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ON AN INFINITE-DIMENSIONAL DIFFERENTIAL EQUATION IN VECTOR DISTRIBUTION WITH DISCONTINUOUS REGULAR FUNCTIONS IN A RIGHT HAND SIDE

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ABSTRACT

An infinite-dimensional differential equation in vector distribution in a Hilbert space is studied in case of an unbounded operator and discontinuous regular functions in a right-hand side. A unique solution (*vibrosolution*) is defined for such an equation, and the necessary and sufficient existence conditions for a vibrosolution are proved. An equivalent equation with a measure, which enables us to directly compute jumps of a vibrosolution at discontinuity points of a distribution function, is also obtained. The application of the obtained results to control theory is discussed in the conclusion.

Key words: Infinite-Dimensional Equation, Discontinuous Right-hand Side, Distribution.

AMS (MOS) subject classifications: 34K35.

1. Introduction

This paper studies an infinite-dimensional differential equation in vector distribution, whose right-hand side also contains discontinuous regular (not generalized) functions. It should be noted that a solution to a differential equation in distribution cannot be defined as a conventional solution (using the Lebesgue-Stieltjes integral) owing to multiplication of the distribution by a discontinuous regular function. Thus, the basic problems are to introduce an appropriate solution (*vibrosolution*), obtain the existence and uniqueness conditions for a vibrosolution, and design an equivalent equation with a measure, which enables us to directly compute jumps of a vibrosolution at discontinuity points of a distribution function.

Infinite-dimensional equations in vector distribution appear, for example, when solving the ellipsoidal guaranteed estimation problem [14] over discontinuous observations [2], or considering infinite-dimensional (solid state) impulsive Lagrangian systems [4]. The definition of a unique vibrosolution to a differential equation is first introduced in the background paper [9] and is shown again in Section 3. Finite-dimensional differential equations in scalar distribution with discontinuous regular functions in right-hand sides are studied in [1]. Finite-dimensional equations in vector distribution are then considered in [3]. This paper generalizes the results obtained in [1, 3] to the case of infinite-dimensional differential equations in vector distribution. The substantiation of existence and uniqueness conditions is based [7, 8] on the representation of a solution to a differential equation in a Hilbert space as a Fourier sum of solutions to finite-dimensional differential equations.

The paper is organized as follows. The problem statement is given in Section 1. In Section 2 a solution to an infinite-dimensional equation in vector distribution is introduced as a *vibrosolution*, that is defined as a unique limit. Sections 3 and 4 present the necessary and sufficient existence conditions for a vibrosolution, respectively. By definition, existence of a vibrosolution yields its uniqueness. In Section 5 an equivalent equation with a measure is designed. The application of the obtained results to control theory is discussed in the conclusion.

2. Problem Statement

Let us consider an infinite-dimensional differential equation in vector distribution with an unbounded operator in a right-hand side

$$\dot{x}(t) = Ax(t) + f(x,u,t) + \beta(x,u,t)b(x,u,t)\dot{u}(t), \quad x(t_0) = x_0, \tag{1}$$

where $x(t) \in H$; A is a generator of a strongly continuous semigroup such that $(-A)$ is a strongly positive operator and has a compact inverse operator A^{-1}; $f(x,u,t) \in H$, $b(x,u,t) \in L(R^m \to H)$ are bounded continuous functions defined in the space $H \times R^m \times R$, $L(\mathcal{A} \to \mathfrak{B})$ is a space of linear continuous operators from a space $\mathcal{A}$ to a space $\mathfrak{B}$; $\beta(x,u,t) \in R$ is a scalar piecewise continuous in x, u, t function such that its continuity domain is locally connected; $u(t) = (u_1(t), \ldots, u_m(t)) \in R^m$ is a vector bounded variation function which is non-decreasing in the following sense: $u(t_2) \geq u(t_1)$ as $t_2 \geq t_1$, if $u_i(t_2) \geq u_i(t_1)$ for $i = 1, \ldots, m$.

