Journal of Applied Mathematics and Stochastic Analysis
7, Number 4, 1994, 581-586.

ALMOST PERIODIC SOLUTIONS TO SYSTEMS OF PARABOLIC EQUATIONS

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(Received February, 1994; revised May, 1994)

ABSTRACT

In this paper we show that the second-order differential solution is $\mathbb{L}^2$-almost periodic, provided it is $\mathbb{L}^2$-bounded, and the growth of the components of a nonlinear function of a system of parabolic equation is bounded by any pair of consecutive eigenvalues of the associated Dirichlet boundary value problems.

Key words: Almost Periodic Solutions, System of Nonlinear Parabolic Equations.

AMS (MOS) subject classifications: 35B15, 35K55, 35K99.

1. Introduction

Foais et al. [2] proved that if a solution of some system of parabolic equations in $C^2(\bar{\Omega})$ and L^2-bounded satisfying certain conditions then it is a L^2-almost periodic solution.

Recall that a continuous function $f: \mathbb{R} \rightarrow X$ is X-almost periodic if for every ϵ there is a relatively dense subset $T_\epsilon \subset \mathbb{R}$ such that

$$\sup_t \| f(t+\tau) - f(t) \|_X < \epsilon, \quad \forall \tau \in T_\epsilon,$$

where X is some Banach space.

Recently, Corduneanu [1] and Yang [4] extended the results of Foias to nonlinear parabolic equations. In this paper we extend the results of Corduneanu [1] and Yang [4] to the following system of nonlinear parabolic equations,

$$\begin{cases} \partial_t u = \Delta u + f(t,x,u) \\ u \mid_{\partial\Omega} = 0, \\ u(0,x) = u_0, \end{cases} \tag{1}$$

where u, and $f \in \mathbb{R}^m$ are m-vector valued functions, $\partial_t = \dfrac{\partial}{\partial_t}$, and Ω is some bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$. Moreover, we assume that $f: \mathbb{R} \times \Omega \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ satisfies the following conditions (cf. [1, 4]):

(CI) $f(t,x,u)$ is continuous and $\mathbb{L}^2$-almost periodic in t, and uniformly continuous with respect to u_j.

(CII) The matrix $\mathbf{D}(f) = (f_{i,j})$ is diagonalizable, with eigenvalues μ_j, and for every $j = 1,2,\ldots,m$, there exists some integer $i(j)$ such that $\lambda_{i(j)-1} < \mu_j < \lambda_{i(j)}$.

Here $f_{i,j} = \dfrac{\partial f_i}{\partial u_j}$, and $\mathbb{L}^2 = L^2(\Omega) \times \cdots \times L^2(\Omega)$, m-times. We call matrix $\mathbf{D}(f)$ diagonalizable if there exists a nonsingular matrix M such that $M\mathbf{D}(f)M^{-1} = I$, at every triple (t,x,u), where I is the identity matrix. Similarly, M is nonsingular if $det\, M \neq 0$ and μ is an eigenvalue of matrix $\mathbf{D}(f)$ if $det(\mathbf{D}(f) - \mu) = 0$. Notice that condition (CII) implies that $\mu_j > 0$, since λ_j is the eigenvalues of Laplacian in the domain Ω corresponding to the eigenfunction ϕ_j, which satisfies

$$\begin{cases} \Delta\phi_j + \lambda_j\phi_j = 0 \\ \phi_j \mid_{\partial\Omega} = 0. \end{cases} \tag{2}$$

We arrange λ_j in the ascending order

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \ldots \text{ for } j = 1,2,\ldots.$$

To simplify the notation, we use λ_0 to denote 0, and the function space $\mathbb{C}^2(\bar{\Omega}) = C^2(\bar{\Omega}) \times \cdots \times C^2(\bar{\Omega})$, m-times.

2. Main Result

Before we prove the main theorem of this paper we first derive a useful a priori estimate of the following problem,

$$\begin{cases} \partial_t w - (\Delta + D)w = v, \\ u \mid_{\partial\Omega} = 0, \end{cases} \tag{3}$$

where w, v are m-vector valued functions, $D = (\delta_{i,j}\nu_j)$ is a diagonal matrix, and $\delta_{i,j}$ is the Kronecker delta, ν_j are positive real numbers satisfying $\lambda_{i(j)-1} < \nu_j < \lambda_{i(j)}$. Here $i(j)$ is the same as in the condition (CII).

