ON CERTAIN NEW GRONWALL-OU-IANG TYPE INTEGRAL INEQUALITIES IN TWO VARIABLES AND THEIR APPLICATIONS

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Some new Gronwall-Ou-Iang type integral inequalities in two independent variables are established. These integral inequalities can be applied as tools to the study of certain class of integral and differential equations. Some applications to a terminal value problem are also indicated.

1. Introduction

In his study of boundedness of solutions to linear second order differential equations, Ou-Iang [12] established and applied the following useful nonlinear integral inequality.

Theorem 1.1. *Let u and h be real-valued, nonnegative and continuous functions defined on* $\mathbb{R}_+ = [0,\infty)$ *and let* $c \geq 0$ *be a real constant. Then the nonlinear integral inequality*

$$u^2(x) \leq c^2 + 2\int_0^x h(s)u(s)ds, \quad x \in \mathbb{R}_+, \tag{1.1}$$

implies

$$u(x) \leq c + \int_0^x h(s)ds, \quad x \in \mathbb{R}_+. \tag{1.2}$$

As indicated by Pachpatte [15], this result has been frequently used by authors to obtain global existence, uniqueness and stability of solutions of various nonlinear integral and differential equations. On the other hand, Theorem 1.1 has also been extended and generalized by many authors; see, for example, the reference [2, 3, 6, 7, 8, 9, 13, 14, 15, 17, 18]. Like Gronwall type inequalities, (1.1) is also used to obtain *a priori* bounds to the unknown function. Therefore, integral inequalities of this type are usually known as Gronwall-Ou-Iang type inequalities.

In recent years, Pachpatte [16] discovered some new integral inequalities involving functions in two independent variables. These inequalities are applied to study the boundedness and uniqueness of the solutions of the following terminal value problem

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for the hyperbolic partial differential equation (1.3) with conditions (1.4)

$$D_1D_2u(x,y) = h(x,y,u(x,y)) + r(x,y), \tag{1.3}$$

$$u(x,\infty) = \sigma_\infty(x), \qquad u(\infty,y) = \tau_\infty(y), \qquad u(\infty,\infty) = k. \tag{1.4}$$

Recently, Cheung [2] and Dragomir-Kim [4, 5] established additional new Gronwall-Ou-Iang type integral inequalities involving functions of two independent variables, and Meng and Li [10] generalized the results of Pachpatte to certain new inequalities. Our main aim here, motivated by the works of Cheung, Dragomir-Kim and Meng-Li, is to establish some new and more general Gronwall-Ou-Iang type integral inequalities with two independent variables which are useful in the analysis of certain classes of partial differential equations.

2. Main results

In what follows, we define $\mathbb{R} = (-\infty,\infty)$, $\mathbb{R}_1 = [1,\infty)$, $\mathbb{R}_+ = [0,\infty)$, and for any $k \in \mathbb{N}$, $\mathbb{R}_+^k = (\mathbb{R}_+)^k$. Denote by $C^i(M,S)$ the class of all i-times continuously differentiable functions defined on set M with range in set S ($i = 1,2,\dots$) and $C^0(M,S) = C(M,S)$. The first-order partial derivatives of a function $z(x,y)$ for $x,y \in \mathbb{R}$ with respect to x and y are denoted as usual by $D_1z(x,y)$ and $D_2z(x,y)$, respectively. We also assume that all improper integrals appeared in the sequel are always convergent.

We need the following lemmas in the discussion of our main results.

Lemma 2.1 [11]. *Let $u(t)$, $k(t)$ be nonnegative continuous functions and $a(t)$, $b(t)$ be Riemann integrable functions on $J = [\alpha,\beta]$ with $a(t)$, $b(t)$ and $k(t)$ being nonnegative on J.*

(i) *If*

$$u(t) \le a(t) + b(t)\int_\alpha^t k(s)u(s)ds \tag{2.1}$$

for all $t \in J$, then

$$u(t) \le a(t) + b(t)\int_\alpha^t a(s)k(s)\exp\left(\int_s^t b(m)k(m)dm\right)ds, \quad t \in J. \tag{2.2}$$