Let $S_t(\cdot): H \to H$ be a strongly continuous semigroup generated by an operator A, and $D(A) \subset H$ be a definition domain. The following conditions imposed on an initial value and a right-hand side of the equation (1): 1) $S_t(x_0) \in D(A)$, 2) $S_{t-s}(f(x,u,s) + \beta(x,u,s)b(x,u,s)\dot{w}(s)) \in D(A)$, $s \leq t$, are assumed to hold for any absolutely continuous non-decreasing function $w(s) \in R^m$.

3. Definition of a Solution

Let us note that a solution to the equation (1) cannot be defined as a conventional solution owing to multiplication of distribution $\dot{u}(t)$ by a discontinuous in t function $b(x(t), u(t), t)$.

If $u(t) \in R^m$ is an absolutely continuous function, then an absolutely continuous solution to the equation (1) is defined in the sense of Filippov [6]. Following [6], a function $\kappa(x,u,t)$ is said to be piecewise continuous in a finite domain $G \subset H \times R^{m+1}$ if

1) a domain G consists of a finite number of continuity domains G_i, $i = 1, \ldots, m$, with boundaries Γ_i,
2) a function $\kappa(x,u,t)$ has finite one-side limiting values along boundaries Γ_i,
3) the set consisting of all boundaries Γ_i has zero measure.

A function $\kappa(x,u,t)$ is said to be piecewise continuous in $H \times R^{m+1}$ if it is piecewise continuous in each finite domain $G \subset H \times R^{m+1}$.

If $u(t) \in R^m$ is an absolutely continuous function, then a solution to the equation (1) is defined [6] as an absolutely continuous solution to the differential inclusion $\dot{x}(t) \in Ax(t) + F(x,t)$, where $F(x,t)$ is a minimum convex closed set containing all limiting values $f(x^*, u(t), t) + \beta(x^*, u(t), t)b(x^*, u(t), t)\dot{u}(t)$ as $x^* \to x$, $t = const$, while points $(x^*, u(t), t)$ are not included in a discontinuity set of the function $f(x, u(t), t) + \beta(x, u(t), t)b(x, u(t), t)\dot{u}(t)$.

The existence and uniqueness conditions for an absolutely continuous solution to the equation (1) are given in the next lemma that is a direct corollary to theorem 1 [11].

Lemma: *Let the above conditions hold, and the functions* $f(x,u,t)$, $\beta(x,u,t)b(x,u,t)$ *satisfy the one-sided Lipschitz condition in* x

$$(x - y, f(x,u,t) - f(y,u,t)) \leq m_1(t,u)(x - y, x - y),$$

$$((\beta(x,u,t)b(x,u,t) - \beta(y,u,t)b(y,u,t))^*(x - y)) \leq m_2(t,u)(x - y, x - y),$$

where functions $m_1(t,u) \in R$, $m_2(t,u) \in R^m$ *are integrable in* t, u; $B^*: H \to R^m$ *is an operator adjoint to an operator* $B: R^m \to H$.

Then there exists a unique absolutely continuous solution to the equation (1) *corresponding to an absolutely continuous function* $u(t) \in R^m$.

In case of an arbitrary non-decrease function $u(t) \in R^m$, a solution to the equation (1) is defined as a *vibrosolution* [9]. A vibrosolution is expected to be a function discontinuous at discontinuity points of the function $u(t)$.

Definition 1: The convergence in the Hilbert space H

$$* - \lim x^k(t) = x(t), \quad t \geq t_0,$$

is said to be the *$*$-weak convergence* if the following conditions hold

1) $\lim \| x^k(t_0) - x(t_0) \| = 0$, $t \geq t_0$,
2) $\lim \| x^k(t) - x(t) \| = 0$, $t \geq t_0$, in all continuity points of the function $x(t)$,
3) $\sup_k Var_\infty[t_0, T]x^k(t) < \infty$ for any $T \geq t_0$, where a variation of a function $x(t) \in H$ is defined by

$$Var_\infty[a,b]x(t) = \| x(t) \| + \sup_\tau \sum_{i=1}^{N} \| x(t_i) - x(t_{i-1}) \|, \tag{2}$$

and *supremum* is over all possible partitions $\tau = (a = t_0, t_1, \ldots, t_N = b)$, $\| \cdot \|$ is the norm in the space H.

