APPLICATIONS OF W. A. KIRK'S FIXED-POINT THEOREM TO GENERALIZED NONLINEAR VARIATIONAL-LIKE INEQUALITIES IN REFLEXIVE BANACH SPACES

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We introduce and study a new class of generalized nonlinear variational-like inequalities, which includes these variational inequalities and variational-like inequalities due to Bose, Cubiotti, Dien, Ding, Ding and Tarafdar, Noor, Parida, Sahoo, and Kumar, and Yao, and others as special cases. By applying Kirk's fixed-point theorem and Ding-Tan minimax inequality, we establish the existence theorems of solutions for the generalized nonlinear variational-like inequalities in reflexive Banach spaces.

1. Introduction and preliminaries

In what follows, let $\mathbb{R} = (-\infty, +\infty)$, let B be a Banach space with norm $\|\cdot\|$, let B^* be the topological dual space of B, and let $\langle u, v\rangle$ be the pairing between $u \in B^*$ and $v \in B$. Let D be a nonempty closed convex subset of B and $a, b : D \times D \to \mathbb{R}$ satisfy the following conditions:

$$a \text{ is a continuous function which is linear in both arguments;} \tag{1.1}$$

$$\begin{aligned}&\text{there exist constants } \alpha > 0,\ \beta > 0 \text{ satisfying } a(x,x) \geq \alpha\|x\|^2\\ &\text{and } a(x,y) \leq \beta\|x\|\|y\|, \quad \forall x, y \in D;\end{aligned} \tag{1.2}$$

$$\begin{aligned}&b(x,y) \text{ is linear in the first argument and is convex}\\ &\text{in the second argument, respectively;}\end{aligned} \tag{1.3}$$

$$\text{there exists a constant } \gamma > 0 \text{ satisfying } b(x,y) \leq \gamma\|x\|\|y\|, \quad \forall x, y \in D; \tag{1.4}$$

$$b(x,y) - b(x,z) \leq b(x, y-z), \quad \forall x, y, z \in D. \tag{1.5}$$

Ding and Tarafdar [7] and Ding [4] introduced and studied the following general nonlinear variational inequality problem and general nonlinear variational-like inequality problem, respectively.

Find $u \in D$ such that

$$a(u, x-u) + b(u,x) - b(u,u) \geq \langle Tu, g(x) - g(u)\rangle, \quad \forall x \in D. \tag{1.6}$$

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Find $u \in D$ such that

$$\langle Tu - Au, \eta(x,u)\rangle + b(u,x) - b(u,u) \geq 0, \quad \forall x \in D. \tag{1.7}$$

They obtained the existence uniqueness theorems of solutions for problems (1.6) and (1.7) in nonempty closed convex subsets of reflexive Banach spaces. Cubiotti [2] established the existence of solution for problem (1.6) in nonempty convex and weakly compact subsets of reflexive Banach spaces. It is well known in the literature that problems (1.6) and (1.7) can characterize a wide class of problems arising in control and optimization, mathematical programming, mechanics, engineering, economics equilibrium, and free boundary-valued problems, and so forth On the other hand, Bose [1], Dien [3], Ding [5], Fang et al. [8], Liu et al. [10], Noor [11], Parida et al. [12], and Yao [13] investigated some special cases of problems (1.6) and (1.7) or a few similar problems in Euclidean spaces, Hilbert spaces, and Banach spaces, respectively. In 1965, Kirk [9] showed the following nice result.

Lemma 1.1 (see [9]). *Let D be a nonempty bounded closed convex subset of a reflexive Banach space B, and suppose that D has normal structure. If $T : D \to D$ is nonexpansive, that is,*

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x \in D, \tag{1.8}$$

then T has a fixed point in D.

Although the result due to Kirk has various applications in different fields, to our knowledge, it never has any applications in variational inequality theory. The main purpose of this paper is to provide a few new applications of Kirk's fixed-point theorem in variational-like inequalities. That is, by applying Kirk's fixed-point theorem, we study the existence of solutions for the following generalized nonlinear variational-like inequality problem.

