COMMON FIXED POINT THEOREMS FOR LEFT REVERSIBLE AND NEAR-COMMUTATIVE SEMIGROUPS AND APPLICATIONS

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We prove some common fixed point theorems for left reversible and near-commutative semigroups in compact and complete metric spaces, respectively. As applications, we get the existence and uniqueness of solutions for a class of nonlinear Volterra integral equations.

1. Introduction

Recently, Y.-Y. Huang and C.-C. Hong [15, 16], T.-J. Huang and Y.-Y. Huang [14], and Y.-Y. Huang et al. [17] obtained a few fixed point theorems for left reversible and near-commutative semigroups of contractive self-mappings in compact and complete metric spaces, respectively. These results subsume some theorems in Boyd and Wong [1], Edelstein [3], and Liu [20].

In this paper, motivated by the results in [14, 15, 16, 17], we establish common fixed point theorems for certain left reversible and near-commutative semigroups of self-mappings in compact and complete metric spaces. As applications, we use our main results to show the existence and uniqueness of solutions of nonlinear Volterra integral equations. Our results generalize, improve, and unify the corresponding results of Fisher [4, 5, 6, 7, 8, 9, 10, 11, 12], Hegedus and Szilagyi [13], Y.-Y. Huang and C.-C. Hong [16], T.-J. Huang and Y.-Y. Huang [14], Y.-Y. Huang et al. [17], Liu [18, 19, 20], Ohta and Nikaido [21], Rosenholtz [22], Taskovic [23], and others.

Recall that a semigroup F is said to be *left reversible* if, for any $s,t \in F$, there exist $u,v \in F$ such that $su = tv$. It is easy to see that the notion of left reversibility is equivalent to the statement that any two right ideals of F have nonempty intersection. A semigroup F is called *near commutative* if, for any $s,t \in F$, there exists $u \in F$ such that $st = tu$. Clearly, every commutative semigroup is near commutative, and every near-commutative semigroup is left reversible, but the converses are not true.

Throughout this paper, (X,d) denotes a metric space, $\mathbb{N}$, $\mathbb{R}^+$, and $\mathbb{R}$ denote the sets of positive integers, nonnegative real numbers, and real numbers, respectively. Let F be a semigroup of self-mappings on X and let f be a self-mapping on X. For $A,B \subseteq X$, $x,y \in X$, define

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\begin{aligned}
\delta_d(A,B) &= \sup\{d(a,b) : a \in A,\ b \in B\},\\
\delta_d(A) &= \delta_d(A,A), \qquad \delta_d(x,A) = \delta_d(\{x\},A),\\
Fx &= \{x\} \cup \{gx : g \in F\},\\
O_f(x) &= \{f^n x : n \in \{0\} \cup \mathbb{N}\}, \qquad O_f(x,y) = O_f(x) \cup O_f(y),\\
C_f &= \{h : h : X \longrightarrow X,\ fh = hf\},\\
H_f &= \{h : h : X \longrightarrow X,\ h(\cap_{n\in\mathbb{N}} f^n x) \subseteq \cap_{n\in\mathbb{N}} f^n X\},\\
H_F &= \{h : h : X \longrightarrow X,\ h(\cap_{g\in F} gX) \subseteq \cap_{g\in F} gX\},\\
\Phi &= \{\phi : \phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \text{is upper semicontinuous from the right},\\
&\qquad \phi(0) = 0,\ \phi(t) < t \text{ for } t > 0\}.
\end{aligned}
\tag{1.1}
$$

$\overline{A}$ denotes the closure of A. Clearly, $H_f \supseteq C_f \supseteq \{f^n : n \in \mathbb{N}\} \cup \{i_X\}$, where i_X is the identity mapping on X. The mapping f is called a *closed mapping* if, $y = fx$ whenever $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} fx_n = y$ for some $x, y \in X$. It is simple to check that the composition of two closed self-mappings in compact metric spaces is closed. The mapping f is called a *local contraction* if, for each $x \in X$, there is an open set U containing x and a real number $m < 1$ such that $d(fz, fy) \le md(z, y)$ for all $z, y \in U$. The mapping f is said to have *diminishing orbital diameters* if, for every $x \in X$ with $\delta_d(O_f(x)) > 0$, there exists $n \in \mathbb{N}$ such that $\delta_d(O_f(f^n x)) < \delta_d(O_f(x))$. Clearly, f has diminishing orbital diameters if and only if $\lim_{n\to\infty} \delta_d(O_f(f^n x)) < \delta_d(O_f(x))$ for all $x \in X$ with $\delta_d(O_f(x)) > 0$. The semigroup F is said to have *diminishing orbital diameters* if, for any $x \in X$ with $\delta_d(Fx) > 0$, there exists $g \in F$ such that $\delta_d(Fgx) < \delta_d(Fx)$.

2. Common fixed points for left reversible semigroups in compact metric spaces

Let F be a left reversible semigroup. We define a relation $\ge$ on F by $a \ge b$ if and only if $a \in bF \cup \{b\}$.

It is easy to verify that $(F, \ge)$ is a directed set. We need the following lemma for our main theorems.