Definition 2: The left-continuous function $x(t)$ is said to be a *vibrosolution* to the equation (1) if the *$*$-weak convergence* of an arbitrary sequence of absolutely continuous non-decreasing functions $u^k(t) \in R^m$ to a non-decreasing function $u(t) \in R^m$

$$* - \lim u^k(t) = u(t)$$

implies the analogous convergence

$$* - \lim x^k(t) = x(t)$$

of corresponding solutions $x^k(t)$ to the equation

$$\dot{x}^k(t) = Ax^k(t) + f(x^k, u^k, t) + \beta(x^k, u^k, t)b(x^k, u^k, t)\dot{u}^k(t), \quad x^k(t_0) = x_0,$$

and the unique limit $x(t)$ occurs regardless of a choice of an approximating sequence $\{u^k(t)\}$, $k = 1, 2, \ldots$.

4. Existence of a Solution. Necessary Conditions

As in case of a finite-dimensional differential equation in distribution [1, 3], existence of a vibrosolution to an equation (1) is closely related to the solvability of a certain associated system in differentials.

Theorem 1: *Let the lemma conditions hold.*

If a unique vibrosolution to the equation (1) exists, then a system of differential equations in differentials in the space H

$$\frac{d\xi}{du} = \beta(\xi, u, s)b(\xi, u, s), \quad \xi(\omega) = z, \tag{3}$$

is solvable inside a cone of positive directions $K = \{u \in R^m: u_i \geq \omega_i,\ i = 1, \ldots, m\}$ *with arbitrary initial values* $\omega \in R^m$, $\omega \geq u(t_0), z \in H$, *and* $s \geq t_0$.

Proof: Consider a vibrosolution $x(t)$ to an equation (1) with an initial value $x(s) = z$ and a function $u(t) = \omega + (v - \omega)\chi(t - s)$, where $\chi(t - s)$ is a Heaviside function, $\omega, v \in R^m$, $v \geq \omega$, and $s \geq t_0$. By virtue of the theorem conditions this vibrosolution exists. Let us prove that under the theorem conditions the Kurzweil equality [10]

$$x(s+) = y(1) \tag{4}$$

holds as $x(s+) = \lim x(t)$, $t \to s+$, where a limit is regarded in the norm of the space H, and $y(\tau)$ is a solution to the equation

$$\frac{dy}{d\tau} = \beta(y, \omega + (v - \omega)\tau, s)b(y, \omega + (v - \omega)\tau, s)(v - \omega), \quad y(0) = z, \quad 0 \leq \tau \leq 1. \tag{5}$$

By virtue of the given lemma and the theorem conditions, an absolutely continuous solution to the equation (5) $y(\tau)$ exists and is unique, if $v \geq \omega$.

Following the proof of theorem 1 [12], it is readily verified that under the theorem conditions, the functions $y^k(\tau) = x^k(s + \tau/k)$, $0 \leq \tau \leq 1$, $k = 1, 2, \ldots$, where $x^k(t)$ are vibrosolutions to equations (1) with initial values $x^k(s) = z$ and absolutely continuous functions $u^k(t) = \omega$, if $t \leq s$, $u^k(t) = \omega + k(v - \omega)(t - s)$, if $s \leq t \leq s + 1/k$, and $u^k(t) = v$, if $t \geq s + 1/k$, in right-hand sides, are solutions to the equations

