with $1 \le m \le n-1$. Then a necessary and sufficient condition for the BVP (1.1), (1.3) with $i \neq 0$ to have a solution $y(x)$ on $(-\infty, \infty)$ is that there exist solutions $u(x)$, $v(x)$ of (1.1) on $(-\infty, \infty)$ satisfying the conditions (1.3) with $i \neq 0$, m,*

$$u(x) \ge v(x) \text{ on } (-\infty, \infty),$$

and

$$(-1)^m u^{(m)}(x_1) \le (-1)^m y_m \le (-1)^m v^{(m)}(x_1).$$

In the sufficiency part, the solution $y(x)$ satisfies $u(x) \geq y(x) \geq v(x)$ on $(-\infty, \infty)$.

Proof:

Necessity: This is obvious since we can choose $u(x) = v(x) = y(x)$.

Sufficiency: If $(-1)^m y_m = (-1)^m u^{(m)}(x_1)$ (or $(-1)^m v^{(m)}(x_1)$) we can choose $y(x) = u(x)$ (or $v(x)$) and there is nothing to prove.

Suppose the inequality

$$(-1)^m u^{(m)}(x_1) < (-1)^m y_m < (-1)^m v^{(m)}(x_1) \tag{3.1}$$

holds. Then we have $u(x) > v(x)$ on $(-\infty, x_1)$ as in Theorem 2.1. Furthermore, if $u(x') = v(x')$ for some $x' > x_1$ we can arrive at a contradiction by virtue of the hypothesis UL and a lemma analogous to Lemma 2.1 for left focal BCs. Hence $u(x) > v(x)$ holds for all $x \neq x_1$.

If for each $j \geq 1$, $v_j(x)$ is the solution of the BVP (1.1), (1.3) with $i \neq 0$ and $y(x_1 - j) = v(x_1 - j)$ then as in Theorem 2.1, we have $v(x) < v_j(x) < v_{j+1}(x) < u(x)$ on $[x_1 - j, x_1]$. Similarly, for each $j \geq 1$ we can obtain a solution $u_j(x)$ of (1.1), (1.3) with $i \neq 0$ and $y(x_1 - j) = u(x_1 - j)$ with the property that $u_{j+1}(x) < u_j(x) \leq u(x)$ on $x_1 - j \leq x \leq x_1$. Moreover, by the hypothesis UR and the theorem of [3] it follows that for each j, $v_j(x) < u_j(x)$ on $[x_1 - j, x_1)$. Thus we have for each $j \geq 1$, $v(x) < v_j(x) < v_{j+1}(x) < u_{j+1}(x) < u_j(x) < u(x)$ on $[x_1 - j, x_1)$. Now since $u_j(x)$, $v_j(x)$ are solutions of equation (1.1) it follows by Lemma 2.1 and Kamke's convergence theorem [p. 14, 1] that there exists subsequences of $\{u_j(x)\}$, $\{v_j(x)\}$ called again $\{u_j(x)\}$, $\{v_j(x)\}$ such that $u_j(x) \to u_0(x)$, $v_j(x) \to v_0(x)$ uniformly on compact subintervals of $(-\infty, x_1)$; consequently $u_0(x), v_0(x)$ are solutions of the BVP (1.1), (1.3) with $i \neq 0$ satisfying $v(x) \leq v_0(x) \leq u_0(x) \leq u(x)$ on $(-\infty, x_1]$. Similarly, using the hypothesis UL, the results analogous to the theorem in [3], theorem 3 of [2], and Lemma 2.1 for left focal BVPs, we can obtain a pair of solutions $w_0(x)$, $z_0(x)$ of (1.1), (1.3) with $i \neq 0$ satisfying $v(x) \leq w_0(x) \leq z_0(x) \leq u(x)$ on $[x_1, \infty)$.

Now the four quantities $u_0(x_1)$, $v_0(x_1)$, $w_0(x_1)$ and $z_0(x_1)$ can be ordered in one of the following ways,

(*i*) $v_0(x_1) \leq u_0(x_1) \leq w_0(x_1) \leq z_0(x_1)$

(*ii*) $w_0(x_1) \leq z_0(x_1) \leq v_0(x_1) \leq u_0(x_1)$

(*iii*) $w_0(x_1) \leq v_0(x_1) \leq u_0(x_1) \leq z_0(x_1)$

(*iv*) $v_0(x_1) \leq w_0(x_1) \leq z_0(x_1) \leq u_0(x_1)$.

In any case let $y(x)$ be the solution of the IVP (1.1), (1.3) with $y(x_1) = c_0$, where c_0 is the average of the middle two quantities in the appropriate ordering stated above. They $y(x)$ is the desired solution.

Remark 1: It follows from theorem 3 of [2] and the theorem in [3] that Theorems 2.3 and 3.1 will be true if we replace (1.1) by

$$y^{(n)} = f(x, y, \ldots, y^{(n-1)}), \tag{1.1)'}$$

provided the hypothesis A is replaced by the hypothesis A' and the additional hypothesis C (compactness of solutions of (1.1)') holds where A' and C are as follows:

A': f is continuous on $\mathbb{R}^{n+1}$.

C: If $\{y_k(x)\}$ is a sequence of solutions of (1.1) and $[c,d]$ is a compact subinterval of (a,b) such that $\{y_k(x)\}$ is uniformly bounded on $[c,d]$, then there exists a subsequence $\left\{y_{k_j}(x)\right\}$ such that $\left\{y_{k_j}^{(i)}(x)\right\}$ converges uniformly on $[c,d]$, $0 \leq i \leq n-1$.

Remark 2: In the case $n = 3$, the hypothesis C can be omitted in view of the comments on page 990 of [2]; while in the case $n = 2$, the hypotheses U and C can be omitted in view of theorem 3.1 of [8].

REFERENCES

[1] P. Hartman, "*Ordinary Differential Equations*", Wiley, New York (1964).

[2] J. Henderson, "Existence of solutions of right focal point boundary value problems for ordinary differential equations", *J. of Nonlinear Analysis: Theory, Methods and Appl.* **5**, (1981), pp. 989-1002.

[3] ________, "Uniqueness of solutions of right focal point boundary value problems for ordinary differential equations, *J. of Differential Equations* **41**, (1981), pp. 218-227.

[4] A. Kolmogorov, "On inequalities between the upper bounds of the successive derivatives of an arbitrary function on an infinite interval, *Amer. Math. Soc. Transl. Ser.* **14**, (1949), pp. 3-19.

[5] K. Schrader, "Existence theorems for second order boundary value problems", *J. of Differential Equations* **5**, (1969), pp. 572-584.

[6] ________, "Second and third order boundary value problems", *Proc. of Amer. Math. Soc.* **32**, (1972), pp. 247-252.

[7] S. Umamaheswaram, "Boundary value problems for higher order differential equations", *J. of Differential Equations* **18**, (1975), pp. 188-201.

[8] S. Umamaheswaram and M. Venkata Rama, "Focal subfunctions and second order differential inequalities", *Rocky Mountain Journal of Math.* (to appear).