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AN APPROACH TO THE STOCHASTIC CALCULUS IN THE NON-GAUSSIAN CASE

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ABSTRACT

We introduce and study a class of operators of stochastic differentiation and integration for non-Gaussian processes. As an application, we establish an analog of the Itô formula.

Key words: Non-Gaussian Stochastic Process, Stochastic Integral, Stochastic Derivative, Itô's Formula.

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1. Introduction

Operators of stochastic differentiation D and an extended integration $I = D^*$ play an important role in stochastic calculus. In the Gaussian case and for certain special martingales, D and I can be defined with the aid of an orthogonal expansion (cf., T. Sekiguchi, Y. Shiota [3]). Also, D and I can be defined by means of the usual differentiation with respect to the admissible translation of the probability measure (A.A. Dorogovtsev [2]). In all these situations there are some common features. In this article we consider a general scheme in which the operators D and I are constructed for a non-Gaussian case. Since I plays the role of stochastic integration, an analog of the Itô formula is also established.

2. Stochastic Derivative and the Logarithmic Process

Let $\{\xi(t); t \in [0,1]\}$ be a random process defined on a probability space $(\Omega, \mathfrak{F}, P)$. A subset K of $\mathbb{R}^n$ is said to have the *conic property* if for every $x \in K$, there exists a cone, C_x, with the non-empty interior and a neighborhood, U_x of x such that $x \in U_x \cap C_x \subset K$.

Suppose that the support of any finite-dimensional distribution of ξ has the conic property.

Let λ be the Lebesgue measure on the Borel σ-algebra $\mathfrak{B}([0,1])$.

Definition 1: A family of the random elements $\{\zeta(t); t \in [0,1]\}$ from $L_2(\Omega \times [0,1], P \times \lambda)$ is called a *differentiation rule* if

1) $\forall t \in [0,1]: \zeta(t) \cdot \chi_{(t,1]} = 0 \pmod P$,
2) for every tuple $t_1, \ldots, t_n \in [0,1]$, $a_1, \ldots, a_n \in \mathbb{R}$, $n \geq 1$, $G \in \mathfrak{F}$, such that

$$(a_1 \xi(t_1) + \ldots + a_n \xi(t_n))\chi_G = 0 \pmod P,$$

the following equality holds

$$(a_1\zeta(t_1)+\ldots+a_n\zeta(t_n))\chi_G=0 \pmod{P\times\lambda}.$$

Definition 2: Let $\varphi:\mathbb{R}^n\to\mathbb{R}$ be bounded, continuously differentiable and have a bounded derivative. For a random variable

$$\alpha=\varphi(\xi(t_1),\ldots,\xi(t_n)),\ \ t_1,\ldots,t_n\in[0,1],$$

the sum

$$\varphi_1'(\xi(t_1),\ldots,\xi(t_n))\zeta(t_1)+\ldots+\varphi_n'(\xi(t_1),\ldots,\xi(t_n))\cdot\zeta(t_n)$$

is called a *stochastic derivative* of α and denoted by $D\alpha$ (so $D\xi(t)=\zeta(t)$).

In the sequel, denote the set of all random variables from Definition 2 by $\mathcal{M}$. $\mathcal{M}$ is a linear subset of L_2 $(\Omega,\mathfrak{F},\ P)$. Also for $t\in[0,1]$, denote by $\mathcal{M}_t$ the subset of $\mathcal{M}$ which is only from $\{\xi(s),\ 0\le s\le t\}$. Obviously, $\mathcal{M}_1=\mathcal{M}_0$.

Lemma 1: *D is well-defined on* $\mathcal{M}$.

Proof: Consider $\varphi,\psi:\mathbb{R}^n\to\mathbb{R}$ which satisfy the conditions in Definition 2, and let $t_1,\ldots,t_n$ be such that

$$\varphi(\xi(t_1),\ldots,\xi(t_n))=\psi(\xi(t_1),\ldots,\xi(t_n)) \pmod{P}.$$

Then, it follows from the assumption about ξ that for all $i=1,\ldots,n$,

$$\varphi_i'(\xi(t_1),\ldots,\xi(t_n))=\psi_i'(\xi(t_1),\ldots,\xi(t_n)) \pmod{P}.$$

Thus, the corresponding sums in Definition 2 are equal. The lemma is proved.

