POSITIVE SOLUTIONS OF SECOND-ORDER SINGULAR BOUNDARY VALUE PROBLEM WITH A LAPLACE-LIKE OPERATOR

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By use of the concavity of solution for an associate boundary value problem, existence criteria of positive solutions are given for the Dirichlet BVP $(\Phi(u'))' + \lambda a(t) f(t,u) = 0$, $0 < t < 1$, $u(0) = 0 = u(1)$, where Φ is odd and continuous with $0 < l_1 \le ((\Phi(x) - \Phi(y))/(x-y)) \le l_2$, $a(t) \ge 0$, and f may change sign and be singular along a curve in $[0,1] \times \mathbb{R}^+$.

1. Introduction

For the Sturm-Liouville boundary value problem (BVP)

$$\begin{gathered}(\Phi(u'))' + \lambda a(t) f(t,u) = 0,\\ \alpha_1 u(0) - \beta_1 u'(0) = 0 = \alpha_1 u(1) + \beta_2 u'(1),\end{gathered} \tag{1.1}$$

there has been much work done for some special cases in order to search the existence of positive solutions. For example, Erbe and Wang [3] studied the case for $\Phi(v) = v$, Wang [8] discussed the problem with boundary conditions replaced by nonlinear ones, Sun and Ge [7] dealt with the problem for the existence of multiple positive solutions in case $\alpha_1 = \beta_2 = 0$ and $\beta_1 = \alpha_2 = 1$, Avery et al. [2] researched the existence of twin positive solutions for the case $\Phi(v) = v$, $\alpha_1 = \beta_2 = 1$, $\beta_1 = \alpha_2 = 0$, and He and Ge [6] discussed the existence of multiple positive solutions. In all the above-mentioned articles f is supposed to be nonnegative. When $\Phi(v) = v$, Agarwal et al. [1] as well as Ge and Ren [4] discussed the existence of positive solutions without nonnegativity condition imposed on f. As for the general BVP

$$\begin{gathered}(p(t)\Phi(u'))' + \lambda p(t) f(t,u) = 0, \quad 0 < t < 1,\\ u(0) = 0 = u(1),\end{gathered} \tag{1.2}$$

Hai et al. [5] studied the existence of positive solutions with $f \ge -M$. When Φ is odd and Φ^{-1} is concave, they proved that there are $\lambda^*, \overline{\lambda} > 0$ such that BVP (1.2) has at least one positive solution if $\lambda \in (0,\lambda^*)[\lambda > \overline{\lambda}]$ under the condition $\lim_{u\to\infty} f(t,u)/\Phi(u) = \infty$ uniformly for $t \in [0,1]$. The restriction, Φ^{-1} being concave, excludes the case $\Phi(u) = |u|^{p-2}u$, $1 < p < 2$.

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In this paper, we want to give theorems for the existence of positive solutions for the BVP

$$(\Phi(u'))' + \lambda a(t) f(t,u) = 0, \quad 0 < t < 1,\ \lambda > 0, \qquad u(0) = 0 = u(1), \tag{1.3}$$

without the restriction $f(t,u) \geq -M$ for $(t,u) \in [0,1] \times \mathbb{R}^+$ and without Φ^{-1} being concave.

We suppose throughout this paper that

(H1) $a \in C((0,1),\mathbb{R}^+)$ and for a $\delta \in (0,(1/2))$, $0 < \int_\delta^{1-\delta} a(t)dt \leq \int_0^1 a(t)dt < \infty$;

(H2) Φ is odd, continuous with

$$0 < l_1 \leq \frac{\Phi(x) - \Phi(y)}{x - y} \leq l_2 < \infty, \quad x \neq y. \tag{1.4}$$

Obviously (H2) implies that $\Phi^{-1}(s)$ exists and

$$0 < \frac{1}{l_2} \leq \frac{(\Phi^{-1})(x) - (\Phi^{-1})(y)}{x - y} \leq \frac{1}{l_1} < \infty, \quad x \neq y. \tag{1.5}$$

2. Preliminary lemmas

Lemma 2.1. *Suppose (H1)-(H2) hold. Then for $\lambda M \in \mathbb{R}$,*

$$(\Phi(u'))' + \lambda a(t) M = 0, \quad 0 < t < 1, \qquad u(0) = 0 = u(1) \tag{2.1}$$

has a unique solution

$$w_{\lambda M}(t) = \int_0^t \Phi^{-1}\left(\lambda M\left(c - \int_0^s a(\tau)d\tau\right)\right)ds \tag{2.2}$$

with c satisfying

$$\int_0^1 \Phi^{-1}\left(\lambda M\left(c - \int_0^s a(\tau)d\tau\right)\right)ds = 0. \tag{2.3}$$

Proof. It is easy to show that $u = w(t)$ is a solution to BVP (2.1) if and only if $u(t)$ is expressed in (2.2) with c satisfying (2.3). Now we show that there is only one c which makes (2.3) hold. Without loss of generality, we suppose $\lambda M \geq 0$. Let $H(c) = \int_0^1 \Phi^{-1}(\lambda M(c - \int_0^s a(\tau)d\tau))ds$. Then $H : \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing with

$$H(0) < 0 < H\left(\int_0^1 a(\tau)d\tau\right), \tag{2.4}$$

which implies there is a unique $c_0 \in (0, \int_0^1 a(\tau)d\tau)$ such that $H(c_0) = 0$. □