Lemma 2.1. *Let F be a left reversible semigroup of closed self-mappings in a compact metric space (X,d) and let $A = \cap_{f\in F} fX$. Then*
(i) $\lim_{f\in F} \delta_d(fX) = \delta_d(A)$*;*
(ii) *A is nonempty, compact and $fA = A$ for all $f \in F$.*

Proof. Note that $fX \subseteq gX$ for all $f, g \in F$ with $f \ge g$. Thus $\{\delta_d(fX)\}_{f\in F}$ is a bounded decreasing net in $\mathbb{R}$. Obviously, $\lim_{f\in F} \delta_d(fX)$ exists in $\mathbb{R}$ and

$$\delta_d(A) \le \lim_{f\in F} \delta_d(fX). \tag{2.1}$$

We now prove that fX is a compact subset of X for each $f \in F$. Let x be in X and $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ with $\lim_{n\to\infty} fx_n = x$. The compactness of X ensures that there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that it converges to some point $t \in X$. In view of closedness of f, we conclude immediately that $x = ft \in fX$. Therefore, fX is closed. That is, fX is compact. This means that A is compact.

We next prove that

$$\delta_d(A) \geq \lim_{f \in F} \delta_d(fX). \tag{2.2}$$

Given $f \in F$, there exist $x_f, y_f \in fX$ with $d(x_f, y_f) = \delta_d(fX)$. Since X is compact, we can choose two subnets $\{x_{f_k}\}$ and $\{y_{f_k}\}$ of $\{x_f\}$ and $\{y_f\}$, respectively, such that $x_{f_k} \to x$ and $y_{f_k} \to y$ for some $x, y \in X$. For every $g \in F$ and $f_k \geq g$, we get that $x_{f_k}, y_{f_k} \in gX$. By virtue of closedness of gX, we infer that $x, y \in gX$. This means that $x, y \in A$. Consequently,

$$\lim_{f \in F} \delta_d(fX) = \lim_{f \in F} d(x_f, y_f) = \lim_k d(x_{f_k}, y_{f_k}) = d(x, y) \leq \delta_d(A). \tag{2.3}$$

Thus (i) follows from (2.1) and (2.2).

Let $n \in \mathbb{N}$ and $f_1, f_2, \ldots, f_n \in F$. It follows from the left reversibility of F that there exist $g_1, g_2, \ldots, g_n \in F$ with $f_1 g_1 = f_2 g_2 = \cdots = f_n g_n = h$ for some $h \in F$. Hence, $\cap_{i=1}^n f_i X \supseteq hX \neq \emptyset$. The compactness of X implies that $A \neq \emptyset$.

We last prove that $fA = A$ for all $f \in F$. Let $f \in F$ and $x \in A$. For any $g \in F$, there exist $a, b \in F$ with $fa = gb$. Note that $x \in A \subseteq aX$. Thus there is $y \in X$ with $x = ay$. It follows that $fx = fay = gby \in gX$. This implies that $fA \subseteq \cap_{g \in F} gX = A$ for $f \in F$. For the reverse inclusion, let $f, g \in F$ and $y \in A$. It follows from $y \in fgX$ that there exists $x_g \in gX$ with $fx_g = y$. The compactness X ensures that there exists a convergent subnet $\{x_{g_k}\}$ of $\{x_g\}$ such that $x_{g_k} \to x$ for some $x \in X$. The closedness of f implies that $y = fx$. For any $h, g \in F$ with $g \geq h$, we obtain that hX is closed and that x_g belongs to hX. Thus the limit point x of $\{x_g\}$ lies in hX. That is, $x \in A$. Note that $y = fx \in fA$. Therefore, $A \subseteq fA$ for $f \in F$. This completes the proof. □

Now, we are ready to prove our main theorems.

Theorem 2.2. *Let F and G be left reversible semigroups of closed self-mappings in a compact metric space (X, d). Assume that there exist $f \in F$, $g \in G$ satisfying*

$$d(fx, gy) < \delta_d(\{su : u \in Fx,\ s \in H_F\}, \{tv : v \in Gy,\ t \in H_G\}) \tag{2.4}$$

for all $x, y \in X$ with $fx \neq gy$. Then F and G have a unique common fixed point $w \in X$ and the point w is also a unique fixed point of F and G, respectively. Moreover, if F (resp., G) is near commutative, then F (resp., G) has diminishing orbital diameters.

Proof. Let $A = \cap_{s \in F} sX$ and $B = \cap_{t \in G} tX$. If $\delta_d(A, B) > 0$, then by Lemma 2.1 there exist $a, x \in A$ and $b, y \in B$ with $\delta_d(A, B) = d(a, b)$, $a = fx$, and $b = gy$. It follows from (2.4) that

$$\begin{aligned} \delta_d(A, B) &= d(fx, gy) \\ &< \delta_d(\{su : u \in Fx,\ s \in H_F\}, \{tv : v \in Gy,\ t \in H_G\}) \\ &\leq \delta_d(A, B), \end{aligned} \tag{2.5}$$

which is a contradiction. Therefore, $\delta_d(A, B) = 0$. That is, $A = B = \{w\}$ for some $w \in X$. Note that $fA = gA = A$ for all $f \in F$ and $g \in G$. Thus F and G have a common fixed point w. If v is a fixed point of F or G, then $v \in \cap_{f\in F} fX = \{w\}$ or $v \in \cap_{g\in G} gX = \{w\}$. This means that $v = w$. Hence, F and G have a unique common fixed point $w \in X$ and the point w is also a unique fixed point of F and G, respectively.

Assume that one of F or G, say F, is near commutative. Let x be in X with $\delta_d(Fx) > 0$. So, for any $f, g \in F$, there exists $h \in F$ such that $gf = fh$. It follows that

$$\delta_d(Ffx) = \delta_d(\{fx\} \cup \{gfx : g \in F\}) \le \delta_d(fX). \tag{2.6}$$

It follows from (2.6) and Lemma 2.1 that

$$\lim_{f\in F} \delta_d(Ffx) = \lim_{f\in F} \delta_d(fX) = 0 < \delta_d(Fx), \tag{2.7}$$

which implies that F has diminishing orbital diameters. This completes the proof. □

Using the argument above, we can conclude the following two results.