Find $u \in D$ such that

$$a(u,x-u) + b(u,x) - b(u,u) \geq \langle N(Tu,Au), \eta(x,u)\rangle, \quad \forall x \in D, \tag{1.9}$$

where a and b satisfy (1.1)–(1.5) and b is not necessarily differentiable, $T,A : D \to B$, $N : B \times B \to B^*$, and $\eta : D \times D \to B$ are four nonlinear mappings.

The results proved in this paper represent a significant improvement and refinement of the previously known results in this field.

Remark 1.2. For suitable and appropriate choices of the mappings T, A, N, η, a and b, one can obtain various new and previously known variational inequality problems and variational-like inequality problems in [1, 2, 3, 4, 7, 11, 12, 13] as special cases of the generalized nonlinear variational-like inequality problem (1.9).

Remark 1.3. It follows from (1.5) that

$$b(x,z) - b(x,y) \leq b(x,z-y), \quad \forall x,y,z \in D. \tag{1.10}$$

By virtue of (1.4), (1.5), and (1.10), we derive that

$$|b(x,y)-b(x,z)| \le \gamma\|x\|\|y-z\|, \quad \forall x,y,z \in D, \tag{1.11}$$

which implies that $b(x,y)$ is continuous with respect to the second argument.

Definition 1.4. Let D be a nonempty convex subset of a Banach space B with the dual space B^*. Let $T,A : D \to B$, $N : B\times B \to B^*$, and $\eta : D\times D \to B$ be mappings. The mappings T, A, N, and η are said to have 0-diagonally concave relation on D if the function $\phi : D\times D \to \mathbb{R}\cup\{+\infty\}$ defined by $\phi(x,y) = \langle N(Tx,Ax),\eta(y,x)\rangle$ is 0-diagonally concave in y. That is, for any finite set $\{y_1,y_2,\ldots,y_m\} \subseteq D$ and for any $x = \sum_{i=1}^m t_i y_i$ with $t_i \ge 0$ and $\sum_{i=1}^m t_i = 1$,

$$\sum_{i=1}^m t_i\phi(x,y_i) \le 0. \tag{1.12}$$

Definition 1.5. Let D be a nonempty convex subset of a Banach space B with the dual space B^*. Let $T : D \to B$, $N : B\times B \to B^*$, and $\eta : D\times D \to B$ be mappings.

(i) T is said to be t-strongly η-antimonotone with respect to the first argument of N if there exists a constant $t > 0$ such that

$$\langle N(Tx,u),\eta(y,x)\rangle + \langle N(Ty,u),\eta(x,y)\rangle \ge t\|x-y\|^2, \quad \forall x,y \in D,\ u \in B. \tag{1.13}$$

(ii) T is said to be t-weakly η-antimonotone with respect to the first argument of N if there exists a constant $t > 0$ such that

$$\langle N(Tx,u),\eta(y,x)\rangle + \langle N(Ty,u),\eta(x,y)\rangle \ge -t\|x-y\|^2, \quad \forall x,y \in D,\ u \in B. \tag{1.14}$$

(iii) If $t = 0$ in (1.13), then T is called η-antimonotone with respect to the first argument of N.

(iv) T is said to be t-Lipschitz continuous if there exists a constant $t > 0$ such that

$$\|Tx-Ty\| \le t\|x-y\|, \quad \forall x,y \in D. \tag{1.15}$$

(v) η is said to satisfy the Lipschitz-type condition if there exists a constant $t > 0$ such that

$$\|\eta(x,y)\| \le t\|x-y\|, \quad \forall x,y \in D. \tag{1.16}$$

(vi) N is said to be t-Lipschitz continuous with respect to the first argument if there exists a constant $t > 0$ such that

$$\|N(x,u)-N(y,u)\| \le t\|x-y\|, \quad \forall x,y,u \in B. \tag{1.17}$$

In a similar way, we can define that T is η-antimonotone, t-strongly η-antimonotone, t-weakly η-antimonotone with respect to the second argument of N, respectively, and N is t-Lipschitz continuous with respect to the second argument.

Remark 1.6. From Definition 1.5, we immediately have the following implications. The t-strong η-antimonotonicity $\Rightarrow$ the η-antimonotonicity $\Rightarrow$ the t-weak η-antimonotonicity.

But the converses are not true, see Examples 1.7 and 1.8 below.