It follows that there is $\sigma^* \in (0,1)$ such that $c_0 = \int_0^{\sigma^*} a(\tau)d\tau$ and then (2.2) becomes

$$w_{\lambda M}(t) = \int_0^t \Phi^{-1}\left(\lambda M \int_s^{\sigma^*} a(\tau)d\tau\right)ds. \tag{2.5}$$

Remark 2.2. Let $k_0 = (1/l_1)\int_0^1 a(\tau)d\tau$. It follows from (2.5) that

$$\begin{aligned} \|w_{\lambda M}\| &= \max_{0\le t\le 1} |w_{\lambda M}(t)| \le |\lambda M|\frac{1}{l_1}\int_0^1 a(\tau)d\tau \le |\lambda M|k_0, \\ \|w'_{\lambda M}\| &= \max_{0\le t\le 1} |w'_{\lambda M}(t)| \le |\lambda M|\frac{1}{l_1}\int_0^1 a(\tau)d\tau = |\lambda M|k_0. \end{aligned} \tag{2.6}$$

Remark 2.3. It is easy to see that

$$w_{-\lambda M}(t) = -w_{\lambda M}(t), \qquad \|w_{-\lambda M}\| = \|w_{\lambda M}\|, \qquad \|w'_{-\lambda M}\| = \|w'_{\lambda M}\|, \tag{2.7}$$

and $\Phi(w'_{\lambda M}(t))$ is nonincreasing when $\lambda M \ge 0$.

Lemma 2.4. *$u_\lambda(t)$ and $w_{\lambda M}(t)$ are solutions of BVP (1.3) and BVP (2.1), respectively, with f replaced by $f^* \in C([0,1]\times\mathbb{R},\mathbb{R})$. Let*

$$D = \{(t,x) \in (0,1)\times(-\infty, w_{\lambda M}(t)]\}. \tag{2.8}$$

If $f^*(t,x) \ge M (\le M)$ holds for each $(t,x) \in D$, then

$$u_\lambda(t) \ge w_{\lambda M}(t) \quad (u_\lambda(t) \le w_{\lambda M}(t)), \quad t \in [0,1]. \tag{2.9}$$

Proof. We prove only the case $f^*(t,x) \ge M$.

Suppose the contrary. Then there is $t_0 \in (0,1)$ such that $(t_0, u_\lambda(t_0)), (t_0, w_{\lambda M}(t_0)) \in D$ and $u_\lambda(t_0) - w_{\lambda M}(t_0) < 0$. Without loss of generality we assume $u'_\lambda(t_0) - w'_{\lambda M}(t_0) \le 0$. The condition $u_\lambda(1) = w_{\lambda M}(1) = 0$ implies there is $t_1 \in (t_0, 1]$ such that

$$u_\lambda(t) < w_{\lambda M}(t), \quad t \in [t_0, t_1); \qquad u_\lambda(t_1) = w_{\lambda M}(t_1). \tag{2.10}$$

Then for $t \in (t_0, t_1)$,

$$\begin{aligned} &\Phi(u'_\lambda(t)) - \Phi(w'_{\lambda M}(t)) \\ &\quad = [\Phi(u'_\lambda(t_0)) - \Phi(w'_{\lambda M}(t_0))] - \lambda\int_{t_0}^t a(\tau)[f^*(\tau, u_\lambda(\tau)) - M]d\tau \\ &\quad \le \Phi(u'_\lambda(t_0)) - \Phi(w'_{\lambda M}(t_0)) \le 0, \end{aligned} \tag{2.11}$$

and therefore $u_\lambda'(t) \le w_{\lambda M}'(t)$ which implies

$$u_\lambda(t_1) - w_{\lambda M}(t_1) = u_\lambda(t_0) - w_{\lambda M}(t_0) + \int_{t_0}^{t_1} [u_\lambda'(s) - w_{\lambda M}'(s)]ds < 0, \tag{2.12}$$

a contradiction to (2.10). □

Remark 2.5. If $f(t,x) \ge M(\le M)$ is replaced by $f(t,x) > M(< M)$, then $u_\lambda(t) > w_{\lambda M}(t)$ $(u_\lambda(t) < w_{\lambda M}(t))$.

LEMMA 2.6. *Suppose u, $v \in C^1([0,1],\mathbb{R}^+)$ and $u(0) = u(1) = v(0) = v(1) = 0$. If $\|u\| \ge \|v'\|$ and u is concave, then*

$$u(t) \ge v(t), \quad t \in [0,1]. \tag{2.13}$$

Proof. Suppose there is $\sigma \in (0,1)$ such that $u(\sigma) = \|u\| = L$. Then $v(t) \le L\min\{t,1-t\}$, $t \in [0,1]$. The concavity of u implies $u(t) \ge u(\sigma)\min\{t,1-t\}$. So $u(t) \ge v(t)$ holds for $t \in [0,1]$.