Theorem 2.3. *Let F be a left reversible semigroup of closed self-mappings in a compact metric space (X,d). Assume that there exist $f, g \in F$ satisfying*

$$d(fx, gy) < \delta_d(\{su : u \in Fx \cup Fy,\ s \in H_F\}) \tag{2.8}$$

for all $x, y \in X$ with $fx \ne gy$. Then F has a unique fixed point $w \in X$. Moreover, if F is near commutative, then it has diminishing orbital diameters.

Theorem 2.4. *Let F be a left reversible semigroup of closed self-mappings in a compact metric space (X,d). Assume that there exists $f \in F$ satisfying*

$$d(fx, fy) < \delta_d(\{su : u \in Fx \cup Fy,\ s \in H_F\}) \tag{2.9}$$

for all $x, y \in X$ with $fx \ne fy$. Then F has a unique fixed point $w \in X$. Moreover, if F is near commutative, then it has diminishing orbital diameters.

Corollary 2.5. *Let f be a closed self-mapping of a compact metric space (X,d). Assume that there exist $p, q \in \mathbb{N}$ such that*

$$d(f^p x, f^q y) < \delta_d(\{su : u \in O_f(x,y),\ s \in H_f\}) \tag{2.10}$$

for all $x, y \in X$ with $f^p x \ne f^q y$. Then f has both a unique fixed point $w \in X$ and diminishing orbital diameters.

Proof. Take $F = \{f^n : n \in \mathbb{N}\}$. Then F is a commutative semigroup. Obviously, F has a unique fixed point $w \in X$ if and only if f has a unique fixed point $w \in X$. Note that $Fx = O_f(x)$ for all $x \in X$ and that $H_F = H_f$. Thus Corollary 2.5 follows immediately from Theorem 2.3. This completes the proof. □

Remark 2.6. Theorems 2.3 and 2.4 and Corollary 2.5 extend, improve, and unify [5, Theorem 4], [6, Theorem 2], [7, Theorem 9], [8, Theorem 4], [10, Theorem 5], [11, Theorem 5], [12, Theorem 3], [9, Theorem 5], [14, Theorem 1.1], [17, Theorem 2.2], [18, Theorem 3], [20, Theorem 2], [21, Theorem 4], and so forth.

Theorem 2.7. *Let (X,d) be a compact and connected metric space and let F be a left reversible semigroup of self-mappings in X such that each f in F is a local contraction. Then F has both a unique fixed point $w \in X$ and diminishing orbital diameters. Moreover, for any $x \in X$ and $f \in F$, the sequence of iterates $\{f^n x\}_{n\in\mathbb{N}}$ converges to w.*

Proof. Let f be in F. We now show that f has a unique fixed point in X. The compactness of X ensures that there exist $r_f > 0$ and $m_f < 1$ such that

$$d(fx, fy) \le m_f d(x,y) \tag{2.11}$$

for all $x,y \in X$ with $d(x,y) < r_f$. Condition (2.11) ensures that f is continuous. Assume that $V_1, V_2, \dots, V_{n_f}$ are a fixed finite open cover of X with sets of diameters less than r_f. For any $x,y \in X$, the connectedness of X implies that there is a chain of open sets from x to y, chosen from the sets $V_1, V_2, \dots, V_{n_f}$. This means that $d(x,y) \le n_f r_f$. It follows from (2.11) that $d(fx, fy) \le m_f n_f r_f$. It is easy to check that

$$d(f^k x, f^k y) \le (m_f)^k n_f r_f \tag{2.12}$$

for all $k \in \mathbb{N}$. By choosing k so large that $(m_f)^k n_f < 1$, we infer that $\delta_d(f^k X) < r_f$. So the mapping f restricted to the set $f^k X$, which maps $f^k X$ to itself, is a contraction. Note that $f^k X$ is closed. By the Banach contraction theorem, the restricted mapping has a unique fixed point $w_f \in f^k X$. Obviously, w_f is a unique fixed point of f in X. In view of (2.12), we have

$$\delta_d(f^j X, w_f) \le \delta_d(f^j X) \le (m_f)^j n_f r_f \tag{2.13}$$

for all $j \in \mathbb{N}$. Therefore,

$$\lim_{j\to\infty} \delta_d(f^j X, w_f) = 0, \tag{2.14}$$

which implies that both $\lim_{j\to\infty} f^j x = w_f$ and f has diminishing orbital diameters.

Let f and g be in F. Then there exist $w_f, w_g \in X$ such that $w_f = f w_f$, $w_g = g w_g$, and (2.14) and the following equation hold:

$$\lim_{j\to\infty} \delta_d(g^j X, w_g) = 0. \tag{2.15}$$

Given $j \in \mathbb{N}$, it follows from the left reversibility of F that there are $a_j, b_j \in F$ with $f^j a_j = g^j b_j$. From (2.14) and (2.15), we infer that

$$\begin{aligned} d(w_f, w_g) &\le d(w_f, f^j a_j x) + d(g^j b_j x, w_g) \\ &\le \delta_d(w_f, f^j X) + \delta_d(g^j X, w_g) \\ &\longrightarrow 0 \quad \text{as } j \longrightarrow \infty. \end{aligned} \tag{2.16}$$

This means that $w_f = w_g$. That is, F has a unique fixed point in X. This completes the proof. □

Remark 2.8. Theorem 2.7 is a generalization of [22, Theorem 1].

Now we like to give two concrete examples for Theorems 2.4 and 2.7.