$$\frac{dy^k}{d\tau} = [Ay^k(\tau) + f(y^k, \omega + (v - \omega)\tau, s + \tau/k)]/k$$
$$+ \beta(y^k, \omega + (v - \omega)\tau, s + \tau/k)b(y^k, \omega + (v - \omega)\tau, s + \tau/k)(v - \omega), \quad y^k(0) = z,$$

and the following equality holds

$$x^k(s + 1/k) = y^k(1). \tag{6}$$

By virtue of the theorem on continuous dependence of a solution to a differential inclusion on a right-hand side in a Banach space [11], a sequence of absolutely continuous functions $y^k(\tau)$ converges to an absolutely continuous solution to the equation (5) pointwise in the norm of the space H:

$$\lim \| y^k(\tau) - y(\tau) \| = 0, \quad k \to \infty, \quad \tau \in [0, 1].$$

Thus, $\lim y^k(1) = y(1)$, $k \to \infty$, and by virtue of (6)

$$\lim_{t \to s+} \lim_{k \to \infty} x^k(t) = y(1).$$

Taking into account the equality

$$\lim_{t \to s+} \lim_{k \to \infty} x^k(t) = x(s+),$$

where $x(t)$ is a vibrosolution to the equation (1), the Kurzweil equality (4) is proved.

Define the function $\xi(z,\omega,v,s)$ by

$$\xi(z,\omega,v,s) = y(1) = z + \int_0^1 \beta(y(\tau),\omega + (v-\omega)\tau, s) b(y(\tau), \omega + (v-\omega)\tau, s)(v-\omega) d\tau, \tag{7}$$

where $y(\tau)$ is a solution to the equation (5). Since a solution $y(\tau)$ exists and is unique under the theorem conditions, if $v \geq \omega$, the function $\xi(z,\omega,v,s) \in H$ is uniquely defined inside a cone $K = \{u \in R^m : u_i \geq \omega_i,\ i = 1,\ldots,m\}$. It only remains to prove that

$$\frac{d\xi(z,\omega,v,s)}{dv} = \beta(\xi(z,\omega,v,s),v,s) b(\xi(z,\omega,v,s),v,s).$$

However, the proof of this correlation is quite consistent with the last part of the proof of theorem 1 [12] and can be omitted here. Thus, the function $\xi(z,\omega,v,s)$ defined by (7) is a unique solution to the system of equations in differentials (3) inside a cone K as $s \geq t_0$. Theorem 1 is proved.

5. Existence of a Solution. Sufficient Conditions

Let us prove that under additional conditions imposed on a function $\beta(x,u,t)b(x,u,t)$ the necessary existence conditions for a vibrosolution to an equation (1) coincide with the sufficient ones.

Theorem 2: *Let* 1) *the lemma conditions hold, and, moreover,*

2) $\{\partial b(x,u,t)/\partial x\} \in L(H \to L(R^m \to H))$, $\{\partial b(x,u,t)/\partial t\} \in L(R^m \to H)$ *be bounded continuous defined in the space* $H \times R^m \times R$,

3) *functions* $\partial \beta(x,u,t)/\partial x, \partial \beta(x,u,t)/\partial t$ *be piecewise continuous in* x,u,t *and their continuity domains be locally connected.*

If a system of differential equations in differentials (3) *is solvable inside a cone of positive directions* $K = \{u \in R^m : u_i \geq \omega_i, i = 1,\ldots,m\}$ *with arbitrary initial values* $\omega \in R^m$, $\omega \geq u(t_0)$, $z \in H$, *and* $s \geq t_0$, *then a unique vibrosolution to the equation* (1) *exists.*

Proof: Let $[u^k(t)\}, k = 1,2,\ldots,$ be a sequence of absolutely continuous non-decreasing functions $u^k(t) \in R^m$, which tends to a distribution function $u(t)$ in the sense of the $*$-weak convergence. Consider the equation (1) with absolutely continuous non-decreasing functions $u^k(t)$ in a right-hand side, that is

$$\dot{x}^k(t) = Ax^k(t) + f(x^k,u^k,t) + \beta(x^k,u^k,t) b(x^k,u^k,t)\dot{u}^k(t), \quad x^k(t_0) = x_0. \tag{8}$$