For each $x \in C([0,1],\mathbb{R})$, $f^* \in C([0,1]\times\mathbb{R},\mathbb{R})$, the solution to

$$\begin{gathered}(\Phi(u'))' + \lambda a(t) f^*(t,x(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0 = u(1)\end{gathered} \tag{2.14}$$

can be expressed in the form

$$u(t) = \int_0^t \Phi^{-1}\left(\lambda\left(c - \int_0^s a(\tau) f^*(\tau,x(\tau))d\tau\right)\right)ds \tag{2.15}$$

with c satisfying

$$\int_0^1 \Phi^{-1}\left(\lambda\left(c - \int_0^s a(\tau) f^*(\tau,x(\tau))d\tau\right)\right)ds = 0. \tag{2.16}$$

Since $H(c) = \int_0^1 \Phi^{-1}(\lambda(c - \int_0^s a(\tau) f^*(\tau,x(\tau))d\tau))ds$ is strictly increasing with respect to c and

$$\begin{aligned} &H(c) < 0 \quad \text{when } c < \min_{0\le t\le 1} \int_0^t a(\tau) f^*(\tau,x(\tau))d\tau, \\ &H(c) > 0 \quad \text{when } c > \max_{0\le t\le 1} \int_0^t a(\tau) f^*(\tau,x(\tau))d\tau,\end{aligned} \tag{2.17}$$

there is only one $c_x \in \mathbb{R}$,

$$\min_{0\le t\le 1} \int_0^t a(\tau) f^*(\tau,x(\tau))d\tau < c_x < \max_{0\le t\le 1} \int_0^t a(\tau) f^*(\tau,x(\tau))d\tau \tag{2.18}$$

such that $H(c_x)=0$. So the solution to BVP (2.14) is unique. At the same time, (2.18) implies there is $\sigma_x\in(0,1)$ such that $\int_0^{\sigma_x}a(\tau)f^*(\tau,x(\tau))d\tau=c_x$. Then (2.15) can be written as

$$u(t)=\int_0^t\Phi^{-1}\left(\lambda\int_s^{\sigma_x}a(\tau)f^*(\tau,x(\tau))d\tau\right)ds. \tag{2.19}$$

Furthermore, by $u(1)=0$ and for $t\geq\sigma_x$,

$$\begin{aligned}u(t)&=-\int_0^t\Phi^{-1}\left(\lambda\int_{\sigma_x}^s a(\tau)f^*(\tau,x(\tau))d\tau\right)ds\\&=\int_t^1\Phi^{-1}\left(\lambda\int_{\sigma_x}^s a(\tau)f^*(\tau,x(\tau))d\tau\right)ds,\end{aligned} \tag{2.20}$$

and therefore

$$u(t)=\begin{cases}\displaystyle\int_0^t\Phi^{-1}\left(\lambda\int_s^{\sigma_x}a(\tau)f^*(\tau,x(\tau))d\tau\right)ds, & 0\leq t\leq\sigma_x,\\ \displaystyle\int_t^1\Phi^{-1}\left(\lambda\int_{\sigma_x}^s a(\tau)f^*(\tau,x(\tau))d\tau\right)ds, & \sigma_x\leq t\leq 1\end{cases} \tag{2.21}$$

for each $\sigma_x\in\Sigma_x=\{\sigma\in[0,1]:\int_0^\sigma a(\tau)f^*(\tau,x(\tau))d\tau=c_x\}$. □

Lemma 2.7. *The constant $c=c_x$ determined by (2.16) is continuous with respect to $x\in C([0,1],\mathbb{R})$.*

Proof. Suppose the contrary. Then there are $x_n\in C([0,1],\mathbb{R})$ which converge to $x(t)$ uniformly in $[0,1]$ and c_n, determined by (2.16) with x replaced by x_n, converging to $c_0\neq c_x$. Applying Lebesgue's dominating convergence theorem, we have

$$\int_0^1\Phi^{-1}\left(\lambda\left(c_0-\int_0^s a(\tau)f^*(\tau,x(\tau))d\tau\right)\right)ds=0 \tag{2.22}$$

as $n\to\infty$ in

$$\int_0^1\Phi^{-1}\left(\lambda\left(c_n-\int_0^s a(\tau)f^*(\tau,x_n(\tau))d\tau\right)\right)ds=0. \tag{2.23}$$

The uniqueness of solution to (2.16) implies $c_0=c_x$, a contradiction.