Example 2.9. Let $X = \{0, 2/3\} \cup \{1/n : n \in \mathbb{N}\}$ with the usual metric d. Define $f, g : X \to X$ by

$$f0 = f\frac{2}{3} = g0 = g\frac{2}{3} = g1 = 0, \qquad f\frac{1}{n} = \frac{1}{n+1}, \qquad g\frac{1}{n+1} = \frac{1}{n+2} \tag{2.17}$$

for all $n \in \mathbb{N}$. Obviously, $gf = f^2$, $fg = g^2$, $gf1 = 1/3 \neq 0 = fg1$, (X,d) is a compact metric space, and f and g are closed. Let F be the semigroup generated by f and g. Now for any $a, b \in F$, there exist $a_1, \dots, a_k$, $b_1, \dots, b_n \in \{f, g\}$ with $a = a_1 \cdots a_k$ and $b = b_1 \cdots b_n$. Hence, $ab = (b_n)^{n+k} = b(b_n)^k$. Thus F is near commutative, and therefore it is left reversible also. But it is not commutative. Note that $\cap_{s \in F} sX = \{0\}$. It is easy to verify that

$$d(fx, fy) \leq \frac{1}{2} < 1 = \delta_d(\{su : u \in Fx \cup Fy,\ s \in H_F\}) \tag{2.18}$$

for all $x, y \in X$ with $fx \neq fy$. Hence, F satisfies the conditions of Theorem 2.4 and clearly 0 is the unique fixed point of F.

Example 2.10. Let $X = [0,1]$ with the usual metric d. Define $f, g : X \to X$ by

$$fx = \frac{1}{2}x, \qquad gx = \frac{2}{3}x \tag{2.19}$$

for all $x \in X$. Then (X,d) is a compact and connected metric space, $fg = gf$, and any one of f and g is a local contraction. Let F be the semigroup generated by f and g. It is easy to see that every element in F is a local contraction and that F is commutative. It follows from Theorem 2.7 that F has a unique fixed point.

3. Common fixed points for near-commutative semigroups in complete metric spaces

Lemma 3.1 (see [15]). *If ϕ is in Φ, then* $\lim_{n\to\infty} \phi^n(t) = 0$.

Lemma 3.2 (see [2]). *If $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is an upper semicontinuous function with $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$, then there exists a strictly increasing continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\psi(0) = 0$ and $\phi(t) \leq \psi(t) < t$ for $t > 0$.*

Theorem 3.3. *Let F and G be near-commutative semigroups of closed self-mappings in a complete metric space (X,d). Assume that the following conditions are satisfied:*

(i) *for any $x \in X$, Fx and Gx are bounded;*

(ii) *there exists $\phi \in \Phi$ such that, for any $f \in F$, $g \in G$, there are $n_f, m_g \in \mathbb{N}$ satisfying*

$$d(f^p x, g^q y) \leq \phi(\delta_d(Fx, Gy)) \tag{3.1}$$

for all $x, y \in X$ and $p \geq n_f$, $q \geq m_g$.

Then F and G have both a unique common fixed point $w \in X$ and diminishing orbital diameters. Moreover, for each $f \in F \cup G$ and $x \in X$, the sequence of iterates $\{f^n x\}_{n \in \mathbb{N}}$ converges to w.

Proof. We assert that, for any $f \in F, g \in G, x, y \in X$, and $k \geq \max\{n_f, m_g\}$,

$$\delta_d(Ff^k x, Gg^k y) \leq \phi(\delta_d(Fx, Gy)). \tag{3.2}$$

Take $u \in Ff^k x$ and $v \in Gg^k y$. Then there are $s \in F$ and $t \in G$ with $u = sf^k x$ and $v = tg^k y$. The near commutativity of F and G ensures that there exist $a \in F$ and $b \in G$ such that $sf^k = f^k a$ and $tg^k = g^k b$. It follows from (3.1) that

$$\begin{aligned} d(u,v) &= d(sf^k x, tg^k y) = d(f^k ax, g^k by) \\ &\leq \phi(\delta_d(Fax, Gby)) \leq \phi(\delta_d(Fx, Gy)), \end{aligned} \tag{3.3}$$

which implies that

$$\begin{aligned} \delta_d(Ff^k x, Gg^k y) &= \sup\{d(u,v) : u \in Ff^k x,\ v \in Gg^k y\} \\ &\leq \phi(\delta_d(Fx, Gy)) \end{aligned} \tag{3.4}$$

for any $x, y \in X$ and $k \geq \max\{n_f, m_g\}$. That is, (3.2) holds. For any $x, y \in X$ and $k \in \{0\} \cup \mathbb{N}$, put $a_k = \delta_d(Ff^{k\max\{n_f,m_g\}} x, Gg^{k\max\{n_f,m_g\}} y)$. It follows from (3.2), (i), and Lemma 3.1 that

$$\begin{aligned} a_k &\leq \phi(\delta_d(Ff^{(k-1)\max\{n_f,m_g\}} x, Gg^{(k-1)\max\{n_f,m_g\}} y)) \\ &= \phi(a_{k-1}) \leq \cdots \leq \phi^k(a_0) = \phi^k(\delta_d(Fx, Gy)) \\ &\longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \end{aligned} \tag{3.5}$$

Let n be in $\mathbb{N}$. Then there exist $k, r \in \{0\} \cup \mathbb{N}$ and $r < \max\{n_f, m_g\}$ such that $n = k\max\{n_f, m_g\} + r$. In view of (3.5), we have

$$\begin{aligned} \delta_d(Ff^n x, Gg^n y) &\leq \delta_d(Ff^{k\max\{n_f,m_g\}} x, Gg^{k\max\{n_f,m_g\}} y) \\ &\leq \phi(a_{k-1}) \leq a_{k-1} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.6}$$