Definition 3: A random process ξ is said to have a *logarithmic derivative* with respect to a differentiation rule ζ if there exist a random process $\{\rho_\Delta,\Delta\in\mathfrak{B}\}$ indexed by the Borel subsets of $[0,1]$ such that

1) $\forall\Delta\in\mathfrak{B}$, $M\rho_\Delta^2<+\infty$;
2) $\forall\alpha\in\mathcal{M}$ and $\forall\Delta\in\mathfrak{B}$;

$$M\int_\Delta D\alpha(\tau)d\tau=M\alpha\cdot\rho_\Delta.$$

In the sequel, suppose that the process ξ satisfies the conditions in Definition 3.

Definition 4: Denote for $t\in[0,1]$,

$$m(t)=\rho_{[0,t]}.$$

The process $\{m(t);t\in[0,1]\}$ is called the *logarithmic process.*

Let for $t\in[0,1]$, $\mathfrak{F}_t=\sigma(\{\xi(s);s\le t\})$. Note, that analogous processes were considered in different situation in A. Benassi [1].

Lemma 2: *For* $0\le s\le t\le 1$,

$$M(m(t)-m(s)/\mathfrak{F}_s)=0 \quad (mod\,P).$$

Proof: For $\alpha\in\mathcal{M}_s$ consider

$$M(m(t)-m(s))\cdot\alpha=M\rho_{[0,t]}\cdot\alpha-M_{\rho[0,s]}\cdot\alpha$$

$$= M\int_0^t D\alpha(\tau)d\tau - M\int_0^s D\alpha(\tau)d\tau$$

$$= M\int_{(s,t]} D\alpha(\tau)d\tau$$

$$= \sum_{i=1}^n M\int_{(s,t]} \varphi_i'(\xi(\tau_1),\ldots,\xi(\tau_n))\cdot\zeta(\tau_i)(\tau)d\tau = 0(\text{mod}\,P).$$

Since the set $\mathcal{M}_s$ is dense in $L_2(\Omega,\mathfrak{F}_s,P)$ then the statement of the lemma follows.

For further considerations the following result will be useful.

Lemma 3: *The operator D can be closed as a linear operator from $\mathcal{M}\subset L_2(\Omega,\mathfrak{F},P)$ to $L_2(\Omega\times[0,1],P\times\lambda)$.*

Proof: Consider a sequence $\{\alpha_n;n\geq 1\}\subset\mathcal{M}$, such that there exists $\nu\in L_2(\Omega\times[0,1],P\times\lambda)$ for

$$M\alpha_n^2\rightarrow 0,\quad n\rightarrow\infty,$$

$$M\int_0^1 (D\alpha_n(\tau)-\nu(\tau))^2\lambda(d\tau)\rightarrow 0,\quad n\rightarrow\infty.$$

Then, for every $\Delta\in\mathfrak{B}$ and $\beta\in\mathcal{M}$,

$$M\beta\cdot\int_\Delta \nu(\tau)d\tau = \lim_{n\rightarrow\infty} M\beta\cdot\int_\Delta D\alpha_n(\tau)d\tau$$

$$= \lim_{n\rightarrow\infty}(M\int_\Delta D(\alpha_n\beta)(\tau)d\tau - M\alpha_n\int_\Delta D\beta(\tau)d\tau$$

$$= \lim_{n\rightarrow\infty}(M\alpha_n\beta\cdot\rho_\Delta - M\alpha_n\int_\Delta D\beta(\tau)d\tau$$

$$= \lim_{n\rightarrow\infty} M\alpha_n(\beta\cdot\rho_\Delta - \int_\Delta D\beta(\tau)d\tau) = 0\quad(\text{mod}\,P).$$

So,

$$\int_\Delta \nu(\tau)d\tau = 0\ \ (\text{mod}\,P).$$

Since Δ was arbitrary,

$$\nu = 0\ \ (\text{mod}\,P\times\lambda).$$

The lemma is proved.