Example 1.7. Let $B = \mathbb{R}$, $D = [0,+\infty)$, $T : D \to \mathbb{R}$, $N : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, and $\eta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be defined by $Tx = \sin 3x$, for all $x \in D$, $N(x,y) = x$, for all $x, y \in \mathbb{R}$, and

$$\eta(x,y) = \begin{cases} x-y & \text{if } xy < 1,\ (x,y) \in D\times D, \\ x^2y^2(x-y) & \text{if } 1 \le xy < 2,\ (x,y) \in D\times D, \\ 2(x-y) & \text{if } 2 \le xy,\ (x,y) \in D\times D, \end{cases} \tag{1.18}$$

respectively. It is clear that $\eta(x,y) = -\eta(y,x)$ for all $x, y \in D$, and T is 12-weakly η-antimonotone with respect to the first argument of N. But it is not η-antimonotone with respect to the first argument of N because

$$\left\langle N\left(T\frac{\pi}{6},1\right),\eta\left(0,\frac{\pi}{6}\right)\right\rangle + \left\langle N(T0,1),\eta\left(\frac{\pi}{6},0\right)\right\rangle = -\frac{\pi}{6} < 0. \tag{1.19}$$

Example 1.8. Let B, D, and N be as in Example 1.7, and define $T : D \to \mathbb{R}$ and $\eta : D\times D \to \mathbb{R}$ by

$$\begin{gathered} Tx = 1-x, \quad \forall x \in D, \\ \eta(x,y) = \begin{cases} x-y & \text{if } xy < 1,\ (x,y) \in D\times D, \\ xy(x-y) & \text{if } 1 \le xy < 2,\ (x,y) \in D\times D, \\ (n+1)^{-1}(x-y) & \text{if } n \le xy < n+1,\ (x,y) \in D\times D, \end{cases} \end{gathered} \tag{1.20}$$

where $n \ge 2$ is any positive integer. It is easy to verify that $\eta(x,y) = -\eta(y,x)$ for all $x, y \in D$, and T is η-antimonotone with respect to the first argument of N. We claim that T is not t-strongly η-antimonotone with respect to the first argument of N. Otherwise, there exists some $t > 0$ satisfying (1.13). For any positive integer $n \ge 2$, we select $x_n = n$, $y_n = 1$, and $u_n = 0$. Then (1.13) yields that

$$\begin{aligned} t(x_n-y_n)^2 &\le \langle N(Tx_n,u_n),\eta(y_n,x_n)\rangle + \langle N(Ty_n,u_n),\eta(x_n,y_n)\rangle \\ &= \langle x_n-y_n,\eta(x_n,y_n)\rangle = (n+1)^{-1}(x_n-y_n)^2, \end{aligned} \tag{1.21}$$

which implies that

$$t \le (n+1)^{-1}, \quad \forall n \ge 2. \tag{1.22}$$

Letting $n \to \infty$ in the above inequality, we infer that $t \le 0$, which is a contradiction.

Lemma 1.9 (see [6]). *Let D be a nonempty convex subset of a topological vector space and let $\phi : D \times D \to \mathbb{R} \cup \{\pm\infty\}$ such that*

(i) *for each $x \in D$, $\phi(x,\cdot)$ is lower semicontinuous on each nonempty compact subset of D;*

(ii) *for each nonempty finite set $\{x_1, x_2, \ldots, x_m\} \subseteq D$ and for each $y = \sum_{i=1}^m t_i x_i$ with $t_i \geq 0$ and $\sum_{i=1}^m t_i = 1$,*

$$\min_{1 \leq i \leq m} \phi(x_i, y) \leq 0; \tag{1.23}$$

(iii) *there exist a nonempty compact convex subset X of D and a nonempty compact subset K of D such that for each $y \in D - K$, there is an $x \in \operatorname{co}(X \cup \{y\})$ with $\phi(x,y) > 0$.*

Then there exists an $u \in K$ satisfying $\phi(x,u) \leq 0$ for all $x \in D$.

2. Existence theorems

Now we use Kirk's fixed-point theorem and Ding-Tan minimax inequality to study the existence of solutions of the generalized nonlinear variational-like inequality problem (1.9).