This implies that

$$\begin{aligned} &\max\{\delta_d(Ff^n x), \delta_d(Gg^n y)\} \\ &\quad \leq \max\{\delta_d(Ff^n x, g^{n+1} y) + \delta_d(g^{n+1} y, Ff^n x), \delta_d(Gg^n y, f^{n+1} x) + \delta_d(f^{n+1} x, Gg^n y)\} \\ &\quad \leq 2\delta_d(Ff^n x, Gg^n y) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.7}$$

Therefore, F and G have diminishing orbital diameters. Note that $\{f^{n+i} x\}_{i \in \mathbb{N}} \subseteq Ff^n x$ and $\{g^{n+i} y\}_{i \in \mathbb{N}} \subseteq Gg^n y$. So $\{f^n x\}_{n \in \mathbb{N}}$ and $\{g^n y\}_{n \in \mathbb{N}}$ are Cauchy sequences. Since (X, d) is complete, $\{f^n x\}_{n \in \mathbb{N}}$ and $\{g^n y\}_{n \in \mathbb{N}}$ converge to some points $w, b \in X$, respectively. Consequently, $w \in \cap_{n \in \mathbb{N}} \overline{Ff^n x}$ and $b \in \cap_{n \in \mathbb{N}} \overline{Gg^n y}$. It follows from (3.6) that

$$d(w, b) \leq \delta_d(Ff^n x, Gg^n y) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.8}$$

That is, $w = b$. The closedness of f and g implies that $w = fw$ and $b = gb$. Hence, f and g have a common fixed point w. By arbitrariness of f and g, we conclude that F and G have a common fixed point w.

Suppose that F and G have also a common fixed point $v \in X$. By virtue of (3.1), for any $f \in F$ and $g \in G$, we have

$$d(w,v) = d(f^{n_f}w, g^{m_g}v) \leq \phi(\delta_d(Fw,Gv)) = \phi(d(w,v)), \tag{3.9}$$

which implies that $w = v$. This completes the proof. □

Theorem 3.4. *Let F be near-commutative semigroup of closed self-mappings in a complete metric space (X,d). Assume that the following conditions are satisfied:*

(iii) *for any $x \in X$, Fx is bounded;*

(iv) *there exists $\phi \in \Phi$ such that, for any $f \in F$, there is $n_f \in \mathbb{N}$ satisfying*

$$d(f^p x, f^q x) \leq \phi(\delta_d(Fx)) \tag{3.10}$$

for all $x \in X$ and $p, q \geq n_f$.

Then F has both a fixed point in X and diminishing orbital diameters. Moreover, for each $f \in F$ and $x \in X$, the sequence of iterates $\{f^n x\}_{n\in\mathbb{N}}$ converges to some fixed point of F.

Proof. It follows from (3.10) that (3.1) is satisfied for $F = G$, $x = y$, $f = g$, and $n_f = m_g$. Thus Theorem 3.4 follows from Theorem 3.3. This completes the proof. □

Theorem 3.5. *Let F be near-commutative semigroup of closed self-mappings in a complete metric space (X,d). Assume that condition (iii) and the following condition (v) hold:*

(v) *there exists $\phi \in \Phi$ such that, for any $f \in F$, there is $n_f \in \mathbb{N}$ satisfying*

$$d(f^p x, f^q y) \leq \phi(\delta_d(Fx \cup Fy)) \tag{3.11}$$

for all $x, y \in X$ and $p, q \geq n_f$.

Then F has both a unique common fixed point $w \in X$ and diminishing orbital diameters. Moreover, for each $f \in F$ and $x \in X$, the sequence of iterates $\{f^n x\}_{n\in\mathbb{N}}$ converges to w.

Proof. Note that (3.11) implies that both (3.10) is satisfied and F has at most one fixed point in X. Thus Theorem 3.5 follows from Theorem 3.4. This completes the proof. □

Remark 3.6. It follows from Lemma 3.2 that Theorem 3.5 extends [16, Theorem 2.1].

Theorem 3.7. *Let F and G be near-commutative semigroups of self-mappings in a complete metric space (X,d). Assume that condition (i) and the following condition hold:*

(vi) *there exists $\phi \in \Phi$ such that, for any $f \in F$, $g \in G$ and $x, y \in X$,*

$$d(fx, gy) \leq \phi(\delta_d(Fx, Gy)). \tag{3.12}$$

Then F and G have both a unique common fixed point $w \in X$ and diminishing orbital diameters. Moreover, for each $f \in F \cup G$ and $x \in X$, the sequence of iterates $\{f^n x\}_{n\in\mathbb{N}}$ converges to w.

Proof. Let f and g be in F and G, respectively. As in the proof of Theorem 3.3, we infer easily that there exists a unique point $w \in X$ such that

$$\lim_{n\to\infty} f^n x = \lim_{n\to\infty} g^n y = w, \qquad \lim_{n\to\infty} \delta_d(Ff^n x) = \lim_{n\to\infty} \delta_d(Gg^n y) = 0 \tag{3.13}$$

for all $x, y \in X$. Thus, F and G have diminishing orbital diameters and

$$\lim_{n\to\infty} f^n x = \lim_{n\to\infty} g^n w = w, \qquad \lim_{n\to\infty} \delta_d(Ff^n w) = \lim_{n\to\infty} \delta_d(Gg^n w) = 0. \tag{3.14}$$