Lemma 1: *Let $w, v \in \mathbb{C}^2(\bar{\Omega})$ be $\mathbb{L}^2$-bounded satisfying problem (3). If ν_j satisfies the assumption above, then*

$$\sup_t \int_\Omega w_j^2(t,x)dx \leq max\{(\nu_j - \lambda_{i(j)-1})^{-2}, (\nu_j - \lambda_{i(j)})^{-2}\} \sup_t \int_\Omega v_j^2(t,x)dx. \tag{4}$$

Proof: It is well known that $\{\phi_j\}_{j=1}^{\infty}$ form an orthogonal basis of $L^2(\Omega)$, thus we have

$$\begin{aligned} w_j(t,x) &= \sum\nolimits_k a_{j,k}\phi_k, \\ v_j(t,x) &= \sum\nolimits_k b_{j,k}\phi_k. \end{aligned} \qquad \text{for } j = 1,2,\ldots,m \tag{5}$$

The Parseval formula and the assumption of $L^2(\Omega)$-boundedness imply that

$$\sum_{j,k} a_{j,k}^2 = \int_\Omega w^2 \leq c,$$
$$\sum_{j,k} b_{j,k}^2 = \int_\Omega v^2 \leq c, \tag{6}$$

for some positive constant c which is independent of t.

Substituting equation (5) into equation (3) yields

$$a'_{j,k}(t) + (\lambda_k - \nu_j)a_{j,k}(t) = b_{j,k}(t), \tag{7}$$

for $j = 1,\ldots,m$, $k = 1,2,\ldots$. Thus for any $t_0 \in \mathbb{R}$ we have

$$a_{j,k}(t) = e^{-(\nu_j - \lambda_k)(t_0 - t)} a_{j,k}(t_0) + \int_{t_0}^{t} e^{-(\nu_j - \lambda_k)(s - t)} b_{j,k}(s)\,ds.$$

Since $\lambda_{i(j)-1} < \nu_j < \lambda_{i(j)}$, we have $\nu_j - \lambda_k > 0$ for $k \leq i(j) - 1$. Thus for $t_0 > t$, the following is true

$$|a_{j,k}(t)| \leq e^{-(\nu_j - \lambda_k)(t_0 - t)} |a_{j,k}(t_0)| + \frac{1 - e^{-(\nu_j - \lambda_k)(t_0 - t)}}{\nu_j - \lambda_k} |b_{j,k}(t)|.$$

Using (5), (6), and the fact that $a_{j,k}$, $b_{j,k}$ are bounded functions of t, and letting $t_0 \rightarrow \infty$, the above inequality yields

$$\sup_t |a_{j,k}(t)| \leq \frac{1}{\nu_j - \lambda_k} \sup_t |b_{j,k}(t)|.$$

Similarly, the above inequality is true for $k \geq i(j)$ which implies

$$\sup_t |a_{j,k}(t)| \leq \alpha_j \sup_t |b_{j,k}(t)|, \tag{8}$$

where

$$\alpha_j = max\left\{\frac{1}{\nu_j - \lambda_{i(j)-1}}, \frac{1}{\lambda_{i(j)} - \nu_j}\right\}.$$

Thus the assertion of the lemma holds.

Theorem 2: *If u is a $\mathbb{C}^2(\bar{\Omega})$, $\mathbb{L}^2$-bounded solution of problem (1), and if $f_{j,i}$ is a continuous function satisfying conditions (CI) and (CII), then u is $\mathbb{L}^2$-almost periodic.*

Proof: Let u be a solution of equation (1) , then for a given $\tau \in \mathbb{R}$ we define the vector valued function $w = u(t + \tau, x) - u(t, x)$. Then w satisfies the following equation,

$$\begin{cases} w_t - \Delta w = f(t + \tau, w, x, u(t + \tau, x)) - f(t, x, u(t, x)), \\ w\,|_{\partial\Omega} = 0. \end{cases}$$

