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On Weighted Hardy and Poincaré-type Inequalities for Differences

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A criterion is obtained for the Hardy-type inequality

$$\left(\int_0^a |f(x)|^p v(x)dx\right)^{1/p} \le c_1 \left\{\left(v(a)\int_0^a |f(x)|^p dx\right)^{1/p} + \left(\int_0^a\int_0^a |f(x)-f(y)|^p w(|x-y|)dxdy\right)^{1/p}\right\}$$

to be valid for $0 < a \le A \le \infty$ and $0 < p < \infty$. This weakens a criterion previously found by the first two authors and comes close to being necessary as well as sufficient. When an inequality in the criterion is reversed, a Poincaré-type inequality is derived in some cases.

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1 INTRODUCTION

In [2] the problem of existence of a bounded linear extension of $W_p^{\lambda(\cdot)}(\Omega)$ into $W_p^{\nu(\cdot)}(R^n)$, for spaces with some "generalized" smoothness defined by functions λ and ν respectively, was investigated in the case of domains Ω admitting arbitrarily strong degeneration. A central role in the analysis was played by the following Hardy-type inequality: for all $f \in L_p(0, a)$

$$\left(\int_0^a |f(x)|^p v(x)dx\right)^{1/p} \le c_1 \left\{ \left(v(a)\int_0^a |f(x)|^p dx\right)^{1/p} \right.$$
$$\left. + \left(\int_0^a \int_0^a |f(x)-f(y)|^p w(|x-y|)dxdy\right)^{1/p} \right\}, \tag{1.1}$$

where $c_1 > 0$ is independent of f and a. It was assumed that $0 < a \le A \le \infty$, $0 < p < \infty$, w is a non-negative measurable function on $(0, A)$ which is such that for all $x \in (0, A)$

$$v(x) := B + \int_x^A w(t)dt \quad < \infty \tag{1.2}$$

for some $B \in [0, \infty)$, and there exists $\alpha \in (1, 2)$ such that

$$v(x) \le \alpha v(2x), \quad x \in (0, A/2). \tag{1.3}$$

When $a = A = \infty$, the inequality becomes

$$\left(\int_0^\infty |f(x)|^p v(x)dx\right)^{1/p}$$
$$\le c_1 \left(\int_0^\infty \int_0^\infty |f(x)-f(y)|^p w(|x-y|)dxdy\right)^{1/p} \tag{1.4}$$

for all $f \in L_p(0, \infty)$. Special cases of the inequality, and analogous ones, were previously studied by Yakovlev [8,9]; see also Grisvard [4], Kufner and Persson [5], Kufner and Triebel [6] and Triebel [7]; a discussion of these earlier works and further references may be found in [2]. Necessary conditions are also given in [2] for the validity of (1.1), and ample evidence is provided that the sufficiency condition (1.3) is close to being necessary. It is also worthy of note for subsequent reference that (1.3) is shown in [2, Remark 2.4] to imply that

$$\int_0^x v(\xi)d\xi \le c_2 x v(x), \quad x \in (0, A), \tag{1.5}$$

for some $c_2 > 0$, and this is clearly necessary for (1.1) as the choice $f = 1$ indicates.

In this paper we adapt the techniques in [2] to obtain (1.1) under a weaker condition than (1.3), thereby narrowing still further the gap between sufficiency and necessary conditions. Moreover, we prove that when a condition which, in some sense, is converse to that which replaces (1.3) is assumed, a Poincaré-type inequality for differences is obtained.

2 A HARDY-TYPE INEQUALITY

THEOREM 2.1 *Let $0 < p < \infty, 0 < A \leq \infty, 0 \leq B < \infty$, and let w be a non-negative measurable function on $(0, A)$ which is such that (1.2) and (1.5) are satisfied for all $x \in (0, A)$. Furthermore, suppose that there exist $\mu > 0$ and $0 < \gamma < 1$ such that*

$$v\left(\frac{x}{\mu+1}\right) - v\left(\frac{x}{\mu}\right) \leq \gamma v(x), \quad x \in (0, A_0), \tag{2.1}$$

where $A_0 = \min\{1, \mu\}A$. Then, for all $a \in (0, A]$ and all measurable f on (0,a), (1.1) is satisfied, where c_1 is independent of f and a.