Denote the closure of D by the same symbol. The domain of D is denoted by W^1.

3. Integral with Respect to the Logarithmic Process and the Procedure of Approximation

Definition 5: The adjoint operator

$$I = D^*: L_2(\Omega\times[0,1];P\times\lambda)\rightarrow L_2(\Omega,\mathfrak{F},P)$$

is called a *stochastic integration with respect to the process m.* The domain of I is dented by $\mathfrak{D}$.

In the following, suppose that

$$\forall \Delta \in \mathfrak{B}: \rho_\Delta \in W^1,$$

and, that the correspondence $\Delta \mapsto \rho_\Delta$ can be extended by the bounded linear operator A: $L_2([0,1],\lambda) \rightarrow W^1$ (the inner product in W^1 is defined in the usual way, as a sum of L_2-products of random variables and their stochastic derivatives). Note that under this assumption, each $\varphi \in L_2([0,1])$ also belongs to $\mathfrak{D}$ and

$$I(\varphi = A(\varphi).$$

To have I act on random elements of $L_2([0,1])$, i.e., to define an extended stochastic integral with respect to the process m, we need the following.

Let $\{K_n; n \geq 1\}$ be a sequence of symmetric kernels defined on $[0,1]^2$ such that

1) $K_n \in L_2([0,1]^2, \lambda \times \lambda)$,
2) $\forall \varphi \in L_2([0,1],\lambda)$,

$$K_n(\varphi) \rightarrow \varphi, \quad n \rightarrow \infty,$$

where K_n is an integral operator in $L_2([0,1],\lambda)$ with the kernel K_n. Denote for $n \geq 1$,

$$h_n(s,r) = D(\int_0^1 K_n(s,\tau)dm(\tau))(r).$$

It follows form the existence of the operator A that

$$\forall n \geq 1; h_n \in L_2([0,1]^2, \lambda \times \lambda) \pmod{P}.$$

Consider the following sequences of integral operators with random kernels:

$$\forall \varphi \in L_2([0,1],\lambda) \text{ and } \forall n \geq 1;$$

$$B_n(\varphi)(t) = \int_0^1 \varphi(s) \int_0^1 h_n(s,\tau)K_n(t,\tau)d\tau ds,$$

$$C_n(\varphi)(t) = \int_0^t \varphi(s) \int_0^1 h_n(s,\tau)K_n(t,\tau)d\tau ds.$$

Suppose that for the every φ there exist

$$L_2 - \lim_{n \rightarrow \infty} B_n(\varphi) = B(\varphi) \text{ and } L_2 - \lim_{n \rightarrow \infty} C_n(\varphi) = C(\varphi).$$

Then the operators B and C are strong random linear operators (A.V. Skorokhod [4]) which are continuous in L_2-sense.

Definition 6: A random element x from $L_2([0,1],\lambda)$ is said to *belong to the domain* of B (or C) if the sequence $\{B_n(x); n \geq 1\}$ converges in L_2-sense ($\{C_n(x); n \geq 1\}$ respectively).

The following statement can be verified.

Lemma 4: *Let H be a separable real Hilbert space embedded into $L_2([0;1],\lambda)$ by the Hilbert-Schmidt operator, and let x be an essentially bounded random element of H. Then, $x \in \mathfrak{D}(B)$ and*

$x \in \mathfrak{D}(C)$.

Now, consider the stochastic integration. Suppose that the differentiation rule is such that the highest derivatives are symmetric, i.e.,

$$D^2\alpha(\tau_1,\tau_2) = D^2\alpha(\tau_2,\tau_1) \quad (\mathrm{mod}\, P \times \lambda \times \lambda).$$

The space of random variables which have kth stochastic derivative will be denoted by W^k.