Applying the mean value theorem to f_j with respect to the component u_i and letting $\alpha_{j,i}$ be a constant in the interval $(0, 1)$, we have that

$$\psi_{j,i} = \alpha_{j,i} u_i(t + \tau, x) + (1 - \alpha_{j,i}) u_i(t, x),$$

satisfies

$$f_j(t,x,u_1(t,x),\ldots,u_{i-1}(t,x),u_i(t+\tau,x),\ldots,u_m(t+\tau,x))$$
$$-f_j(t,x,u_1(t,x),\ldots,u_i(t,x),u_{i+1}(t+\tau,x),\ldots,u_m(t+\tau,x))$$
$$=f_{j,i}(t,x,u_1(t,x),\ldots,\psi_i,u_{i+1}(t+\tau,x),\ldots,u_m(t+\tau,x))w_i(t,x).$$

Let the vector valued functions $\Psi_{j,i}$ be

$$\Psi_{j,i}=(u_1(t,x),\ldots,u_{i-1}(t,x),\psi_{j,i},u_{i+1}(t+\tau,x),\ldots,u_m(t+\tau,x)),$$

then w_j satisfies

$$\begin{aligned}\partial_t w_j-\Delta w_j &= \sum\nolimits_i f_{j,i}(t+\tau,x,\Psi_i)w_i\\ &+f_j(t+\tau,x,u(t,x))-f_j(t,x,u(t,x)),\end{aligned} \tag{9}$$

and boundary condition

$$w_j\,|_{\partial\Omega}=0.$$

Since w_j are $L^2(\Omega)$-bounded, we have

$$w_j(t,x)=\sum_k a_{j,k}(t)\phi_k,$$

for $j=1,\ldots,m$. The condition (CII) implies that for every j there exist two constants $\overline{\theta}_j$ and $\widehat{\theta}_j$, and some integer $i(j)\geq 1$ such that

$$\lambda_{i(j)-1}<\overline{\theta}_j\leq\mu_j\leq\widehat{\theta}_j<\lambda_{i(j)}.$$

Recall that λ_0 is 0.

Equation (9) can be rewritten as

$$\partial_t w_j-(\Delta w_j+\nu_j)=\sum_{\substack{i=1\\ i\neq j}} f_{j,i}(t+\tau,x,\psi_i)w_i+(f_{j,j}-\nu_j)w_j$$
$$+f_j(t+\tau,x,u(t,x))-f_j(t,x,u(t,x)),$$

where $\overline{\theta}_j<\nu_j<\widehat{\theta}_j$. Observe that the ν_j are real; and they will be determined later. By inequality (8), we immediately obtain

$$\sup_t((1-k_j)\,|\,w_j\,|\,-\sum_{\substack{i=1\\ i\neq j}}\beta_{j,i}\,|\,w_i\,|\,)\leq\gamma_j\sup_t|\,v_j\,|\,,$$

where

$$k_j=\frac{|\,f_{j,j}-\nu_j\,|}{\alpha_j},\ \beta_{j,i}=\frac{|\,f_{j,i}\,|}{\alpha_j},\ \gamma_j=\frac{1}{\alpha_j},\ \alpha_j=max\left\{\frac{1}{\nu_j-\lambda_{i(j)-1}},\frac{1}{\lambda_{i(j)}-\nu_j}\right\},$$

for $j=1,\ldots,m$. Let $\epsilon_i>0$, satisfy

$$\sup_t\{(1-k_j)\,|\,w_j\,|\,=\sum_{\substack{i=1\\ i\neq j}}\beta_{j,i}\,|\,w_i\,|\,\}=\gamma_j\sup_t|\,v_j\,|\,+\epsilon_i.$$

Let $\xi = (\epsilon_1, \ldots, \epsilon_m)$, and rewrite the above equation as

$$M \cdot w = \xi + G \cdot v, \tag{10}$$

where

$$w = (\sup_t | w_1 |, \ldots, \sup_t | w_m |), \quad v = (\sup_t | v_1 |, \ldots, \sup_t | v_m |),$$

$M = (m_{i,j})$, $G = (\delta_{i,j}\gamma_j)$ are $m \times m$ matrices, where $m_{i,j} = \beta_{i,j}$, for $j \neq i$, and $m_{j,j} = 1 - k_j$. Since $\overline{\theta}_j < \nu_j < \widehat{\theta}_j$, and using condition (CII), we may choose suitable ν_j such that $1 - k_j > 0$, and M is diagonalizable. By linear algebra we have

$$w = \xi + G \cdot w.$$

Since $\xi = (\epsilon_1, \ldots, \epsilon_m)$ and $\epsilon_i > 0$, we have

$$\sup_t \int_\Omega w_j^2(t,x)dx \leq c_j \sup_t \int_\Omega v_j^2(t,x)dx, \quad \text{for } j = 1, \ldots, m,$$

and some constant $c_j > 0$. This completes the proof of the theorem.