(ii) *If*

$$u(t) \le a(t) + b(t)\int_t^\beta k(s)u(s)ds \tag{2.3}$$

for all $t \in J$, then

$$u(t) \le a(t) + b(t)\int_t^\beta a(s)k(s)\exp\left(\int_t^\beta b(m)k(m)dm\right)ds, \quad t \in J. \tag{2.4}$$

Lemma 2.2. *Let $u(x,y)$, $a(x,y)$, $c(x,y)$ and $d(x,y)$ be nonnegative continuous functions defined for $x,y \in \mathbb{R}_+$ and $w(u)$ be a nonnegative, nondecreasing continuous function for $u \in \mathbb{R}_+$ with $w(u) > 0$ for $u > 0$.*

(i) *Assume that* $a(x,y)$ *and* $c(x,y)$ *are nondecreasing in* x *and nonincreasing in* y *for* $x,y\in\mathbb{R}_+$. *If*

$$u(x,y)\le a(x,y)+c(x,y)\int_0^x\int_y^\infty d(s,t)w(u(s,t))\,dt\,ds \tag{2.5}$$

for all $x,y\in\mathbb{R}_+$, *then*

$$u(x,y)\le G^{-1}\left[G(a(x,y))+c(x,y)\int_0^x\int_y^\infty d(s,t)dt\,ds\right] \tag{2.6}$$

for all $0\le x\le x_1$, $y_1\le y<\infty$, *where*

$$G(r):=\int_{r_0}^r\frac{dr}{w(r)},\qquad r\ge r_0>0, \tag{2.7}$$

G^{-1} *is the inverse function of* G, *and* $x_1,y_1\in\mathbb{R}_+$ *are chosen so that*

$$G(a(x,y))+c(x,y)\int_0^x\int_y^\infty d(s,t)dt\,ds\in\operatorname{Dom}(G^{-1}). \tag{2.8}$$

(ii) *Assume that* $a(x,y)$ *and* $c(x,y)$ *are nonincreasing in each variable* $x,y\in\mathbb{R}_+$. *If*

$$u(x,y)\le a(x,y)+c(x,y)\int_x^\infty\int_y^\infty d(s,t)w(u(s,t))\,dt\,ds \tag{2.9}$$

for all $x,y\in\mathbb{R}_+$, *then*

$$u(x,y)\le G^{-1}\left[G(a(x,y))+c(x,y)\int_x^\infty\int_y^\infty d(s,t)dt\,ds\right] \tag{2.10}$$

for all $0\le x\le x_2$, $y_2\le y<\infty$, *where* G *and* G^{-1} *are defined as in* (i), *and* $x_2,y_2\in\mathbb{R}_+$ *are chosen so that*

$$G(a(x,y))+c(x,y)\int_x^\infty\int_y^\infty d(s,t)dt\,ds\in\operatorname{Dom}(G^{-1}). \tag{2.11}$$

Proof. (i) Fixing any numbers $\overline{x}_1$ and $\overline{y}_1$ with $0<\overline{x}_1\le x_1$ and $y_1\le\overline{y}_1<\infty$, from (2.5) we have

$$u(x,y)\le a(\overline{x}_1,\overline{y}_1)+c(\overline{x}_1,\overline{y}_1)\int_0^x\int_y^\infty d(s,t)w(u(s,t))\,dt\,ds \tag{2.12}$$

for $0\le x\le\overline{x}_1$, $\overline{y}_1\le y<\infty$.