In particular, (1.4) is satisfied for all $f \in L_p(0, \infty)$.

Proof Let $1 \leq p < \infty$ initially. As in [2] we start with the inequality

$$|f(x)| \leq |f(y)| + |f(x) - f(y)|.$$

For $\varepsilon \in (0, a/[\mu + 1])$, this gives

$$\begin{aligned}
&\left(\int_{\varepsilon}^{a/(\mu+1)} \int_{(\mu+1)x}^{a} |f(x)|^p w(y-x)dydx\right)^{1/p} \\
&\quad \leq \left(\int_{(\mu+1)\varepsilon}^{a} \int_{0}^{y/(\mu+1)} |f(x)|^p w(y-x)dxdy\right)^{1/p} \\
&\quad\quad + \left(\int_0^a \int_0^y |f(x) - f(y)|^p w(y-x)dxdy\right)^{1/p}
\end{aligned}$$

and so

$$\begin{aligned}
&\left(\int_{\varepsilon}^{a/(\mu+1)} |f(x)|^p \int_{\mu x}^{a-x} w(t)dtdx\right)^{1/p} \\
&\quad \leq \left(\int_{\varepsilon}^{a} |f(y)|^p \int_{\mu y/(\mu+1)}^{y} w(t)dtdy\right)^{1/p} + \Lambda
\end{aligned}$$

where

$$\Lambda^p = (1/2) \int_0^a \int_0^a |f(x) - f(y)|^p w(|x-y|)dxdy.$$

Hence we have

$$\left(\int_{\varepsilon}^{a/(\mu+1)} |f(x)|^p[v(\mu x) - v(a-x)]dx\right)^{1/p}$$
$$\leq \left(\int_{\varepsilon}^{a} |f(x)|^p[v(\frac{\mu x}{\mu+1}) - v(x)]dx\right)^{1/p} + \Lambda$$
$$\leq \left(\int_{\varepsilon}^{a/(\mu+1)} |f(x)|^p[v(\frac{\mu x}{\mu+1}) - v(x)]dx\right)^{1/p}$$
$$+ \left(v(\frac{\mu a}{(\mu+1)^2}) \int_{a/(\mu+1)}^{a} |f(x)|^p dx\right)^{1/p} + \Lambda.$$

From (2.1) it follows that

$$v\left(\frac{\mu x}{\mu+1}\right) - v(x) \leq \gamma v(\mu x), \quad 0 < x \leq a/(\mu+1)$$

and consequently

$$\left(\int_{\varepsilon}^{a/(\mu+1)} |f(x)|^p[v(\mu x) - v(a-x)]dx\right)^{1/p}$$
$$\leq \gamma \left(\int_{\varepsilon}^{a/(\mu+1)} |f(x)|^p v(\mu x)dx\right)^{1/p}$$
$$+ \left(v(\frac{\mu a}{(\mu+1)^2}) \int_{a/(\mu+1)}^{a} |f(x)|^p dx\right)^{1/p} + \Lambda.$$

Also $v(a-x) \leq v\left(\frac{\mu a}{\mu+1}\right)$ in $[0, a/(\mu+1)]$, and so, since $\alpha^{1/p} + \beta^{1/p} \leq 2^{1-1/p}(\alpha+\beta)^{1/p}$, we have

$$\left(\int_{\varepsilon}^{a/(\mu+1)} |f(x)|^p v(\mu x)dx\right)^{1/p}$$
$$\leq \left(\int_{\varepsilon}^{a/(\mu+1)} |f(x)|^p[v(\mu x) - v(a-x)]dx\right)^{1/p}$$
$$+ \left(\int_{\varepsilon}^{a/(\mu+1)} |f(x)|^p v(a-x)dx\right)^{1/p}$$

$$\leq \gamma^{1/p}\left(\int_{\varepsilon}^{a/(\mu+1)} |f(x)|^p v(\mu x)dx\right)^{1/p}$$
$$+\left(2^{p-1} v\left(\frac{\mu a}{(\mu+1)^2}\right)\int_0^a |f(x)|^p dx\right)^{1/p} + \Lambda. \tag{2.2}$$