It should be noted that the theorem conditions (1)-(3), the lemma of Section 2, and the theorem 1 [11] yield existence and uniqueness of an absolutely continuous solution to the equation (8). As follows from [7, 8], this solution can be represented as a Fourier sum in the space H on the complete orthonormal basis $\{c_i\}_{i=0}^{\infty}$ generated by eigenfunctions of the operator A

$$x^k(t) = \sum_{i=0}^{\infty} x_i^k(t)c_i. \tag{9}$$

Scalar functions $x_i^k(t)$ satisfy the equations

$$dx_i^k(t) = \lambda_i x_i^k(t)dt + f_i(x_i^k, u^k, t) + \beta(x_i^k, u^k, t)b_i(x_i^k, u^k, t)du^k(t), \quad x_i^k(0) = x_{i0}, \tag{10}$$

$\{\lambda_i\}_{i=0}^{\infty}$ is a countable set [7] of eigenvalues of the operator A, and $f_i(x,u,t) \in R$, $b_i(x,u,t) \in R^m$, $x_{i0} \in R$, $i = 0,1,2,\ldots$, are Fourier coefficients for a function $f(x,u,t)$, an operator $b(x,u,t)$, and an initial value x_0 on the basis $\{c_i\}_{i=0}^{\infty}$, respectively:

$$f(x,u,t) = \sum_{i=0}^{\infty} f_i(x,u,t)c_i, \quad b(x,u,t) = \sum_{i=0}^{\infty} b_i(x,u,t)c_i, \quad x_0 = \sum_{i=0}^{\infty} x_{i0}c_i. \tag{11}$$

The convergence of the Fourier series (11) is regarded in the norms of the corresponding Hilbert spaces.

Consider an infinite ($i = 0,1,2,\ldots$) number of finite-dimensional equations (10) which contain an arbitrary non-decreasing function $u(t) \in R^m$ in right-hand sides

$$dx_i(t) = \lambda_i x_i(t)dt + f_i(x_i, u, t) + \beta(x_i, u, t)b_i(x_i, u, t)du(t), \quad x_i(0) = x_{i0}, \tag{12}$$

whose solutions are thus regarded as vibrosolutions. Since $x_i(t) \in R$ are scalar functions, existence and uniqueness of solutions to the equations (12) are assured of the existence and uniqueness theorem for a vibrosolution [3] by virtue of the inequalities $Re(\lambda_i) < 0$ [7], the theorem conditions (1)-(3), and the solvability of the system of equations in differentials (3) inside a cone K. Then, taking into account the vibrosolution definition given in Section 2, we obtain the pointwise convergence of absolutely continuous solutions $x_i^k(t)$ to the equations (10) to vibrosolutions $x_i(t)$ to the equations (12)

$$\lim \mid x_i^k(t) - x_i(t) \mid = 0, \quad k \to \infty, \quad t \geq t_0, \quad i = 0,1,2,\ldots,$$

in all continuity points of the function $u(t)$ as

$$* - \lim u^k(t) = u(t), \quad t \to \infty,$$

where $u^k(t) \in R^m$ are absolutely continuous non-decreasing functions. Thus,

$$\lim \| \sum_{i=0}^{N} x_i^k(t)c_i - \sum_{i=0}^{N} x_i(t)c_i \| = 0, \quad k \to \infty, \ N < \infty,$$

in all continuity points of the function $u(t)$.