Defining $r_1(x,y)$ as the right-hand side of the last inequality, then $r_1(0,y)=r_1(x,\infty)=a(\overline{x}_1,\overline{y}_1)$,

$$u(x,y)\le r_1(x,y), \tag{2.13}$$

$r_1(x,y)$ is non-increasing in $y \in [\overline{y}_1, \infty)$, and

$$D_1 r_1(x,y) = c(\overline{x}_1,\overline{y}_1)\int_y^\infty d(x,t)w(u(x,t))dt \le c(\overline{x}_1,\overline{y}_1)\int_y^\infty d(x,t)w(r_1(x,t))dt \\ \le c(\overline{x}_1,\overline{y}_1)w(r_1(x,y))\int_y^\infty d(x,t)dt. \tag{2.14}$$

Dividing both sides of (2.14) by $w(r(x,y))$, we obtain

$$\frac{D_1 r_1(x,y)}{w(r_1(x,y))} \le c(\overline{x}_1,\overline{y}_1)\int_y^\infty d(x,t)dt. \tag{2.15}$$

From (2.7) and (2.15) we have

$$D_1 G(r_1(x,y)) \le c(\overline{x}_1,\overline{y}_1)\int_y^\infty d(x,t)dt. \tag{2.16}$$

Now setting $x = s$ in (2.16) and then integrating with respect to s from 0 to x, we obtain

$$G(r_1(x,y)) \le G(r_1(0,y)) + c(\overline{x}_1,\overline{y}_1)\int_0^x\int_y^\infty d(s,t)dt\,ds. \tag{2.17}$$

Noting $G(r_1(0,y)) = G(a(\overline{x}_1,\overline{y}_1))$, we have

$$G(r_1(x,y)) \le G(a(\overline{x}_1,\overline{y}_1)) + c(\overline{x}_1,\overline{y}_1)\int_0^x\int_y^\infty d(s,t)dt\,ds. \tag{2.18}$$

Taking $x = \overline{x}_1$, $y = \overline{y}_1$ in (2.13) and the last inequality, we obtain

$$u(\overline{x}_1,\overline{y}_1) \le r_1(\overline{x}_1,\overline{y}_1), \\ G(r_1(\overline{x}_1,\overline{y}_1)) \le G(a(\overline{x}_1,\overline{y}_1)) + c(\overline{x}_1,\overline{y}_1)\int_0^{\overline{x}_1}\int_{\overline{y}_1}^\infty d(s,t)dt\,ds. \tag{2.19}$$

Since $0 < \overline{x}_1 \le x_1$, $y_1 \le \overline{y}_1 < \infty$ are arbitrary, from (2.19) we have

$$u(x,y) \le r_1(x,y), \tag{2.20}$$

$$G(r_1(x,y)) \le G(a(x,y)) + c(x,y)\int_0^x\int_y^\infty d(s,t)dt\,ds, \tag{2.21}$$

or

$$r_1(x,y) \le G^{-1}\left[G(a(x,y)) + c(x,y)\int_0^x\int_y^\infty d(s,t)dt\,ds\right] \tag{2.22}$$

for all $0 < x \le x_1$, $y_1 \le y < \infty$. Hence by (2.20) and (2.22) we have

$$u(x,y) \le G^{-1}\left[G(a(x,y)) + c(x,y)\int_0^x\int_y^\infty d(s,t)dt\,ds\right] \tag{2.23}$$

for all $0 < x \le x_1$, $y_1 \le y < \infty$. By (2.5), (2.23) holds also when $x = 0$.

(ii) The proof of (ii) is similar to the argument in the proof of Lemma 2.2(i) with suitable modification. We omit the details here. □

THEOREM 2.3. *Let $a(x,y)$, $c(x,y)$, and $w(u)$ be defined as in Lemma 2.2*(i), *and $e(x,y) \in C(\mathbb{R}_+^2,\mathbb{R}_+)$. Let $\varphi(u) \in C^1(\mathbb{R}_+,\mathbb{R}_+)$ with $\varphi'(u) > 0$ for $u > 0$, here φ' denotes the derivative of φ. If*

$$\varphi(u(x,y)) \leq a(x,y) + c(x,y)\int_0^x\int_y^\infty \varphi'(u(s,t))[d(s,t)w(u(s,t)) + e(s,t)]dt\,ds \tag{2.24}$$

for all $x, y \in \mathbb{R}_+$, then

$$u(x,y) \leq G^{-1}\left\{G[\varphi^{-1}(a(x,y)) + E(x,y)] + c(x,y)\int_0^x\int_y^\infty d(s,t)dt\,ds\right\} \tag{2.25}$$

for all $0 \leq x \leq x_3$, $y_3 \leq y < \infty$, where

$$E(x,y) := c(x,y)\int_0^x\int_y^\infty e(s,t)dt\,ds, \tag{2.26}$$

G and G^{-1} are defined as in Lemma 2.2, φ^{-1} is the inverse function of φ, and $x_3, y_3 \in \mathbb{R}_+$ are chosen so that

$$G[\varphi^{-1}(a(x,y)) + E(x,y)] + c(x,y)\int_0^x\int_y^\infty d(s,t)dt\,ds \in \mathrm{Dom}\,(G^{-1}). \tag{2.27}$$