On allowing $\varepsilon \to 0$ this gives

$$\left(\int_0^{a/(\mu+1)} |f(x)|^p v(\mu x)dx\right)^{1/p} \tag{2.3}$$
$$\leq 2^{1-1/p}(1-\gamma^{1/p})^{-1}\left\{v\left(\frac{\mu a}{(\mu+1)^2}\right)\left(\int_0^a |f(x)|^p dx\right)^{1/p} + \Lambda\right\}.$$

In [1] it is proved that (1.5) implies the existence of $\delta \in (0, 1)$ and $c_3 > 0$ such that

$$x^\delta v(x) \leq c_3 y^\delta v(y), \quad 0 < x < y. \tag{2.4}$$

Hence, if $\mu > 1$, we have

$$v(x) \leq c_3 x^{-\delta}(\mu x)^\delta v(\mu x) = c_3 \mu^\delta v(\mu x), \tag{2.5}$$

while $v(x) \leq v(\mu x)$ if $0 < \mu \leq 1$. Similarly

$$v\left(\frac{\mu a}{\mu+1}\right) \leq c_3(1+1/\mu)^\delta v(a). \tag{2.6}$$

Therefore, from (2.3), for some $c > 0$ independent of f and a,

$$\left(\int_0^{a/(\mu+1)} |f(x)|^p v(x)dx\right)^{1/p} \leq c\left\{\left(v(a)\int_0^a |f(x)|^p dx\right)^{1/p} + \Lambda\right\}.$$

Finally (1.1) follows from

$$\left(\int_0^a |f(x)|^p v(x)dx\right)^{1/p} \leq \left(\int_0^{a/(\mu+1)} |f(x)|^p v(x)dx\right)^{1/p}$$
$$+\left(v(a/[\mu+1])\int_{a/(\mu+1)}^a |f(x)|^p dx\right)^{1/p}$$

and (2.5).

The case $0 < p < 1$ can be treated similarly, the only difference being that we start with

$$|f(x)|^p \leq |f(y)|^p + |f(x) - f(y)|^p$$

and, after multiplying by $w(y-x)$, integrate this inequality in the same way as before instead of applying $\|\cdot\|_{L_p}$.

COROLLARY 2.2 *Suppose that for some positive* α_1, α_2 *satisfying*

$$\alpha_1 < 1 + 1/\alpha_2 \tag{2.7}$$

we have that

$$v(x) \leq \alpha_1 v([1 + 1/\mu]x),\ v(x) \leq \alpha_2 v(\mu x), \quad x \in (0, A_1), \tag{2.8}$$

where $A_1 = \min\{1/\mu, \mu/\mu + 1\}A$. *Then (1.5) and (2.1) are satisfied.*

Proof It follows from (2.8) that either $\alpha_1 < 1 + 1/\mu$ or $\alpha_2 < \mu$. The argument in [2, Remark 2.4] can be used to show that (1.5) is satisfied. For instance, suppose $\alpha_2 < \mu$ and $\mu > 1$, the other cases being similar. Then

$$\begin{aligned}\int_0^x v(\xi)d\xi &= \sum_{k=0}^{\infty} \int_{\mu^{-k-1}x}^{\mu^{-k}x} v(\xi)d\xi \leq \sum_{k=0}^{\infty} \alpha_2^k \int_{\mu^{-k-1}x}^{\mu^{-k}x} v(\mu^k\xi)d\xi \qquad \text{(by (2.8),)}\\ &= \sum_{k=0}^{\infty} (\alpha_2/\mu)^k \int_{x/\mu}^{x} v(t)dt \\ &\leq (1-\alpha_2/\mu)^{-1}(1-1/\mu)xv(x/\mu) \leq c_2' x v(x)\end{aligned}$$

which is (1.5). Moreover

$$v(x/(\mu+1)) - v(x/\mu) \leq (\alpha_1 - 1)v(x/\mu) \leq (\alpha_1 - 1)\alpha_2 v(x)$$

and (2.1) follows since $(\alpha_1 - 1)\alpha_2 < 1$ by (2.8).