Lemma 5: *For every bounded $\alpha_1,\dots,\alpha_n \in W^2$ and for every $\varphi_1,\varphi_2,\dots,\varphi_n \in L^2([0;1],\lambda)$, the sum*

$$x = \sum_{i=1}^{n} \alpha_i \varphi_i \in \mathfrak{D}$$

and

$$I(x) = \sum_{i=1}^{n} \alpha_i I(\varphi_i) - \sum_{i=1}^{n} \int_0^1 D\alpha_i(\tau)\varphi_i(\tau)d\tau,$$

$$MI(x) = 0,$$

$$MI(x)^2 = M\left\{ \int_0^1 (Bx)(\tau)x(\tau)d\tau + tr(Dx \cdot Dx) \right\}.$$

Proof: First consider $x = \alpha \cdot \varphi$. For every $\beta \in \mathcal{M}$,

$$M\int_0^1 D\beta(\tau)\cdot x(\tau)d\tau = M\alpha \int_0^1 D\beta(\tau)\varphi(\tau)d\tau$$

$$= M\int_0^1 (D(\alpha\beta)(\tau) - \beta D\alpha(\tau))\varphi(\tau)d\tau$$

$$= M\alpha\beta I(\varphi) - M\beta\int_0^1 D\alpha(\tau)\varphi(\tau)d\tau$$

$$= M\beta[\alpha I(\varphi) - \int_0^1 D\alpha(\tau)\varphi(\tau)d\tau].$$

So, $\alpha\cdot\varphi \in \mathfrak{D}$ and

$$I(\alpha\cdot\varphi) = \alpha\cdot I(\varphi) - \int_0^1 D\alpha(\tau)\varphi(\tau)d\tau.$$

Consequently,

$$I(\sum_{i=1}^{n}\alpha_i\varphi_i) = \sum_{i=1}^{n}\alpha_i I(\varphi_i) - \sum_{i=1}^{n}\int_0^1 D\alpha_i(\tau)\varphi_i(\tau)d\tau$$

$$= \sum_{i=1}^{n}\alpha_i I(\varphi_i) - tr(D\sum_{i=1}^{n}\alpha_i\varphi_i).$$

To prove that $MI(x) = 0$ it is sufficient to see that $D1 = 0$ and use the equation $I = D^*$. Now, consider the following chain of equalities:

$$MI(x)^2 = M\left[\sum_{i_1 i_2 = 1}^{n} \alpha_{i_1}\alpha_{i_2} I(\varphi_{i_1})\cdot I(\varphi_{i_2}) - 2\sum_{i_1 i_2 = 1}^{n} \alpha_{i_1} I(\varphi_{i_1})\int_0^1 D\alpha_{i_2}(\tau)\varphi_2(\tau)d\tau\right.$$

$$+\sum_{i_1 i_2=1}^{n}\int_0^1 D\alpha_{i_1}(\tau)\varphi_{i_1}(\tau)d\tau\cdot\int_0^1 D\alpha_{i_2}(\tau)\varphi_{i_2}(\tau)d\tau\Bigg]$$

$$= M\Bigg[\sum_{i_1 i_2=1}^{n}\alpha_{i_1}\alpha_{i_2}\cdot\int_0^1 D(I(\varphi_{i_1}))(\tau)\cdot\varphi_{i_2}(\tau)d\tau+\sum_{i_1 i_2=1}^{n}\alpha_{i_1}I(\varphi_{i_1})\int_0^1 D\alpha_{i_2}(\tau)\varphi i_{i_2}(\tau)d\tau$$

$$+\sum_{i_1 i_2=1}^{n}\alpha_{i_2}I(\varphi_{i_1})\int_0^1 D\alpha_{i_1}(\tau)\varphi_{i_2}(\tau)d\tau-2\sum_{i_1 i_2=1}^{n}\alpha_{i_1}I(\varphi_{i_1})\int_0^1 D\alpha_{i_2}(\tau)\varphi_{i_2}(\tau)d\tau$$

$$+\sum_{i_1 i_2=1}^{n}\int_0^1 D\alpha_{i_1}(\tau)\varphi_{i_1}(\tau)d\tau\cdot\int_0^1 D\alpha_{i_2}(\tau)\varphi_{i_2}(\tau)d\tau\Bigg]$$