Proof. If $a(x,y) > 0$, fixing any numbers $\overline{x}_3$ and $\overline{y}_3$ $(0 < \overline{x}_3 \leq x_3, y_3 \leq \overline{y}_3 < \infty)$, from (2.24) we have

$$\varphi(u(x,y)) \leq a(\overline{x}_3,\overline{y}_3) + c(\overline{x}_3,\overline{y}_3)\int_0^x\int_y^\infty \varphi'(u(s,t))\Big[d(s,t)w(u(s,t)) + e(s,t)\Big]dt\,ds \tag{2.28}$$

for all $0 \leq x \leq \overline{x}_3$, $\overline{y}_3 \leq y < \infty$. Defining $r_2(x,y)$ as the right-hand side of the last inequality, then

$$r_2(0,y) = r_2(x,\infty) = a(\overline{x}_3,\overline{y}_3), \tag{2.29}$$

$$u(x,y) \leq \varphi^{-1}(r_2(x,y)) \tag{2.30}$$

for all $0 \leq x \leq \overline{x}_3$, $\overline{y}_3 \leq y < \infty$. Since $r_2(x,y)$ is non-increasing in y, by (2.30), we have

$$\begin{aligned} D_1r_2(x,y) &= c(\overline{x}_3,\overline{y}_3)\int_y^\infty \varphi'(u(x,t))[d(x,t)w(u(x,t)) + e(x,t)]dt \\ &\leq c(\overline{x}_3,\overline{y}_3)\int_y^\infty \varphi'(\varphi^{-1}(r_2(x,t)))[d(x,t)w(\varphi^{-1}(r_2(x,t))) + e(x,t)]dt \\ &\leq c(\overline{x}_3,\overline{y}_3)\varphi'(\varphi^{-1}(r_2(x,y)))\int_y^\infty [d(x,t)w(\varphi^{-1}(r_2(x,t))) + e(x,t)]dt. \end{aligned} \tag{2.31}$$

Dividing both sides of (2.31) by $\varphi'(\varphi^{-1}(r_2(x,y)))$, we have

$$\frac{D_1 r_2(x,y)}{\varphi'(\varphi^{-1}(r_2(x,y)))} \leq c(\overline{x}_3,\overline{y}_3)\int_y^\infty [d(x,t)w(\varphi^{-1}(r_2(x,t)))+e(x,t)]dt. \tag{2.32}$$

Observe that for any continuously differentiable and invertible function $\Phi(\xi)$, by the change of variable $\eta=\Phi^{-1}(\xi)$, we have

$$\int \frac{d\xi}{\Phi'(\Phi^{-1}(\xi))} = \int \frac{\Phi'(\eta)}{\Phi'(\eta)}d\eta = \eta + c = \Phi^{-1}(\xi)+c. \tag{2.33}$$

Keeping y fixed in (2.32), setting $x=s$ and integrating with respect to s from 0 to x, and applying (2.33) to the left-hand side we obtain

$$\begin{aligned}\varphi^{-1}(r_2(x,y)) &\leq \varphi^{-1}(r_2(0,y)) + c(\overline{x}_3,\overline{y}_3)\int_0^x\int_y^\infty [d(s,t)w(\varphi^{-1}(r_2(s,t)))+e(s,t)]dt\,ds \\ &= \varphi^{-1}(a(\overline{x}_3,\overline{y}_3)) + c(\overline{x}_3,\overline{y}_3)\int_0^x\int_y^\infty [d(s,t)w(\varphi^{-1}(r_2(s,t)))+e(s,t)]dt\,ds.\end{aligned} \tag{2.34}$$