Take $X=C([0,1],\mathbb{R})$ and define

$$T:X\longrightarrow X \tag{2.24}$$

by

$$(Tx)(t)=\int_0^t\Phi^{-1}\left(\lambda\left(c_x-\int_0^s a(\tau)f^*(\tau,x(\tau))d\tau\right)\right)ds \tag{2.25}$$

with c_x satisfying

$$\int_0^1\Phi^{-1}\left(\lambda\left(c_x-\int_0^s a(\tau)f^*(\tau,x(\tau))d\tau\right)\right)ds=0, \tag{2.26}$$

or equivalently,

$$(Tx)(t) = \begin{cases} \int_0^t \Phi^{-1}\left(\lambda \int_s^{\sigma_x} a(\tau) f^*(\tau, x(\tau))\, d\tau\right) ds, & 0 \le t \le \sigma_x, \\ \int_t^1 \Phi^{-1}\left(\lambda \int_{\sigma_x}^s a(\tau) f^*(\tau, x(\tau))\, d\tau\right) ds, & \sigma_x \le t \le 1, \end{cases} \tag{2.27}$$

where $\sigma_x \in \Sigma_x$. Obviously $u(t) = (Tx)(t)$ is the solution of (2.14). □

Lemma 2.8. *$T : X \to X$ is completely continuous.*

Proof. Because Φ^{-1}, f are both continuous, $a(t)$ is integrable on $(0,1)$, and c_x is continuous with respect to x, it is easy to show that T is continuous in X. Given a bounded set $\Omega \subset X$, (2.25) implies $T\Omega$ is bounded. Differentiating (2.25) with respect to t, one has

$$(Tx)'(t) = \Phi^{-1}\left(\lambda\left(c_x - \int_0^t a(\tau) f^*(\tau, x(\tau))\, d\tau\right)\right). \tag{2.28}$$

Obviously there is $L > 0$ independent of individual $x \in \Omega$ such that

$$\left|(Tx)'(t)\right| \le L, \quad x \in \Omega,\ t \in [0,1] \tag{2.29}$$

which implies $T\Omega$ is equicontinuous. Then the complete continuity of $T : X \to X$ follows from the Arzela-Ascoli theorem. □

Now we define furthermore

$$T^* : X \longrightarrow X \tag{2.30}$$

by

$$(T^* x)(t) = w_{\lambda \widetilde{M}}(t) + \left(T\left(x - w_{\lambda \widetilde{M}}\right)\right)(t), \tag{2.31}$$

where $\widetilde{M}$ is an arbitrary constant.

From Lemma 2.8 the following result holds.

Lemma 2.9. *$T^* : X \to X$ is completely continuous.*

Obviously, $u(t) = x(t) - w_{\lambda \widetilde{M}}(t)$ is a solution to BVP (1.3) if and only if x is a fixed point of $T^* : X \to X$.

3. Main results

Let $\widetilde{M} = (l_2/l_1)M$, let $A = (1/l_1)\max_{0 \le c \le 1}[\int_0^c \int_s^c a(\tau) d\tau\, ds + \int_c^1 \int_c^s a(\tau) d\tau\, ds]$, let $B = (1/l_2)\min_{\delta \le c \le 1-\delta}[\int_\delta^c \int_s^c a(\tau) d\tau\, ds + \int_c^{1-\delta} \int_c^s a(\tau) d\tau\, ds]$, and let $d = (\widetilde{M}/l_1)[\int_0^\delta \int_s^1 a(\tau) d\tau\, ds + \int_{1-\delta}^1 \int_0^s a(\tau) d\tau\, ds]$, where $M > 0$ is a constant. Condition (H1) implies $A, B > 0$.

Let also $X = C([0,1],\mathbb{R})$ with the norm $\|\cdot\|$ defined by $\|u\| = \max_{0 \le t \le 1} |u(t)|$, and $K = \{u \in X : u(t) \ge 0$ is concave on $[0,1]\}$. Then K is a cone in Banach space X.

Suppose in addition

(H3) $f(t,u) \geq -M$ is continuous for $(t,u) \in [0,1] \times [w_\alpha(t), \infty)$, where α, $M > 0$ are two constants.

Then let

$$f^*(t,u) = \begin{cases} f(t,u), & u \geq w_\alpha(t), \\ f(t, w_\alpha(t)), & u < w_\alpha(t). \end{cases} \tag{3.1}$$

Clearly $f^*(t,u) \geq -M$ is continuous on $[0,1] \times \mathbb{R}$.

Define $T^* : K \to K$ as (2.31).

Lemma 3.1. $T^*(K) \subset K$.

Proof. For $y \in K$,

$$T^* y = w_{\lambda\widetilde{M}}(t) + T(y - w_{\lambda\widetilde{M}})(t), \tag{3.2}$$

where $w_{\lambda\widetilde{M}}$ and $T(y - w_{\lambda\widetilde{M}})$ satisfy, respectively, (2.1) and (2.14). Applying Lemma 2.4 we get from $f^*(t, (y - w_{\lambda\widetilde{M}})(t)) \geq -M \geq -(l_2/l_1)M = -\widetilde{M}$ that

$$T(y - w_{\lambda\widetilde{M}})(t) \geq w_{-\lambda\widetilde{M}}(t) = -w_{\lambda\widetilde{M}}(t), \quad 0 \leq t \leq 1, \tag{3.3}$$