$$= M\Bigg[\sum_{i_1 i_2=1}^{n}\alpha_{i_1}\alpha_{i_2}\int_0^1 D(I(\varphi_{i_1}))(\tau)\varphi_{i_2}(\tau)d\tau+\sum_{i_1 i_2=1}^{n}\int_0^1 D\alpha_{i_2}(\tau)\varphi_{i_1}(\tau)d\tau\cdot\int_0^1 D\alpha_{i_1}(\tau)\varphi_{i_2}(\tau)d\tau$$

$$+\sum_{i_1 i_2=1}^{n}\alpha_{i_2}\int_0^1\int_0^1 D^2\alpha_{i_1}(\tau_1,\tau_2)\varphi i_{i_1}(\tau_1)\varphi_{i_2}(\tau_2)d\tau_1 d\tau_2$$

$$-\sum_{i_1 i_2=1}^{n}\int_0^1 D\alpha_{i_1}(\tau)\varphi_{i_1}(\tau)d\tau\cdot\int_0^1 D\alpha_{i_2}(\tau)\varphi_{i_2}(\tau)d\tau$$

$$-\sum_{i_1 i_2=1}^{n}\alpha_{i_1}\int_0^1\int_0^1 D^2\alpha_{i_2}(\tau_1,\tau_2)\varphi_{i_1}(\tau_1)\varphi_{i_2}(\tau_2)d\tau_1 d\tau_2$$

$$+\sum_{i_1 i_2=1}^{n}\int_0^1 D\alpha_{i_1}(\tau)\varphi_{i_1}(\tau)d\tau\cdot\int_0^1 D\alpha_{i_2}(\tau)\varphi_{i_2}(\tau)\varphi_{i_2}(\tau)d\tau$$

$$= M\Bigg[\sum_{i_1 i_2=1}^{n}\alpha_{i_1}\alpha_{i_2}\int_0^1 D(I(\varphi_{i_1}))(\tau)\varphi_{i_2}(\tau)d\tau+\sum_{i_1 i_2=1}^{n}\int_0^1 D\alpha_{i_2}(\tau)\varphi_{i_1}(\tau)d\tau\int_0^1 D\alpha_{i_1}(\tau)\varphi_{i_2}(\tau)d\tau\Bigg]$$

$$= M\sum_{i_1 i_2=1}^{n}\alpha_{i_1}\alpha_{i_2}\int_0^1 D(I(\varphi_{i_1}))(\tau)\varphi_{i_2}(\tau)d\tau+tr(Dx)^2.$$

Note that, due to the previous lemma, $x\in\mathfrak{D}(B)$, and

$$B_n(x)=\sum_{i=1}^{n}\alpha_i\int_0^1\varphi_i(s)\int_0^1 D\left(\int_0^1 K_n(s,r)dm(r)\right)(\tau)K_n(\cdot,\tau)ds\,d\tau,\quad n\geq 1.$$

So, from the assumption about the operator A, it follows that

$$B(x) = \lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i \int_0^1 K_n(\cdot,\tau)\cdot D\left(\int_0^1\left(\int_0^1 \varphi_i(s)K_n(s,r)ds\right)dm(r)\right)(\tau)d\tau$$

$$= \sum_{i=1}^{n} \alpha_i \cdot D(I(\varphi_i)).$$

Consequently,

$$\sum_{i_1 i_2 = 1}^{n} \alpha_{i_1}\alpha_{i_2} \int_0^1 D(I(\varphi_{i_1}))(\tau)\varphi_{i_2}(\tau)d\tau = \int_0^1 B(x)(\tau)x(\tau)d\tau.$$

The lemma is proved.

From this lemma and from the fact that I is a closed operator, it follows that every random element x that satisfies the conditions of Lemma 4 and has a stochastic derivative belongs to $\mathfrak{D}$, and the equalities from Lemma 5 are valid.

The famous particular case of this situation is as follows. Let H be a Sobolev space of the first order on $[0,1]$. Then elements of H have usual derivatives with respect to parameters from $[0,1]$. Suppose that x satisfies the conditions of Lemma 4 and that Dx is a.s. a nuclear operator. Then,

$$I(x) = x(1)m(1) - \int_0^1 m(t)x'(t)dt - trDx.$$

Note also that in this case,

$$I(x) = P - \lim_{n\to\infty}\left\{\int_0^1 x(t)\int_0^1 K_n(t,\tau)dm(\tau)dt - \int_0^1\int_0^1 Dx(t)(\tau)K_n(t,\tau)d\tau dt\right\}. \quad (1)$$

This expansion enables one to establish the Itô formula.