Theorem 2.1. *Let D be a nonempty bounded closed convex subset of a reflexive Banach space B with the dual space B^*, and suppose that D has normal structure. Assume that $a : D \times D \to \mathbb{R}$ and $b : D \times D \to \mathbb{R} \cup \{\pm\infty\}$ satisfy (1.1)–(1.5). Let $T, A : D \to B$, $N : B \times B \to B^*$, and $\eta : D \times D \to B$ satisfy that*

$$T,\ A,\ N,\ \text{and } \eta \text{ have 0-diagonally concave relation on } D; \tag{2.1}$$

$$T,\ A,\ N,\ \text{and } \eta \text{ are continuous}; \tag{2.2}$$

$$\begin{gathered}\eta \text{ satisfies the Lipschitz-type condition with constant } \delta > 0 \text{ and} \\ \eta(x,y) = -\eta(y,x), \quad \forall x, y \in D;\end{gathered} \tag{2.3}$$

$$T \text{ is } \xi\text{-strongly } \eta\text{-antimonotone with respect to the first argument of } N; \tag{2.4}$$

$$A \text{ is } \zeta\text{-weakly } \eta\text{-antimonotone with respect to the second argument of } N; \tag{2.5}$$

$$\alpha + \xi = \zeta + \gamma. \tag{2.6}$$

Then the generalized nonlinear variational-like inequality problem (1.9) has a solution $u \in D$.

Proof. First of all, we show that the following problem (2.7) has a unique solution, that is, for each fixed $u \in D$, there exists a unique $v \in D$ satisfying

$$a(v, w - v) + b(u, w) - b(u, v) \geq \langle N(Tv, Av), \eta(w, v) \rangle, \quad \forall w \in D. \tag{2.7}$$

Define a function $\phi : D \times D \to \mathbb{R}$ by

$$\phi(w,x) = \langle N(Tx, Ax), \eta(w,x) \rangle + b(u,x) - b(u,w) - a(x, w - x), \quad \forall w, x \in D, \tag{2.8}$$

where u is fixed in D. Making use of (1.1), (1.3), (2.2), and Remark 1.3, we infer that for each $w \in D$, the function $x \to \phi(w,x)$ is weakly lower semicontinuous on D. Now we claim that $\phi(w,x)$ satisfies the condition (ii) of Lemma 1.9. Otherwise, there exists a finite set $\{w_1,w_2,\dots,w_m\}\subseteq D$ and $x=\sum_{i=1}^m t_iw_i$ with $t_i\geq 0$, and $\sum_{i=1}^m t_i=1$ such that $\phi(w_i,x)>0$ for $i=1,2,\dots,m$. From (1.1), (1.3), (2.1), and (2.3), we conclude that

$$0\geq\left\langle N(Tx,Ax),\sum_{i=1}^m t_i\eta(w_i,x)\right\rangle>\sum_{i=1}^m t_ib(u,w_i)-b(u,x)+a\left(x,\sum_{i=1}^m t_iw_i-x\right)\geq 0, \tag{2.9}$$

which is a contradiction. Therefore, the condition (ii) of Lemma 1.9 is satisfied. Putting $X=\{u\}$,

$$\begin{aligned}\rho&=1+\gamma^{-1}\big[(\beta+\gamma)\|u\|+\delta\big|\big|N(Tu,Au)\big|\big|\big],\\ K&=\{y\in D:\|y-u\|\leq\rho\}.\end{aligned} \tag{2.10}$$

Then K and X are both weakly compact convex subsets of D. Taking into account assumptions (1.1), (1.2), (1.11), and (2.3)–(2.6), we know that for given $x\in D-K$, there exists $u\in\operatorname{co}(X\cup\{x\})$ satisfying

$$\begin{aligned}\phi(u,x)&=\langle N(Tx,Ax),\eta(u,x)\rangle+b(u,x)-b(u,u)-a(x,u-x)\\&=a(u-x,u-x)-a(u,u-x)+\langle N(Tx,Ax)-N(Tu,Ax),\eta(u,x)\rangle\\&\quad+\langle N(Tu,Ax)-N(Tu,Au),\eta(u,x)\rangle+\langle N(Tu,Au),\eta(u,x)\rangle+b(u,x)-b(u,u)\\&\geq\alpha\|u-x\|^2-\beta\|u\|\|u-x\|+\xi\|x-u\|^2-\zeta\|x-u\|^2\\&\quad-\delta\big|\big|N(Tu,Au)\big|\big|\|u-x\|-\gamma\|u\|\|u-x\|\\&=\|u-x\|\big[\gamma\|u-x\|-(\beta+\gamma)\|u\|-\delta\big|\big|N(Tu,Au)\big|\big|\big]>0.\end{aligned} \tag{2.11}$$