Since $w \in (\cap_{n\in\mathbb{N}}\overline{Ff^n w}) \cap (\cap_{n\in\mathbb{N}}\overline{Gg^n w})$, it follows that

$$\max\{\delta_d(Ff^n w, w), \delta_d(Gg^n w, w)\} \le \max\{\delta_d(Ff^n w), \delta_d(Gg^n w)\} \tag{3.15}$$

for all $n \in \mathbb{N}$. Using (3.14) and (3.15), we have

$$\lim_{n\to\infty} \delta_d(Ff^n w, w) = \lim_{n\to\infty} \delta_d(Gg^n w, w) = 0. \tag{3.16}$$

Let $\epsilon > 0$ be arbitrary. By virtue of (3.14) and (3.16), there exists $k \in \mathbb{N}$ such that, for all $n \ge k$,

$$\max\{d(w, f^{n+1}w), d(w, g^{n+1}w), \delta_d(Ff^n w, w), \delta_d(Gg^n w, w)\} < \epsilon. \tag{3.17}$$

For any $h \in G$, from (3.12) and (3.17), we immediately conclude that

$$\begin{aligned} d(w, hw) &\le d(w, f^{n+1}w) + d(f^{n+1}w, hw) \le \epsilon + \phi(\delta_d(Ff^n w, Gw)) \\ &\le \epsilon + \phi(\delta_d(Ff^n w, w) + \delta_d(w, Gw)) \le \epsilon + \phi(\epsilon + \delta_d(w, Gw)), \end{aligned} \tag{3.18}$$

which implies that

$$\delta_d(w, Gw) \le \epsilon + \phi(\epsilon + \delta_d(w, Gw)). \tag{3.19}$$

Letting $\epsilon \to 0$ in the above inequality, we obtain that $\delta_d(w, Gw) \le \phi(\delta_d(w, Gw))$. This means that $\delta_d(w, Gw) = 0$. That is, $Gw = \{w\}$. Similarly, $Fw = \{w\}$. Therefore, F and G have a common fixed point w. The uniqueness of common fixed point of F and G follows immediately from (3.12). This completes the proof. □

From Theorems 3.5 and 3.7, we have the following.

Theorem 3.8. *Let F be a near-commutative semigroup of self-mappings in a complete metric space (X, d). Assume that condition (iii) and the following condition (vii) hold:*

(vii) *there exists $\phi \in \Phi$ such that, for any $f \in F$ and $x, y \in X$,*

$$d(fx, fy) \le \phi(\delta_d(Fx \cup Fy)). \tag{3.20}$$

Then F has both a unique common fixed point $w \in X$ and diminishing orbital diameters. Moreover, for each $f \in F$ and $x \in X$, the sequence of iterates $\{f^n x\}_{n\in\mathbb{N}}$ converges to w.

Corollary 3.9. *Let f be a self-mapping of a complete metric space (X,d) and satisfy the following:*

(viii) *for each $x \in X$, $O_f(x)$ is bounded;*

(ix) *there exists $\phi \in \Phi$ such that, for any $x,y \in X$,*

$$d(fx,fy) \le \phi(\delta_d(O_f(x,y))). \tag{3.21}$$

Then f has both a unique fixed point $w \in X$ and diminishing orbital diameters. Moreover, the sequence of iterates $\{f^n x\}_{n\in\mathbb{N}}$ converges to w for each $x \in X$.

Proof. Put $F = \{f^n : n \in \mathbb{N}\}$. Condition (3.21) ensures that

$$\begin{aligned} d(f^n x, f^n y) &\le \phi(\delta_d(O_f(f^{n-1}x, f^{n-1}y))) \\ &\le \phi(\delta_d(O_f(x,y))) = \phi(\delta_d(Fx \cup Fy)) \end{aligned} \tag{3.22}$$

for all $n \in \mathbb{N}$ and $x,y \in X$. So Corollary 3.9 follows from Theorem 3.8. This completes the proof. □

Remark 3.10. Theorem 3.8 extends, improves, and unifies [4, Theorem 2], [13, Theorem 5], and [19, Theorem 1].

4. Applications

Throughout this section, let $(X,\|\cdot\|_X)$ be a real Banach space and $I = [a,b] \subseteq \mathbb{R}$. Define

$$\begin{aligned} C(I,X) &= \{f : f : I \longrightarrow X \text{ is continuous}\}, \\ C(I\times I\times X,X) &= \{f : f : I\times I\times X \longrightarrow X \text{ is continuous}\}, \\ \|f\|_C &= \sup_{t\in I} \|f(t)\|_X \end{aligned} \tag{4.1}$$

for all $f \in C(I,X)$. It is easy to verify that $(C(I,X),\|\cdot\|_C)$ is a real Banach space also.

Now we investigate the existence problem of common solutions for nonlinear Volterra integral equations of the from

$$\begin{aligned} x_\alpha(t) &= v(t) + \lambda\int_a^t K_\alpha(t,s,x_\alpha(s))\,ds, \quad \alpha \in A,\ t \in I, \\ y_\beta(t) &= v(t) + \lambda\int_a^t M_\beta(t,s,y_\beta(s))\,ds, \quad \beta \in B,\ t \in I, \end{aligned} \tag{4.2}$$

where $v(t) \in C(I,X)$ is a given function, $\lambda \in \mathbb{R}$ is an arbitrary parameter, K_α and M_β are in $C(I\times I\times X,X)$, A and B are index sets.