and hence

$$(T^* y)(t) \geq 0, \quad 0 \leq t \leq 1. \tag{3.4}$$

At the same time, for $t_1, t_2 \in [0,1]$, $t_1 < t_2$,

$$\begin{aligned} &(T^* y)'(t_2) - (T^* y)'(t_1) \\ &\quad = (T(y - w_{\lambda\widetilde{M}}))'(t_2) - (T(y - w_{\lambda\widetilde{M}}))'(t_1) + w'_{\lambda\widetilde{M}}(t_2) - w'_{\lambda\widetilde{M}}(t_1) \\ &\quad = \Phi^{-1}\left(\lambda \int_{t_2}^{\sigma} a(\tau) f^*(\tau, (y - w_{\lambda\widetilde{M}})(\tau))\, d\tau\right) \\ &\qquad - \Phi^{-1}\left(\lambda \int_{t_1}^{\sigma} a(\tau) f^*(\tau, (y - w_{\lambda\widetilde{M}})(\tau))\, d\tau\right) \\ &\qquad + (\Phi^{-1})\left(\lambda\widetilde{M} \int_{t_2}^{\sigma_0} a(\tau)\, d\tau\right) - (\Phi^{-1})\left(\lambda\widetilde{M} \int_{t_1}^{\sigma_0} a(\tau)\, d\tau\right). \end{aligned} \tag{3.5}$$

Since

$$(\Phi^{-1})\left(\lambda\widetilde{M} \int_{t_2}^{\sigma_0} a(\tau)\, d\tau\right) - (\Phi^{-1})\left(\lambda\widetilde{M} \int_{t_1}^{\sigma_0} a(\tau)\, d\tau\right) \leq -\frac{1}{l_2}\lambda\widetilde{M} \int_{t_1}^{t_2} a(\tau)\, d\tau \leq 0, \tag{3.6}$$

one has

$$(T^* y)'(t_2) - (T^* y)'(t_1) \leq 0 \tag{3.7}$$

if

$$\int_{t_1}^{t_2} a(\tau) f^*(\tau,(y-w_{\lambda\widetilde{M}})(\tau))\,d\tau \ge 0. \tag{3.8}$$

On the other hand, when

$$\int_{t_1}^{t_2} a(\tau) f^*(\tau,(y-w_{\lambda\widetilde{M}})(\tau))\,d\tau < 0, \tag{3.9}$$

it follows that

$$\begin{aligned}
&\Phi^{-1}\left(\lambda\int_{t_2}^{\sigma} a(\tau) f^*(\tau,(y-w_{\lambda\widetilde{M}})(\tau))\,d\tau\right)-\Phi^{-1}\left(\lambda\int_{t_1}^{\sigma} a(\tau) f^*(\tau,(y-w_{\lambda\widetilde{M}})(\tau))\,d\tau\right)\\
&\quad\le -\frac{1}{l_1}\lambda\int_{t_1}^{t_2} a(\tau) f^*(\tau,(y-w_{\lambda\widetilde{M}})(\tau))\,d\tau \le \frac{1}{l_1}\lambda M\int_{t_1}^{t_2} a(\tau)\,d\tau,
\end{aligned} \tag{3.10}$$

and then

$$(T^*y)'(t_2)-(T^*y)'(t_1) \le \frac{1}{l_1}\lambda M\int_{t_1}^{t_2} a(\tau)\,d\tau-\frac{1}{l_2}\lambda\widetilde{M}\int_{t_1}^{t_2} a(\tau)\,d\tau = 0. \tag{3.11}$$

Then $(T^*y)(t)$ is concave. So $T^*K\subset K$.

Lemma 2.9 and Lemma 3.1 imply that $T^*: K\to K$ is completely continuous. □

Theorem 3.2. *Suppose (H1), (H2), and (H3) hold and*

(H4) $f(t,w_\alpha(t))\ge \alpha a(t)$, $t\in(0,1)$,

(H5) *there are* $b>\widetilde{M}k_0$ *and* $c\in(2\widetilde{M}k_0,2b)$ *such that*

$$f(t,u) < \frac{c-2\widetilde{M}k_0}{A}, \quad (t,u)\in[0,1]\times[w_\alpha(t),b]. \tag{3.12}$$

Then BVP (1.3) has at least a positive solution $u=u(t)$ *with*

$$||u+w_{\lambda\widetilde{M}}|| < b, \quad u(t)\ge w_\alpha(t), \quad t\in[0,1], \tag{3.13}$$

if $\lambda\in[1,(2b/c)]$.

Proof. Take $K_b=\{x\in K:\|x\|<b\}$. Then $\overline{K_b}$ is a closed convex set in X. Each $y\in\partial K_b$,

$$\begin{aligned}
(T^*y)(t) &\le |w_{\lambda\widetilde{M}}(t)|+\int_0^t \Phi^{-1}\left(\lambda\int_s^{\sigma} a(\tau) f^*(\tau,(y-w_{\lambda\widetilde{M}})(\tau))\,d\tau\right)ds\\
&\le \lambda\widetilde{M}k_0+\int_0^t \Phi^{-1}\left(\lambda\int_s^{\sigma} a(\tau) f^*(\tau,(y-w_{\lambda\widetilde{M}})(\tau))\,d\tau\right)ds,
\end{aligned} \tag{3.14}$$

where σ is taken from $\Sigma_{y-w_{\lambda\widetilde{M}}}$ such that

$$T(y-w_{\lambda\widetilde{M}})(\sigma)=\max_{0\le t\le 1} T(y-w_{\lambda\widetilde{M}})(t). \tag{3.15}$$