Consider the Fourier sum generated by the functions $x_i(t)$ on the basis $\{c_i\}_{i=0}^{\infty}$

$$\sum_{i=0}^{\infty} x_i(t)c_i. \tag{13}$$

Let us prove that the series (13) converges in the norm of the space H. Indeed, the following inequalities [5] hold

$$\| \sum_{i=N}^{\infty} x_i(t)c_i \|$$

$$= \| \sum_{i=N}^{\infty} \{x_{i0}\exp(\lambda_i(t)) + \int_{t_0}^{t} \exp(\lambda_i(t-s))[f_i(x_i,u,s)ds + \beta(x_i,u,s)b_i(x_i,u,s)du(s)]\}c_i \|$$

$$\leq \sum_{i=N}^{\infty} \| x_{i0}\exp(Re(\lambda_i(t))) \| + \sum_{i=N}^{\infty} \| \int_{t_0}^{t} \exp(Re(\lambda_i(t-s)))f_i(x_i,u,s)ds \|$$
$$+ \sum_{i=N}^{\infty} \| \int_{t_0}^{t} \exp(Re(\lambda_i(t-s)))\beta(x_i,u,s)b_i(x_i,u,s)du(s) \| < \infty,$$

since functions $f_i(x,u,t)$ and $\beta(x,u,t)b_i(x,u,t)$ are bounded and satisfy the one-sided Lipschitz condition, $\lim Re(\lambda_i) = -\infty$ as $i \to \infty$ [7], $t - s \geq 0$, the Fourier series (11) converge, and the latter integral is with a bounded variation function $u(t)$. Thus, the Fourier series (13) converges, i.e., there exists an H-valued function $x(t) \in H$ such that

$$\lim \| x(t) - \sum_{i=0}^{N} x_i(t)c_i \| = 0, \quad N \to \infty.$$

Let us finally prove that the function $x(t) \in H$ obtained as a Fourier sum (13) is a vibrosolution to the equation (1). For any $\epsilon > 0$ there exist a number N_1 such that the inequality

$$\| x(t) - \sum_{i=0}^{N_1} x_i(t)c_i \| < \epsilon/3,$$

holds by virtue of the convergence of a Fourier series (13), and for any $\epsilon > 0$ there exists a number N_2 such that the inequality

$$\| x^k(t) - \sum_{i=0}^{N_2} x_i^k(t)c_i \| < \epsilon/3,$$

holds by virtue of convergence of a Fourier series (9), Moreover, for any $\epsilon > 0$ there exist a number $N = \max(N_1, N_2)$ and a number K such that for any $k \geq K$ the inequality

$$\| \sum_{i=0}^{N} x_i(t)c_i - \sum_{i=0}^{N} x_i^k(t)c_i \| < \epsilon/3,$$

holds in all continuity points of the function $u(t)$, since $x_i(t)$ is a vibrosolution to the equation (12) and $\{x_i^k(t)\}$, $k = 1,2,\ldots$, is a sequence of approximating solutions to the equations (10). Thus, for any $\epsilon > 0$ there exist a number N and number K such that for any $k \geq K$ we obtain

$$\| x(t) - x^k(t) \| \leq \| x(t) - \sum_{i=0}^{N} x_i(t)c_i \| \tag{14}$$
$$+ \| \sum_{i=0}^{N} x_i(t)c_i - \sum_{i=0}^{N} x_i^k(t)c_i \| + \| x^k(t) - \sum^{N} x_i^k(t)c_i \| < \epsilon,$$

in all continuity points of the function $u(t)$. The inequalities (14) yield the convergence in the norm of the space H

$$\lim \| x^k(t) - x(t) \| = 0, \ \ k \to \infty,$$

in all continuity points of the function $u(t)$. Moreover,

$$x^k(t_0) = x(t_0) = x_0$$

by virtue of coincidence of initial values of the equations (1) and (8), and the inequality

$$\sup Var_{\infty}[t_0, t]x^k(T) < \infty \text{ for any } T \geq t_0$$

holds by virtue of the uniform boundedness of variations of absolutely continuous functions $x_i^k(t)$, $\sup Var[t_0, T]x_i^k(t) < \infty$ for all $i = 0, 1, 2, \ldots$ and any $T \geq t_0$, and the convergence of the Fourier series (9). Thus, the $*$-weak convergence in the space H