Remark 2.3 Theorem 2.1 in [2] is the special case $\mu = 1, \alpha_2 = 1$ of Corollary 1. Note that (2.8) reduces to a single inequality in one other case, namely

$$v(x) \leq \alpha v\left(\frac{1+\sqrt{5}}{2}x\right), \quad x \in \left(0, \frac{2A}{1+\sqrt{5}}\right) \tag{2.9}$$

where $\alpha < \frac{1+\sqrt{5}}{2}$, the golden ratio.

Remark 2.4 Since

$$\int_0^a \int_0^a |f(x) - f(y)|^p w(|x-y|)dxdy = 2\int_0^a \|\Delta_h f\|^p_{L_p(0,a-h)} w(h)dh,$$

where $\Delta_h f(x) = f(x+h) - f(x)$, the inequality (1.1) may be written as

$$\left(\int_0^a |f(x)|^p v(x)dx\right)^{1/p} \le c_8 \left\{ \left(v(a)\int_0^a |f(x)|^p dx\right)^{1/p} + \left(\int_0^a \|\Delta_h f\|^p_{L_p(0,a-h)} w(h)dh\right)^{1/p}\right\}.$$

In [3] Burenkov and Goldman have proved that (1.5) is necessary and sufficient for the validity of a rougher inequality

$$\left(\int_0^a |f(x)|^p v(x)dx\right)^{1/p} \le c_9 \left\{ \left(v(a)\int_0^a |f(x)|^p dx\right)^{1/p} + \left(\int_0^a \omega_{h,p}(f)^p w(h)dh\right)^{1/p}\right\},$$

where

$$\omega_{h,p}(f) = \sup_{0\le t\le h} \|\Delta_t f\|_{L_p(0,a-t)},$$

the modulus of continuity of f.

3 A POINCARÉ-TYPE INEQUALITY

THEOREM 3.1 *Let* $0 < p < \infty, 0 < A \le \infty, 0 \le B < \infty$ *and let* w *be a non- negative measurable function on* $(0, A)$ *which satisfies (1.2) for all* $x \in (0, A)$. *Suppose there exist* $\mu > 0$ *and* $\gamma > 1$ *such that*

$$v(x/[\mu+1]) - v(x/\mu) \ge \gamma v(x), \quad x \in (0, A_0), \tag{3.1}$$

where $A_0 = \min\{1, \mu\}A$, *and that if* $\mu \ne 1$ *there exist* $c_4 \ge 1$ *and* $c_5 \in (0, 1)$ *such that*

$$v(\mu x) \le c_4 v(x), \quad x \in (0, A) \tag{3.2}$$

if $\mu < 1$ *and*

$$v(x) \le \begin{cases} c_4 v(\mu x), & x \in (0, A/\mu) \\ c_5 v(\mu x/[\mu+1]), & x \in [A/\mu, A) \end{cases} \tag{3.3}$$

if $\mu > 1$. *Then, for all* $a \in (0, A]$ *and* f *such that* $f - b \in L_p(0, a; v(x)dx)$ *for some* $b \in \mathbb{C}$

$$\left(\int_0^a |f(x) - b|^p v(x)dx\right)^{1/p} \le c_6 \left(\int_0^a \int_0^a |f(x) - f(y)|^p w(|x-y|)dxdy\right)^{1/p}, \tag{3.4}$$

where c_6 *is independent of* f, b *and* a.

Proof It is clearly enough to prove the theorem for $b = 0$. Let $1 \le p < \infty$: the modifications necessary for $0 < p < 1$ are as in the proof of Theorem 2.1. On starting with the inequality

$$|f(y)| \le |f(x)| + |f(x) - f(y)|$$

and following the initial steps of the proof of Theorem 1 with $\varepsilon = 0$, we obtain

$$\left(\int_0^a |f(x)|^p [v(\mu x/(\mu+1)) - v(x)]dx\right)^{1/p} \le \left(\int_0^{a/(\mu+1)} |f(x)|^p [v(\mu x) - v(a-x)]dx\right)^{1/p} + \Lambda. \tag{3.5}$$

If $0 < \mu \le 1$, then

$$\left(\int_0^a |f(x)|^p [v(\mu x/(\mu+1)) - v(x)]dx\right)^{1/p} \le \left(\int_0^a |f(x)|^p v(\mu x)dx\right)^{1/p} + \Lambda$$

and, by (3.1), this gives

$$\left(\int_0^a |f(x)|^p v(x)dx\right)^{1/p} \le \left(\int_0^a |f(x)|^p v(\mu x)dx\right)^{1/p} \le (\gamma^{1/p} - 1)^{-1}\Lambda$$

since, by (3.2), $f \in L_p(0, a; v(\mu x)dx)$. Thus (3.4) follows.