It follows from (2.27) that

$$(T^* y)(t) \leq \lambda \widetilde{M} k_0 + \frac{\lambda(c-2\widetilde{M}k_0)}{A}\frac{1}{l_1}\int_0^{\sigma}\int_s^{\sigma} a(\tau)d\tau\, ds,$$
$$(T^* y)(t) \leq \lambda \widetilde{M} k_0 + \frac{\lambda(c-2\widetilde{M}k_0)}{A}\frac{1}{l_1}\int_{\sigma}^{1}\int_{\sigma}^{s} a(\tau)d\tau\, ds, \tag{3.16}$$

then

$$\begin{aligned}(T^* y)(t) &\leq \lambda \widetilde{M} k_0 + \frac{\lambda(c-2\widetilde{M}k_0)}{2A}\frac{1}{l_1}\left[\int_0^{\sigma}\int_s^{\sigma} a(\tau)d\tau\, ds + \int_{\sigma}^{1}\int_{\sigma}^{s} a(\tau)d\tau\, ds\right] \\ &< \lambda \widetilde{M} k_0 + \frac{\lambda(c-2\widetilde{M}k_0)}{2} = \frac{\lambda c}{2}.\end{aligned} \tag{3.17}$$

When $\lambda \leq (2b/c)$, we have

$$||T^* y|| < b = \|y\|. \tag{3.18}$$

Hence T^* has a fixed point $y = y(t)$ in K_b. Obviously $u = y - w_{\lambda\widetilde{M}}$ is a solution to BVP (2.14). When $\lambda \geq 1$, one has $\lambda f^*(t,x) \geq \alpha a(t)$, $(t,x) \in [0,1] \times (-\infty, w_\alpha(t)]$. And Lemma 2.4 implies $u(t) \geq w_\alpha(t)$, $0 \leq t \leq 1$. So $u(t)$ is also a solution to BVP (1.3). □

Corollary 3.3. *In Theorem 3.2 if (H5) is replaced by*
(H5)′ $\lim_{u\to+\infty}((f(t,u))/u) = 0$ *uniformly in* $t \in [0,1]$,
then BVP (1.3) has at least one solution $u = u(t)$ *with*

$$u(t) \geq w_\alpha(t), \quad \|u\| < \infty \tag{3.19}$$

when $\lambda \geq 1$.

Proof. For $\lambda \in [1,\infty)$, take $\varepsilon \in (0,(1/\lambda A))$. Then (H5)′ implies there is $b > (2\widetilde{M}k_0/\varepsilon A)$ such that

$$\frac{f(t,u)}{b} < \varepsilon \quad \text{for } (t,u) \in [0,1] \times [w_\alpha(t), b], \tag{3.20}$$

that is,

$$f(t,u) < \varepsilon b = \frac{c-2\widetilde{M}k_0}{A} \quad \text{for } (t,u) \in [0,1] \times [w_\alpha(t), b], \tag{3.21}$$

where $c = 2\widetilde{M}k_0 + \varepsilon bA$. Applying Theorem 3.2, we see that BVP (1.3) has a positive solution $u(t) \geq w_\alpha(t)$ since $\lambda \leq (1/\varepsilon A) = (2b/2b\varepsilon A) < (2b/(2\widetilde{M}k_0 + b\varepsilon A)) = (2b/c)$. □

Theorem 3.4. *Suppose (H1), (H2), and (H3) hold and in addition*
(H6) *there are* $b > 2k_0 \max\{\alpha, \widetilde{M}\}$ *and* $c \in (4\widetilde{M}k_0, 2b)$ *such that*

$$f(t,u) < \frac{c-2\widetilde{M}k_0}{A}, \quad (t,u) \in [0,1] \times [w_\alpha(t), b], \tag{3.22}$$

(H7) *there are $a > \delta a > b$ and $r > (ac/b)$ such that*

$$f(t,u) > \frac{r+d}{B}, \quad (t,u) \in [0,1] \times \left[\delta a - \frac{2b}{c}\widetilde{M}k_0, a\right]. \tag{3.23}$$

Then BVP (1.3) has a solution $u = v(t)$ with

$$v(t) > w_\alpha(t), \qquad b < \|v + w_{\lambda\widetilde{M}}\| < a \tag{3.24}$$

when $\lambda \in [(2a/r),(2b/c)]$.

Proof. It can be shown as in the proof of Theorem 3.2 that for $y \in \partial K_b$, we have

$$\|T^* y\| < \|y\|, \qquad \lambda \in \left(0, \frac{2b}{c}\right]. \tag{3.25}$$

For $y \in \partial K_a$, the concavity of y implies

$$y(t) \geq a\delta, \quad (y - w_{\lambda\widetilde{M}})(t) \geq a\delta - \frac{2b}{c}\widetilde{M}k_0,\ t \in [\delta, 1-\delta] \tag{3.26}$$

for $0 < \lambda \leq (2b/c)$. Take σ which satisfies (3.15).