Consequently, all the assumptions of Lemma 1.9 are satisfied. Thus, there exists $v\in D$ such that $\phi(w,v)\leq 0$ for all $w\in D$. That is, problem (2.7) has a solution $v\in D$.

Suppose that problem (2.7) has another solution $x\in D$ different from v. It follows that

$$a(x,w-x)+b(u,w)-b(u,x)\geq\langle N(Tx,Ax),\eta(w,x)\rangle,\quad\forall w\in D. \tag{2.12}$$

Taking $w=x$ in (2.7) and $w=v$ in (2.12), respectively, and adding these inequalities, by (1.1), (1.2), (2.4), and (2.5), we get that

$$\begin{aligned}\alpha\|v-x\|^2&\leq a(v-x,v-x)\\&\leq\langle N(Tv,Av)-N(Tx,Ax),\eta(v,x)\rangle\\&\leq\langle N(Tv,Av)-N(Tx,Av),\eta(v,x)\rangle+\langle N(Tx,Av)-N(Tx,Ax),\eta(v,x)\rangle\\&\leq-\xi\|v-x\|^2+\zeta\|v-x\|^2=(\zeta-\xi)\|v-x\|^2.\end{aligned} \tag{2.13}$$

Taking into account (2.6) and (2.13), we conclude immediately that

$$0<\gamma\|v-x\|^2=(\alpha+\xi-\zeta)\|v-x\|^2\leq 0, \tag{2.14}$$

which is a contradiction, and hence $v = x$, that is, problem (2.7) has a unique solution $v \in D$. It follows that there exists a mapping $f : D \to D$ such that for each $u \in D$, $f(u)$ is the unique solution of problem (2.7).

Next, we show that f is nonexpansive. By the definition of f, we have

$$a(f(u), w - f(u)) + b(u, w) - b(u, f(u)) \geq \langle N(Tf(u), Af(u)), \eta(w, f(u))\rangle, \quad (2.15)$$
$$a(f(x), w - f(x)) + b(x, w) - b(x, f(x)) \geq \langle N(Tf(x), Af(x)), \eta(w, f(x))\rangle \quad (2.16)$$

for any $u, x, w \in D$. Taking $w = f(x)$ in (2.15) and $w = f(u)$ in (2.16) and adding these inequalities, we get that

$$\begin{aligned} &a(f(u) - f(x), f(u) - f(x)) + b(u, f(u)) + b(x, f(x)) - b(u, f(x)) - b(x, f(u)) \\ &\quad \leq \langle N(Tf(u), Af(u)) - N(Tf(x), Af(x)), \eta(f(u), f(x))\rangle. \end{aligned} \quad (2.17)$$

By virtue of (1.2)–(1.5), (2.4), (2.5), and (2.17), we infer that

$$\begin{aligned} &\alpha\|f(u) - f(x)\|^2 \\ &\quad \leq a(f(u) - f(x), f(u) - f(x)) \\ &\quad \leq b(u - x, f(x)) - b(u - x, f(u)) \\ &\qquad + \langle N(Tf(u), Af(u)) - N(Tf(x), Af(x)), \eta(f(u), f(x))\rangle \\ &\quad \leq b(u - x, f(x) - f(u)) \\ &\qquad + \langle N(Tf(u), Af(u)) - N(Tf(x), Af(u)), \eta(f(u), f(x))\rangle \\ &\qquad + \langle N(Tf(x), Af(u)) - N(Tf(x), Af(x)), \eta(f(u), f(x))\rangle \\ &\quad \leq \gamma\|u - x\|\|f(x) - f(u)\| - \xi\|f(x) - f(u)\|^2 + \zeta\|f(x) - f(u)\|^2. \end{aligned} \quad (2.18)$$