Theorem (The Itô formula): *Let a function $F:[0,1]\times\mathbb{R}\to\mathbb{R}$ have a continuous bounded derivative of the first and second order, and let the random process x satisfy the conditions:*

1) *x has the second stochastic derivative;*
2) *for every $\tau\in[0,1]$, x and $Dx(\cdot)(\tau)$ satisfy all integrability conditions (considered above);*
3) *$Dx(\cdot)(\cdot)\in C([0,1]^2)$ $(mod P)$;*
4) *x, Dx and D^2x are bounded.*

Then, the following random process

$$z(t) = \int_0^t x(\tau)dm(\tau), \quad t\in[0,1]$$

is well-defined and it holds true that

$$F(t,z(t)) = F(0,0) + \int_0^t F_1'(s,z(s))ds$$

$$+ \int_0^t F_2'(s,z(s))x(s)dm(s) + \int_0^t x(s)F_{22}''(s,z(s))C(x)(s)ds$$

$$+ \int_0^t x(s)F_{22}''(s,z(s))\cdot\int_0^s Dx(r)(s)dm(r)ds.$$

The proof follows directly from the expansion (1) and approximation arguments.

4. Examples

Example 1: (Wiener case) Let $\xi(t) = w(t),\ \ t \in [0,1]$ be a Wiener process. Consider the differentiation rule of the form $\zeta(t) = \chi_{[0,t]}, t \in [0,1]$. Then the stochastic derivative D which is obtained from this rule is a well-known stochastic derivative of L_2-integrable Wiener functionals (T. Sekiguchi, Y. Shiota [3]) and $m(t) = w(t),\ \ t \in [0,1]$.

Now the operator B is the identity operator and $C = \frac{1}{2}B$. Then, from the previous theorem we can obtain the Itô formula for the extended stochastic integral in the Gaussian case:

$$F(t, z(t)) = F(0,0) + \int_0^t F'_1(s, z(s))ds + \int_0^t F'_2(s, z(s))dw(s)$$

$$+\frac{1}{2}\int_0^t F''_{22}(s, z(s)) \cdot x(s)^2 ds + \int_0^t x(s)F''_{22}(s, z(s)) \cdot \int_0^s Dx(r)(s)dw(r)ds.$$

Example 2: Let the distribution of the process ξ in the space $C([0,1])$ be absolutely continuous with respect to the Wiener measure with the density p. Suppose, that

1) $0 < \inf p \le \sup p < +\infty$,

2) p has a bounded continuous derivative on $C([0,1])$.

Consider the differentiation rule from Example 1: $\zeta(t) = \chi_{[0,t]},\ t \in [0,1]$. Then the stochastic derivative of the random variable α from the family $\mathfrak{I}\mathcal{M}$ ($\mathcal{M}$) is of type

$$D\alpha = D\varphi(\xi(t_1), \ldots, \xi(t_n)) = \sum_{i=1}^n \varphi'_i \chi_{[0,t_i]}.$$

Hence, for the Borel subset

$$M\int_\Delta D\alpha(\tau)d\tau = M\sum_{i=1}^n \varphi'_i \langle \delta_{t_i}, \int_0^{\cdot} \chi_\Delta(\tau)d\tau\rangle.$$