(A) $\sigma \in [\delta, 1-\delta]$.

By use of expressions (2.27) and (2.31), we get

$$\begin{aligned}
\|T^* y\| &> T(y - w_{\lambda\widetilde{M}})(\sigma) \\
&= \int_0^\sigma \Phi^{-1}\left(\lambda \int_s^\sigma a(\tau) f^*(\tau,(y - w_{\lambda\widetilde{M}})(\tau))\,d\tau\right) ds \\
&\geq \int_\delta^\sigma \Phi^{-1}\left(\lambda \int_s^\sigma a(\tau) f^*(\tau,(y - w_{\lambda\widetilde{M}})(\tau))\,d\tau\right) ds \\
&\quad - \int_0^\delta \Phi^{-1}\left(\lambda\widetilde{M} \int_s^\sigma a(\tau)\,d\tau\right) ds \\
&> \frac{\lambda(r+d)}{l_2 B} \int_\delta^\sigma \int_s^\sigma a(\tau)\,d\tau\,ds - \frac{\lambda\widetilde{M}}{l_1} \int_0^\delta \int_s^\sigma a(\tau)\,d\tau\,ds, \\
\|T^* y\| &> \frac{\lambda(r+d)}{l_2 B} \int_\sigma^{1-\delta} \int_\sigma^s a(\tau)\,d\tau\,ds - \frac{\lambda\widetilde{M}}{l_1} \int_{1-\delta}^1 \int_\sigma^s a(\tau)\,d\tau\,ds.
\end{aligned} \tag{3.27}$$

It follows that

$$2\|T^* y\| > \lambda(r+d) - \lambda d = \lambda r, \tag{3.28}$$

$$\|T^* y\| > a = \|y\| \quad \text{for } \lambda \geq \frac{2a}{r}. \tag{3.29}$$

(B) $\sigma \in (0,\delta) \cup (1-\delta, 1)$.

Without loss of generality we suppose $\sigma \in (0,\delta)$. Then

$$\begin{aligned} \|T^* y\| &> \frac{\lambda(r+d)}{l_2 B}\int_\delta^{1-\delta}\int_\delta^s a(\tau)\,d\tau\,ds - \frac{\lambda\widetilde{M}}{l_1}\int_{1-\delta}^1\int_\sigma^s a(\tau)\,d\tau\,ds \\ &> \lambda(r+d) - \lambda d = \lambda r, \end{aligned} \tag{3.30}$$

$$\|T^* y\| > 2a > a \quad \text{for } \lambda \geq \frac{2a}{r}. \tag{3.31}$$

Expression (3.25), together with (3.29) or (3.31), implies T^* has a fixed point y, $b < \|y\| < a$, when $\lambda \in [(2a/r),(2b/c)]$. Then $v = y - w_{\lambda\widetilde{M}}$ is a positive solution to BVP (2.14) and

$$\begin{aligned} \|y\| &> b \geq \alpha k_0 + \frac{b}{2} \\ &= \alpha k_0 + \frac{2b}{4\widetilde{M}k_0}\widetilde{M}k_0 > \alpha k_0 + \frac{2b}{c}\widetilde{M}k_0 \\ &\geq \|w'_\alpha + w'_{\lambda\widetilde{M}}\|, \quad \lambda \in \left[\frac{2a}{r},\frac{2b}{c}\right]. \end{aligned} \tag{3.32}$$

Applying Lemma 2.6, we have

$$y(t) \geq w_\alpha(t) + w_{\lambda\widetilde{M}}(t), \quad \lambda \in \left[\frac{2a}{r},\frac{2b}{c}\right], \tag{3.33}$$

and then

$$v(t) = y(t) - w_{\lambda\widetilde{M}}(t) \geq w_\alpha(t), \quad \lambda \in \left[\frac{2a}{r},\frac{2b}{c}\right] \tag{3.34}$$

which implies $v(t)$ is also a positive solution to (1.3) with

$$v(t) \geq w_\alpha(t), \qquad b < \|v + w_{\lambda\widetilde{M}}\| < a. \tag{3.35}$$

□

Corollary 3.5. *In Theorem 3.4, if (H7) is replaced by*
(H7)′ $\lim_{u\to+\infty}(f(t,u)/u) = +\infty$ *uniformly in* $t \in [0,1]$,
then BVP (1.3) has at least a solution $u = v(t)$ *with*

$$v(t) \geq w_\alpha(t), \quad |v| < \infty, \qquad \|v + w_{\lambda\widetilde{M}}\| > b \tag{3.36}$$

when $\lambda \in (0,(2b/c)]$.