$$* - \lim x^k(t) = x(t), \quad k \to \infty, \quad t \geq t_0,$$

is proved. Since $x^k(t)$, $k = 1, 2, \ldots$, are absolutely continuous solutions to the equations (8) that are equations (1) with absolutely continuous non-decreasing functions $u^k(t) \in R^m$ in right-hand sides, the vibrosolution definition implies that the function $x(t) \in H$ is a vibrosolution to the equation (1). Theorem 2 is proved.

Remark: The vibrosolution definition as well as the necessary and sufficient existence conditions can also be stated for nonmonotonic functions $u(t) \in R^m$, assuming that the one-sided Lipschitz condition holds for a function $\beta(x, u, t)b(x, u, t)sign(\dot{u}(t))$ and approximating functions $\beta(x^k, u^k, t)b(x^k, u^k, t)\dot{u}^k(t)$ for any $k = 1, 2, \ldots$.

6. Equivalent Equation with a Measure

It should be noted that only vibrosolutions, which correspond to absolutely continuous functions $u^k(t) \in R^m$, are absolutely continuous solutions to a differential equation in distribution (1). Therefore, it is not clear how to compute jumps of a vibrosolution to a differential equation in distribution at discontinuity points of an arbitrary non-decreasing function $u(t) \in R^m$. Thus, it is helpful to design an equivalent equation with a measure whose conventional (in the sense of the definition of a solution to an ordinary differential equation with a discontinuous right-hand side that is given in Section 2) solution coincides with a vibrosolution to an equation (1), and which enables us to directly compute jumps of a solution at discontinuity points of an arbitrary non-decreasing function $u(t) \in R^m$.

Theorem 3: *Let the theorem 2 conditions hold. Then an equation (1) and an equivalent equation with a measure*

$$dy(t) = Ay(t)dt + f(y, u, t)dt + \beta(y, u, t)b(u, u, t)du^c(t)$$
$$+ \sum_{t_i} G(y(t_i -), u(t_i -), \Delta u(t_i), t_i)d\chi(t - t_i), \quad y(t_0) = x_0. \tag{15}$$

have the same unique solution regarded for in equation (1) as a vibrosolution.

Here $G(z, v, u, s) = \xi(z, v, v + u, s) - z$, where $\xi(z, v, u, s)$ is a solution to a system of equations in differentials (3); $u^c(t)$ is a continuous component of a non-decreasing function $u(t)$, $\Delta u(t_i) = u(t_i +) - u(t_i -)$ is a jump of a function $u(t)$ at t_i, t_i are discontinuity points of a function $u(t)$, $\chi(t - t_i)$ is a Heaviside function.

Proof: A function of jumps $G(z, \omega, u, s)$ is bounded in the norm of the space H as a solution to the system (3) with a right-hand side $\beta(\xi, u, s)b(\xi, u, s)$ satisfying the one-sided Lipschitz condition. Then, by virtue of the lemma of Section 2 and the theorem 1 [11], a solution to the equation with a measure (15) exists and is unique as a bounded variation function with an absolutely continuous component in continuity intervals of the function $u(t)$ and the jumps determined by the function $G(y(t_i -), u(t_i -), \Delta u(t_i), t_i)$ at discontinuity points of the function $u(t)$. As follows from [7, 8], this solution can be represented as a Fourier sum in the space H on the complete orthonormal basis $\{c_i\}_{i=0}^{\infty}$ generated by eigenfunctions of the operator A:

$$\sum_{i=0}^{\infty} y_i(t)c_i. \tag{16}$$

Scalar functions $y_i(t)$ satisfy the equation

$$y_i(t) = \lambda_i y_i(t)dt + f_i(y_i, u, t) + \beta(y_i, u, t)b_i(y_i, u, t)du^c(t) \tag{17}$$
$$+ \sum_{t_i} G_i(y_i(t_i-), u(t_i-), \Delta u(t_i), t_i)d\chi(t-t_i), \quad y_i(t_0) = x_{i0},$$