Here δ_t is Dirac δ-function with respect to the point t. Denote by u_Δ the function

$$u_\Delta(s) = \int_0^s \chi_\Delta(\tau)d\tau,\ \ s \in [0,1],$$

by ν the distribution of ξ, and by μ the Wiener measure. Also, denote by Φ the following function on $C([0,1])$:

$$\forall v \in C([0,1]), \Phi(v) = \varphi(v(t_1), \ldots, v(t_n)).$$

Then,

$$M\int_\Delta D\alpha(\tau)d\tau = \int \langle \Phi'(v); u_\Delta\rangle \nu(dv) = \int \langle \Phi'(v); u_\Delta\rangle p(v)\mu(dv)$$

$$= \int \langle (p(v)\Phi(v))'; u_\Delta\rangle \mu(dv) - \int \langle p'(v); u_\Delta\rangle \cdot \Phi(v)\mu(dv) = \int \Phi(v)p(v) \cdot \int_\Delta dv(\tau)\mu(dv)$$

$$-\int \Phi(v)\langle(\ln p(v))';u_\Delta\rangle p(v)\mu(dv)=\int \Phi(v)\left[\int_\Delta dv(\tau)-\langle(\ln p(v))';u_\Delta\rangle\right]\nu(dv).$$

Here the symbol of integration is used for the integration through all $C([0,1])$, and the integral

$$\int_\Delta dv(\tau)$$

is a measurable linear functional on $C([0,1])$ with respect to the measure $\nu\sim\mu$. Note also that the function

$$\int_\Delta dv(\tau)-\langle(\ln p(v))';u_\Delta\rangle$$

is square-integrable with respect to the measure ν. Consequently, ξ has a logarithmic derivative, and

$$\rho_\Delta=\int_\Delta d\xi(\tau)-\langle(\ln p(\xi))';u_\Delta\rangle.$$

So, the operator D is closed, and for every bounded functional ψ, which has a bounded continuous derivative on $C([0,1])$, the random variable $\psi(\xi)$ belongs to W^1; in particular, $\ln p(\xi)\in W^1$ and

$$\int_\Delta d\ln p(\xi)(\tau)d\tau=\langle(\ln p(\xi))';u_\Delta\rangle.$$

Hence, the logarithmic process is of the form

$$m(t)=\xi(t)-\int_0^t D\ln p(\xi)(\tau)d\tau.$$

Now the second stochastic derivatives are symmetric. So to estimate the second moment of the extended stochastic integral only the operator B is essential. To describe the operators B and C let us find the stochastic derivative of the integral

$$\int_0^1 f(\tau)dm(\tau)=\int_0^1 f(\tau)d\xi(\tau)-\int_0^1 f(\tau)D\ln p(\xi)(\tau)d\tau.$$

Using the approximation by step functions, it can be verified that

$$D\left(\int_0^1 f(\tau)dm(\tau)\right)(s)=f(s)+\int_0^1 f(\tau)\cdot D^2\ln p(\xi)(\tau,s)d\tau,\quad s\in[0,1].$$

Consequently, for the $n\geq 1$,

$$B_n(\varphi)(t)=\int_0^1\varphi(s)\int_0^1\left[K_n(s,\tau)+\int_0^1 K_n(s,r)D^2\ln p(\xi)(r,\tau)dr\right]\cdot K_n(t,\tau)d\tau ds.$$

Hence,

$$B(\varphi)(t)=\varphi(t)+\int_0^1 D^2\ln p(\xi)(t,s)\varphi(s)ds.$$

In a similar way,

$$C(\varphi)(t) = \frac{1}{2}\varphi(t) + \int_0^t D^2 ln\, p(\xi)(s,t)\varphi(s)ds.$$

Now the second moment of the extended stochastic integral and the Itô formula have the form

$$M\left(\int_0^1 x(t)dm(t)\right)^2 = M\int_0^1 x^2(t)dt + M\int_0^1\int D^2 ln\, p(\xi)(t,s)x(t)x(s)dtds + M(tr(Dx)^2,$$

$$F(t,z(t)) = F(0,0) + \int_0^t F_1'(s,z(s))ds + \int_0^t F_2'(s,z(s))x(s)dm(s)$$

$$+\frac{1}{2}\int_0^t F_{22}''(s,z(s))x^2(s)ds + \int_0^t F_{22}''(s,z(s))x(s)\int_0^s D^2 ln\, p(\xi)(\tau,s)x(\tau)d\tau ds$$

$$+\int_0^t x(s)F_{22}''(s,z(s))\cdot\int_0^s Dx(r)(s)dm(r)ds.$$

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