Let $\mu > 1$. If $a \leq A/\mu$, we obtain from (3.5) and (3.1) that

$$\left(\int_0^a |f(x)|^p v(\mu x)dx\right)^{1/p} \leq (\gamma^{1/p} - 1)^{-1}\Lambda,$$

the left-hand side being finite since $v(\mu x) \leq v(x)$. Thus (3.4) follows in this case by (3.3). If $a > A/\mu$, we have from (3.1), (3.3) and (3.5)

$$\left(\gamma \int_0^{A/\mu} |f(x)|^p v(\mu x)dx + \int_{A/\mu}^a |f(x)|^p [v\left(\frac{\mu x}{\mu+1}\right) - v(x)]dx\right)^{1/p}$$
$$\leq \left(\int_0^{A/\mu} |f(x)|^p v(\mu x)dx\right)^{1/p} + \Lambda.$$

The left-hand side is again finite and (3.4) follows from (3.3).

COROLLARY 3.2 *Suppose that for some positive α_1, α_2 satisfying*

$$\alpha_1 > 1 + 1/\alpha_2 \tag{3.6}$$

we have that

$$v(x) \geq \alpha_1 v([1 + 1/\mu]x), \quad v(x) \geq \alpha_2 v(\mu x), \quad x \in (0, A_1), \tag{3.7}$$

where $A_1 = \min\{1/\mu, \mu/\mu + 1\}A$. Then (3.1) is satisfied.

Proof From (3.7), for $x \in (0, A_0)$,

$$v(x/[\mu+1]) - v(x/\mu) \geq (\alpha_1 - 1)v(x/\mu) \geq (\alpha_1 - 1)\alpha_2 v(x)$$

and this yields (3.1) since $(\alpha_1 - 1)\alpha_2 > 1$ by (3.7).

Remark 3.3 On choosing $\mu = 1$ and $\mu = \frac{1+\sqrt{5}}{2}$ in (3.5) we obtain the following two sufficiency conditions for (3.1) to be valid:

$$v(x) \geq \alpha v(2x) \quad for \quad \alpha > 2, \quad x \in (0, A/2), \tag{3.8}$$

$$v(x) \geq \alpha v\left(\left[\frac{1+\sqrt{5}}{2}\right]x\right) \quad for \quad \alpha > \frac{1+\sqrt{5}}{2}, \quad x \in (0, 2A/[1+\sqrt{5}]). \tag{3.9}$$

Remark 3.4 The choice $f(x) = c \neq b$ in (3.4) yields a contradiction unless $c - b \notin L_p(0, a; v(x)dx)$. Hence it is necessary that

$$\int_0^a v(x)dx = \infty. \tag{3.10}$$

Remark 3.5 From (3.7) and (3.8) it follows that

$$\int_x^A v(\xi)d\xi \le c_7 x v(x), \quad x \in (0, A), \tag{3.11}$$

for some $c_7 > 0$. For instance suppose that $\alpha_2 > \mu$, $\mu > 1$ and $A = \infty$, the other cases being similar. Then

$$\begin{aligned}\int_x^\infty v(\xi)d\xi &= \sum_{k=0}^\infty \int_{\mu^k x}^{\mu^{k+1}x} v(\xi)d\xi \le \sum_{k=0}^\infty \alpha_2^{-k} \int_{\mu^k x}^{\mu^{k+1}x} v(\mu^{-k}\xi)d\xi \\ &= \sum_{k=0}^\infty (\mu/\alpha_2)^k \int_x^{\mu x} v(\xi)d\xi \le (1-\mu/\alpha_2)^{-1}(\mu-1)xv(x).\end{aligned}$$

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