Using (2.6) and (2.18), we know that

$$\|f(u) - f(x)\| \leq \frac{\gamma}{\alpha + \xi - \zeta}\|u - x\| = \|u - x\|, \quad \forall u, x \in D, \quad (2.19)$$

that is, f is nonexpansive. It follows from Lemma 1.1 that f has a fixed point $u \in D$, which satisfies the following:

$$a(u, x - u) + b(u, x) - b(u, u) \geq \langle N(Tu, Au), \eta(x, u)\rangle, \quad \forall x \in D. \quad (2.20)$$

That is, $u \in D$ is a solution of the generalized nonlinear variational-like inequality problem (1.9). This completes the proof. □

Theorem 2.2. *Let D, B, B^*, a and b be as in Theorem 2.1. Assume that $T, A : D \to B$, $N : B \times B \to B^*$, and $\eta : D \times D \to B$ satisfy that (2.1), (2.3), (2.4) hold and*

T and η are continuous and A is s-Lipschitz continuous; (2.21)

N is continuous with respect to the first argument; (2.22)

N is t-Lipschitz continuous with respect to the second argument; (2.23)

$$\alpha + \xi = ts\delta + \gamma. \quad (2.24)$$

Then there exists $u \in D$ which solves the generalized nonlinear variational-like inequality problem (1.9).

Proof. For any fixed $u \in D$, define a function $\phi : D \times D \to \mathbb{R}$ by

$$\phi(w,x) = \langle N(Tx,Ax), \eta(w,x)\rangle + b(u,x) - b(u,w) - a(x,w-x), \quad \forall w,x \in D. \tag{2.25}$$

Observe that (2.21) and (2.23) yield that

$$\begin{aligned} &||N(Tx,Ax) - N(Tw,Aw)|| \\ &\quad \le ||N(Tx,Ax) - N(Tw,Ax)|| + ||N(Tw,Ax) - N(Tw,Aw)|| \\ &\quad \le ||N(Tx,Ax) - N(Tw,Ax)|| + ts\|x-w\|, \quad \forall x,w \in D. \end{aligned} \tag{2.26}$$

It follows from (2.21), (2.22), and (2.26) that $N(Tx,Ax)$ is continuous on D. As in the proof of Theorem 2.1, we conclude that conditions (i) and (ii) in Lemma 1.9 hold. Now we verify that condition (iii) in Lemma 1.9 holds also. Putting $X = \{u\}$,

$$\begin{gathered} \rho = 1 + \gamma^{-1}\left[(\beta+\gamma)\|u\| + \delta||N(Tu,Au)||\right], \\ K = \{y \in D : \|y-u\| \le \rho\}. \end{gathered} \tag{2.27}$$

It follows from (1.1), (1.2), (1.11), (2.3), (2.4), (2.21), (2.23), and (2.24) that for given $x \in D - K$, there exists $u \in \mathrm{co}(X \cup \{x\})$ satisfying

$$\begin{aligned} \phi(u,x) &= \langle N(Tx,Ax), \eta(u,x)\rangle + b(u,x) - b(u,u) - a(x,u-x) \\ &= a(u-x,u-x) - a(u,u-x) + \langle N(Tx,Ax) - N(Tu,Ax), \eta(u,x)\rangle \\ &\quad + \langle N(Tu,Ax) - N(Tu,Au), \eta(u,x)\rangle + \langle N(Tu,Au), \eta(u,x)\rangle + b(u,x) - b(u,u) \\ &\ge \alpha\|u-x\|^2 - \beta\|u\|\,\|u-x\| + \xi\|x-u\|^2 - ts\delta\|x-u\|^2 \\ &\quad - \delta||N(Tu,Au)||\,\|u-x\| - \gamma\|u\|\,\|u-x\| \\ &= \|u-x\|\left[\gamma\|u-x\| - (\beta+\gamma)\|u\| - \delta||N(Tu,Au)||\right] > 0. \end{aligned} \tag{2.28}$$

Therefore, Lemma 1.9 ensures that problem (2.7) has a solution $v \in D$. By a similar argument used in the proof of Theorem 2.1, the result follows. This completes the proof. □

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