where scalar functions $G_i(z_i, \omega, u, s)$ are defined as follows

$$G_i(z_i, \omega, u, s) = \xi_i(z_i, \omega, \omega + u, s) - z_i,$$

and functions $\xi_i(z_i, \omega, u, s)$, $u \geq \omega$, $s \geq t_0$, are solutions to the equations in differentials

$$\frac{d\xi_i}{du} = \beta(\xi_i, u, s)b_i(\xi_i, u, s), \quad \xi_i(\omega) = z_i, \quad i = 0, 1, 2, \ldots,$$

inside cones of positive directions $K = \{u \in R^m : u_j \geq \omega_j, j = 1, \ldots, m\}$ with arbitrary initial values $\omega \in R^m$, $z_i \in R$, and $s \geq t_0$.

Consider an equation (1) with an arbitrary non-decreasing function $u(t) \in R^m$ in a right-hand side. Existence and uniqueness of a vibrosolution to such an equation have already been proved in the theorem 2. That vibrosolution can also be represented as a Fourier sum (13) on the basis $\{c_i\}_{i=0}^{\infty}$:

$$x(t) = \sum_{i=0}^{\infty} x_i(t)c_i,$$

where scalar functions $x_i(t)$ satisfy the equations (12)

$$dx_i(t) = \lambda_i x_i(t)dt + f_i(x_i, u, t) + \beta(x_i, u, t)b_i(x_i, u, t)du(t), \quad x_i(0) = x_{i0}.$$

The equations (12) and (17) are finite-dimensional equations whose right-hand sides contain vector distribution and piecewise continuous regular functions satisfying the one-sided Lipschitz condition. Thus, by the virtue of theorem 2 [3], unique solutions $x_i(t)$ and $y_i(t)$ to the equations (12) and (17) coincide as vibrosolutions for any $i = 0, 1, 2, \ldots$. The vibrosolution definition given in Section 2 implies that for any $i = 0, 1, 2 \ldots$, the equalities

$$| x_i(t) - y_i(t) | = 0, \quad t \geq t_0, \tag{18}$$

hold in all continuity points of the function $u(t)$.

Let us finally prove that solutions to the equations (1) and (15) are indistinguishable as vibrosolutions. Let

$$\| x(t) - y(t) \| \neq 0, \quad t \geq t_0,$$

for at least one continuity point of the function $u(t) \in R^m$. Expand the mentioned solutions into Fourier series on the basis $\{c_i\}_{i=0}^{\infty}$:

$$x(t) - y(t) = \sum_{i=0}^{\infty} (x_i(t) - y_i(t))c_i, \quad t \geq t_0.$$

By virtue of the equalities (18) and the uniqueness of the expansion into a Fourier series on the given basis, the equalities

$$\| x(t) - y(t) \| \leq \sum_{i=0}^{\infty} | x_i(t) - y_i(t) | = 0, \quad t \geq t_0,$$

also hold in all continuity points of the function $u(t)$. Thus, the vibrosolution definition given in

Section 2 implies that the functions $x(t) \in H$ and $y(t) \in H$ are indistinguishable as vibrosolutions to the equations (1) and (15). In other words, the equations (1) and (15) have the same unique vibrosolution. Theorem 3 is proved.

7. Conclusion

The vibrosolution definition assumes uniqueness of a vibrosolution to an equation (1). This enables us to apply the obtained sufficient existence conditions for a vibrosolution to filtering equations for an infinite-dimensional process over discontinuous observations, for example [13], in case of simultaneous impulses in all observation channels (a scalar function $u(t)$). However, the results of this paper also enable us to consider a case of non-simultaneous impulses in observation channels (a vector function $u(t)$).

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