Theorem 4.1. *Let $F = \{f_\alpha : \alpha \in A\}$ and $G = \{g_\beta : \beta \in B\}$ be near-commutative semigroups and satisfy the following:*

(i) *for any $x \in C(I,X)$, $\max\{\delta_{\|\cdot\|_X}(Fx), \delta_{\|\cdot\|_X}(Gx)\} < \infty$;*

(ii) *there exists $L > 0$ such that, for any $\alpha \in A$, $\beta \in B$, $x,y \in C(I,X)$, and $t,s \in I$,*

$$\|K_\alpha(t,s,x(s)) - M_\beta(t,s,y(s))\|_X \le L\delta_{\|\cdot\|_X}(Fx,Gy), \tag{4.3}$$

where

$$f_\alpha x(t) = v(t) + \lambda \int_a^t K_\alpha(t,s,x(s))\,ds, \quad \alpha \in A,\ x \in C(I,X),\ t \in I,$$
$$g_\beta y(t) = v(t) + \lambda \int_a^t M_\beta(t,s,y(s))\,ds, \quad \beta \in B,\ y \in C(I,X),\ t \in I. \tag{4.4}$$

Then (4.2) have a unique common solution $w \in C(I,X)$. Moreover, for each $\alpha \in A$, $\beta \in B$, and $x \in C(I,X)$, the sequences defined by

$$(f_\alpha)^n x(t) = v(t) + \lambda \int_a^t K_\alpha(t,s,(f_\alpha)^{n-1} x(s))\,ds, \quad t \in I,\ n \in \mathbb{N},$$
$$(g_\beta)^n x(t) = v(t) + \lambda \int_a^t M_\beta(t,s,(g_\beta)^{n-1} x(s))\,ds, \quad t \in I,\ n \in \mathbb{N}, \tag{4.5}$$

converge to the unique solution w in the norm $\|\cdot\|_C$.

Proof. For any $x \in C(I,X)$, define $\|x\|_* = \sup_{t\in I} e^{-(1+|\lambda|)Lt}\|x(t)\|_X$. It is easy to show that

$$e^{-(1+|\lambda|)Lb}\|x\|_C \le \|x\|_* \le e^{-(1+|\lambda|)La}\|x\|_C. \tag{4.6}$$

Therefore, the norm $\|\cdot\|_*$ and the sup-norm $\|\cdot\|_C$ are equivalent to each other. Obviously, $(C(I,X),\|\cdot\|_*)$ is also a real Banach space. Note that all f_α and g_β are self-mappings of $(C(I,X),\|\cdot\|_*)$. In view of (4.3) and (4.5), we infer that, for all $\alpha \in A$, $\beta \in B$, and $x, y \in C(I,X)$,

$$\begin{aligned}
&||f_\alpha x(t) - g_\beta y(t)||_* \\
&\quad = \sup_{t\in I}\left\{ e^{-(1+|\lambda|)Lt}|\lambda| \left|\left| \int_a^t \big(K_\alpha(t,s,x(s)) - M_\beta(t,s,y(s))\big)\,ds \right|\right|_X \right\} \\
&\quad \le |\lambda| \sup_{t\in I} e^{-(1+|\lambda|)Lt} \int_a^t ||K_\alpha(t,s,x(s)) - M_\beta(t,s,y(s))||_X\,ds \\
&\quad \le |\lambda| \sup_{t\in I} \int_a^t e^{(1+|\lambda|)L(s-t)} e^{-(1+|\lambda|)Ls} ||K_\alpha(t,s,x(s)) - M_\beta(t,s,y(s))||_X\,ds \\
&\quad \le |\lambda| \sup_{t\in I} \int_a^t e^{(1+|\lambda|)L(s-t)} \sup_{s\in I} e^{-(1+|\lambda|)Ls} L\delta_{\|\cdot\|_X}(Fx,Gy)\,ds \\
&\quad \le |\lambda| L \sup_{t\in I} \int_a^t e^{(1+|\lambda|)L(s-t)} \delta_{\|\cdot\|_*}(Fx,Gy)\,ds \\
&\quad \le \frac{|\lambda|}{1+|\lambda|} \delta_{\|\cdot\|_X}(Fx,Gy) \sup_{t\in I}\big[1 - e^{(1+|\lambda|)L(a-t)}\big] \\
&\quad = \frac{|\lambda|}{1+|\lambda|}\big[1 - e^{(1+|\lambda|)L(a-b)}\big]\delta_{\|\cdot\|_X}(Fx,Gy) \\
&\quad = \phi(\delta_{\|\cdot\|_*}(Fx,Gy)),
\end{aligned} \tag{4.7}$$

where $\phi(t) = (|\lambda|/(1+|\lambda|))(1-e^{(1+|\lambda|)L(a-b)})t$ for $t \in \mathbb{R}^+$. It follows from Theorem 3.7 that F and G have a unique common fixed point $w \in C(I,X)$ and the sequences $\{(f_\alpha)^n x\}_{n\in\mathbb{N}}$ and $\{(g_\beta)^n x\}_{n\in\mathbb{N}}$ converge to w in the norm $\|\cdot\|_*$. Therefore, (4.2) have a unique common solution $w \in C(I,X)$ and by (4.6) the sequences $\{(f_\alpha)^n x\}_{n\in\mathbb{N}}$ and $\{(g_\beta)^n x\}_{n\in\mathbb{N}}$ converge to w in the norm $\|\cdot\|_C$. This completes the proof. □

From Theorems 3.8 and 4.1, we have the following.