Proof. For each $\lambda \in (0,(2b/c)]$, take a $\varepsilon \in (0,\lambda)$ and a $N > (2/\varepsilon\delta B)$. Condition (H7)′ implies there is $a_0 > (2b\widetilde{M}k_0/\delta c)$ such that for each $a \geq a_0$,

$$f(t,u) \geq Nu \geq N\left(\delta a - \frac{2b}{c}\widetilde{M}k_0\right), \quad u \in \left[\delta a - \frac{2b}{c}\widetilde{M}k_0, a\right]. \tag{3.37}$$

Take $a > a_0$ large enough such that

$$N\left(\delta a - \frac{2b}{c}\widetilde{M}k_0\right) > \frac{(2a/\varepsilon)+d}{B}, \quad u \in \left[\delta a - \frac{2b}{c}\widetilde{M}k_0, a\right], \tag{3.38}$$

then Theorem 3.4 implies BVP (1.3) has a positive solution $v(t)$ when $\lambda \in [\varepsilon, (2b/c)]$, where $v(t) \geq w_\alpha(t)$, $b < \|v + w_{\lambda\widetilde{M}}\| < \infty$.

It is easy to show the following two theorems. □

Theorem 3.6. *Suppose (H1), (H2), (H3), (H4), (H5), and (H7)′ hold. Then BVP (1.3) has at least two positive solutions $u(t)$ and $v(t)$ when $\lambda \in [1, (2b/c)]$, where*

$$u(t), v(t) \geq w_\alpha(t), \quad \alpha \leq \|u + w_{\lambda\widetilde{M}}\| < b < \|v + w_{\lambda\widetilde{M}}\| < \infty. \tag{3.39}$$

Theorem 3.7. *Suppose (H1), (H2), (H3), (H5)′, (H6), and (H7) hold. Then BVP (1.3) has at least two positive solutions $v(t)$ and $u(t)$, when $\lambda \in [(2a/r), (2b/c)]$, where*

$$v(t), u(t) \geq w_\alpha(t), \qquad b < \|v + w_{\lambda\widetilde{M}}\| < a < \|u + w_{\lambda\widetilde{M}}\| < \infty. \tag{3.40}$$

Remark 3.8. Our theorems can be applied to case that f possesses singularity along a curve in $[0,1] \times \mathbb{R}^+$ since no restriction is imposed on f for $(t,u) \in [0,1] \times (0, w_\alpha(t)]$.

Example 3.9. Let $a(t) = \pi^2 \sin \pi t$, $f(t,x) = (4/(4x + 1 - 2\sin \pi t))$, and

$$\Phi(u) = \begin{cases} u, & |u| \leq \dfrac{3}{2}, \\ u(2 + \sin \pi |u|), & \dfrac{3}{2} < |u| < \dfrac{5}{2}, \\ 3u, & |u| \geq \dfrac{5}{2}. \end{cases} \tag{3.41}$$

Then $w_1(t) = \sin \pi t$ is the unique solution of

$$\begin{gathered} (\Phi(u'))' + \pi^2 \sin \pi t = 0, \qquad u(0) = 0 = u(1), \\ f(t, w_1(t)) = \frac{4}{1 + 2\sin \pi t} \geq \frac{4}{3} > 1 = \alpha. \end{gathered} \tag{3.42}$$

Clearly $\lim_{u\to\infty}(f(t,u)/u) = \lim_{u\to\infty}(4/(4u^2 + u - 2u\sin \pi t)) = 0$ uniformly. Applying Corollary 3.3, we conclude that

$$\begin{gathered} (\Phi(u'))' + \lambda \frac{4\pi^2 \sin \pi t}{4u + 1 - 2\sin \pi t} = 0, \\ u(0) = 0 = u(1) \end{gathered} \tag{3.43}$$

has at least a positive solution $u(t) > \sin\pi t$ when $\lambda > 1$. Since f is singular along with $u = (1/4)(2\sin\pi t - 1) > 1$, $(1/6) < t < (5/6)$, no previous result can be applied to obtain the above conclusion.

Example 3.10. Let a, Φ be the same as those in Example 3.9 and $f(t,x) = (x^2/432\pi^2) - (4/(4x+1-2\sin\pi t))$. Then $l_1 = 1$, $l_2 = 3$ and for $w_1(t) = \sin\pi t$ we have $k_0 = 2\pi$ and

$$f(t,x) > -4, \quad (t,x) \in [0,1] \times [\sin\pi t, \infty),$$
$$A = \max_{0\le x\le 1}\left[\int_0^x ds \int_s^x \pi^2 \sin\pi\tau\, d\tau + \int_x^1 ds \int_x^s \pi^2 \sin\pi\tau\, d\tau\right] = \pi. \tag{3.44}$$

Take $c = 5\widetilde{M}k_0 = 120\pi$, $b = 144\pi$. It follows that

$$\frac{c - 2\widetilde{M}k_0}{A} > 48,$$
$$f(t,x) < \frac{(144\pi)^2}{432\pi^2} = 48 < \frac{c - 2\widetilde{M}k_0}{A} \quad \text{for } (t,x) \in [0,1] \times [w_1(t), b]. \tag{3.45}$$

Based on Corollary 3.5, BVP

$$(\Phi(u'))' + \lambda\pi^2 \sin\pi t\left[\frac{u^2}{432\pi^2} - \frac{4}{4u+1-2\sin\pi t}\right] = 0,$$
$$u(0) = 0 = u(1) \tag{3.46}$$

has at least a positive solution $u(t)$, satisfying $u(t) > \sin\pi t$ for $t \in (0,1)$, when $\lambda \le (2b/c) = (12/5)$.

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