We can easily generalize Theorem 2 to the following system of nonlinear parabolic equations

$$\begin{cases} \partial_t u_j - L_j u_j = f_j(t,x,u), \\ B_j u_j \,|_{\partial\Omega} = 0, \end{cases} \tag{11}$$

where L_j and B_j are elliptic operators and boundary operators respectively satisfying

$$L_j = \sum_{\alpha \leq 2} A_{j,\alpha}(x) D^\alpha,$$

$$B_j u = b_{j,1}\frac{\partial u}{\partial n} + b_{j,0}u.$$

We denote by $D^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $| \alpha | = \alpha_1 + \ldots^s + \alpha_n$.

Furthermore, we assume that the principal parts of L_j be

$$P_j = \sum_{|\alpha| = 2} A_{j,\alpha}(x) D^\alpha,$$

such that $A_{j,\alpha} \in C^{|\alpha|}(\overline{\Omega})$ and $b_{j,i} \in C^1(\overline{\Omega})$ are real, and L_j are self-adjoint operators such that $ker(L_j - c_j) = \{0\}$ for some real c_j. Denote by $\sigma(L_j)$ the spectrum of L_j, for $j = 1, \ldots, m$ and replace the assumption (CII) by the following,

$(CII)'$ The matrix $\mathbf{D}(f) = (f_{i,j})$ is diagonalizable with eigenvalue μ_j and for every $j = 1, 2, \ldots, m$, there exists some integer $i(j)$ such that $\lambda_{j,i(j)-1} < \mu_j < \lambda_{j,i(j)}$, where $\{\lambda_{j,k}\}_{k=1}^\infty$ are the eigenvalues of the operators L_j.

Then by the same argument as used for Theorem 2, we have the following results.

Theorem 3: *If u is a $\mathbb{C}^2(\overline{\Omega})$, $\mathbb{L}^2$-bounded solution of equation* (11), *and if f is a continuous function satisfies conditions (CI), $(CII)'$, then u is $\mathbb{L}^2$-almost periodic.*

Consider the solution u of equation (11) in the sense that $u \in C^1(\mathbb{R}, \mathcal{H}^2)$, where $\mathcal{H}^2 = H_1^2(\Omega) \times, \ldots, H_m^2(\Omega)$, and

$$H_j^2 = \{u \mid u \in W^{2,2}(\Omega) \text{ is real, and } B_j u \,|_{\partial\Omega} = 0\}.$$

Here $W^{2,2}(\Omega)$ is the Sobolev space (cf. Ladyženskaja [3]). Let the operator $\mathfrak{L} = (L_1, \ldots, L_m)$, $\mathfrak{L}: \mathbb{L}^2 \to \mathbb{L}^2$ with domain $D(\mathfrak{L}) = \mathcal{H}^2$. Then we have a similar result as Theorem 2 (cf. Yang [4]).

Theorem 4: *If u is a $C^1(\mathbb{R}, \mathcal{H}^2)$, and $\mathbb{L}^2$-bounded solution of equation, and if f satisfies conditions (CI), $(CII)'$, then u is $\mathbb{L}^2$-almost periodic.*

Remark: The condition (CII) implies the uniqueness of $\mathbb{L}^2$-bounded solution to problem (1).

To prove the uniqueness, we assume that u, v are the solution to problem (1), and let $w_j = u_j - v_j$ then $w = (w_1, \ldots, w_m)$ satisfies

$$\partial_t w_j - \Delta w_j = \sum_i f_{j,i}(t, x, \Psi_{j,i}) w_i.$$

Applying Lemma 1, we see that

$$M \cdot w \leq 0$$

where M is defined as in equation (10). By linear algebra, we obtain $w \equiv 0$.

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