Theorem 4.2. *Let $F = \{f_\alpha : \alpha \in A\}$ be a near-commutative semigroup and satisfy the following:*

(iii) *for any $x \in C(I,X)$, $\delta_{\|\cdot\|_X}(Fx) < \infty$;*

(iv) *there exists $L > 0$ such that, for any $\alpha \in A$, $x, y \in C(I,X)$ and $t, s \in I$,*

$$||K_\alpha(t,s,x(s)) - K_\alpha(t,s,y(s))||_X \le L\delta_{\|\cdot\|_X}(Fx \cup Fy), \tag{4.8}$$

where

$$f_\alpha x(t) = v(t) + \lambda \int_a^t K_\alpha(t,s,x(s))\,ds, \quad \alpha \in A,\ x \in C(I,X),\ t \in I. \tag{4.9}$$

Then the following equations

$$x_\alpha(t) = v(t) + \lambda \int_a^t K_\alpha(t,s,x(s))\,ds, \quad \alpha \in A,\ t \in I, \tag{4.10}$$

have a unique common solution $w \in C(I,X)$. Moreover, for each $\alpha \in A$ and $x \in C(I,X)$, the sequence defined by

$$(f_\alpha)^n x(t) = v(t) + \lambda \int_a^t K_\alpha(t,s,(f_\alpha)^{n-1}x(s))\,ds, \quad t \in I,\ n \in \mathbb{N}, \tag{4.11}$$

converges to the unique solution w in the norm $\|\cdot\|_C$.

By Theorem 4.2 and Corollary 3.9, we have the following.

Corollary 4.3. *Let K be in $C(I \times I \times X, X)$ and satisfy the following:*

(v) *there exists $L > 0$ such that, for any $x, y \in C(I,X)$ and $t, s \in I$,*

$$||K(t,s,x(s)) - K(t,s,y(s))||_X \le L\delta_{\|\cdot\|_X}(O_f(x,y)), \tag{4.12}$$

where

$$fx(t) = v(t) + \lambda \int_a^t K(t,s,x(s))\,ds, \quad x \in C(I,X),\ t \in I; \tag{4.13}$$

(vi) *for every $x \in C(I,X)$, $\delta_{\|\cdot\|_X}(O_f(x)) < \infty$.*

Then the following equation

$$x(t) = v(t) + \lambda \int_a^t K(t,s,x(t))\,ds, \quad t \in I, \tag{4.14}$$

has a unique solution $w \in C(I,X)$. Moreover, for each $x \in C(I,X)$, the sequence defined by

$$f^n x(t) = v(t) + \lambda \int_a^t K(t,s,f^{n-1}x(s))\,ds, \quad t \in I,\ n \in \mathbb{N}, \tag{4.15}$$

converges to the unique solution w in the norm $\|\cdot\|_C$.

Remark 4.4. [23, Theorem 6] is a special case of Corollary 4.3.

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References

[1] D. W. Boyd and J. S. W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc. **20** (1969), 458–464.
[2] T. H. Chang, *Fixed point theorems for contractive type set-valued mappings*, Math. Japon. **38** (1993), no. 4, 675–690.
[3] M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc. **37** (1962), 74–79.
[4] B. Fisher, *A fixed point theorem*, Math. Mag. **48** (1975), no. 4, 223–225.
[5] ______, *On three fixed point mappings for compact metric spaces*, Indian J. Pure Appl. Math. **8** (1977), no. 4, 479–481.
[6] ______, *A fixed point theorem for compact metric spaces*, Publ. Math. Debrecen **25** (1978), no. 3-4, 193–194.
[7] ______, *Theorems on fixed points*, Riv. Mat. Univ. Parma **4** (1978), 109–114.
[8] ______, *Quasi-contractions on metric spaces*, Proc. Amer. Math. Soc. **75** (1979), no. 2, 321–325.
[9] ______, *Results on common fixed points on bounded metric spaces*, Math. Sem. Notes Kobe Univ. **7** (1979), no. 1, 73–80.
[10] ______, *Common fixed points of commuting mappings*, Bull. Inst. Math. Acad. Sinica **9** (1981), no. 3, 399–406.
[11] ______, *Four mappings with a common fixed point*, J. Univ. Kuwait Sci. **8** (1981), 131–139.
[12] ______, *Common fixed points of four mappings*, Bull. Inst. Math. Acad. Sinica **11** (1983), no. 1, 103–113.
[13] M. Hegedus and T. Szilagyi, *Equivalent conditions and a new fixed point theorem in the theory of contractive type mappings*, Math. Japon. **25** (1980), no. 1, 147–157.
[14] T.-J. Huang and Y.-Y. Huang, *Fixed point theorems for left reversible semigroups in compact metric spaces*, Indian J. Math. **37** (1995), no. 2, 103–105.
[15] Y.-Y. Huang and C.-C. Hong, *A note on left reversible semigroups of contractions*, Far East J. Math. Sci. (FJMS) **4** (1996), no. 1, 81–87.
[16] ______, *Common fixed point theorems for semigroups on metric spaces*, Int. J. Math. Math. Sci. **22** (1999), no. 2, 377–386.
[17] Y.-Y. Huang, T.-J. Huang, and J.-C. Jeng, *On common fixed points of semigroups in compact metric spaces*, Indian J. Pure Appl. Math. **27** (1996), no. 11, 1073–1076.
[18] Z. Liu, *Common fixed point theorems in compact metric spaces*, Pure Appl. Math. Sci. **37** (1993), no. 1-2, 83–87.

[19] ______, *Fixed points in bounded complete metric spaces*, Bull. Malaysian Math. Soc. **18** (1995), 9–14.
[20] ______, *On an open question concerning fixed points*, Bull. Calcutta Math. Soc. **87** (1995), no. 2, 191–194.
[21] M. Ohta and G. Nikaido, *Remarks on fixed point theorems in complete metric spaces*, Math. Japon. **39** (1994), no. 2, 287–290.
[22] I. Rosenholtz, *Evidence of a conspiracy among fixed point theorems*, Proc. Amer. Math. Soc. **53** (1975), no. 1, 213–218.
[23] M. R. Taskovic, *A characterization of the class of contraction type mappings*, Kobe J. Math. **2** (1985), no. 1, 45–55.

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