Applying Lemma 2.2(i) to the last inequality, we get

$$\begin{aligned}\varphi^{-1}(r_2(x,y)) \leq G^{-1}\Big\{G\Big[\varphi^{-1}(a(\overline{x}_3,\overline{y}_3)) + c(\overline{x}_3,\overline{y}_3)\int_0^x\int_y^\infty e(s,t)dt\,ds\Big] \\ + c(\overline{x}_3,\overline{y}_3)\int_0^x\int_y^\infty d(s,t)dt\,ds\Big\}\end{aligned} \tag{2.35}$$

for all $0\leq x\leq \overline{x}_3$, $\overline{y}_3\leq y<\infty$. By (2.30), (2.35) and using similar procedures as from (2.19) to (2.23) in the proof of Lemma 2.2(i), we can get the desired bound of $u(x,y)$ in (2.25). By continuity, (2.25) also holds for the case $a(x,y)\geq 0$. □

Theorem 2.4. *Let $a(x,y)$, $c(x,y)$, $w(u)$ be defined as in Lemma 2.2*(ii) *and $\varphi(u)$, $e(x,y)$ defined as in Theorem 2.3. If*

$$\varphi(u(x,y)) \leq a(x,y) + c(x,y)\int_x^\infty\int_y^\infty \varphi'(u(s,t))[d(s,t)w(u(s,t))+e(s,t)]dt\,ds \tag{2.36}$$

for all $x,y\in\mathbb{R}_+$, then

$$u(x,y) \leq G^{-1}\Big\{G[\varphi^{-1}(a(x,y))+\overline{E}(x,y)] + c(x,y)\int_x^\infty\int_y^\infty d(s,t)dt\,ds\Big\} \tag{2.37}$$

for all $x_4\leq x<\infty$, $y_4\leq y<\infty$, where

$$\overline{E}(x,y) := c(x,y)\int_x^\infty\int_y^\infty e(s,t)dt\,ds, \tag{2.38}$$

G and G^{-1} are defined as in Lemma 2.2, φ and φ^{-1} are defined as in Theorem 2.3, and $x_4, y_4 \in \mathbb{R}_+$ are chosen so that

$$G[\varphi^{-1}(a(x,y)) + \overline{E}(x,y)] + c(x,y)\int_x^\infty \int_y^\infty d(s,t)\,dt\,ds \in \operatorname{Dom}(G^{-1}). \tag{2.39}$$

The proof of Theorem 2.4 follows by an argument similar to that in the proof of Theorem 2.3 with suitable modification. We omit the details here.

Theorem 2.5. *Let $a(x,y)$, $c(x,y)$, $e(x,y)$, $w(u)$, $\varphi(u)$, and $\varphi'(u)$ be defined as in Theorem 2.3. Let $b(x,y)$, $d(x,y)$, and $f(x,y) \in C(\mathbb{R}_+^2, \mathbb{R}_+)$ and $b(x,y)$, $d(x,y)$ be nondecreasing in x and non-increasing in y. If*

$$\begin{aligned}\varphi(u(x,y)) \le a(x,y) + b(x,y)\int_\alpha^x c(s,y)\varphi(u(s,y))\,ds \\ + d(x,y)\int_0^x \int_y^\infty \varphi'(u(s,t))[f(s,t)w(u(s,t)) + e(s,t)]\,dt\,ds\end{aligned} \tag{2.40}$$

for all $x, y, \alpha \in \mathbb{R}_+$ with $\alpha \le x$, then

$$u(x,y) \le G^{-1}\Big\{G[\varphi^{-1}(p(x,y)a(x,y)) + p(x,y)E_1(x,y)] + p(x,y)d(x,y)\int_0^x \int_y^\infty f(s,t)\,dt\,ds\Big\} \tag{2.41}$$

for all $0 \le x \le x_5$, $y_5 \le y < \infty$, where

$$p(x,y) := 1 + b(x,y)\int_\alpha^x c(s,y)\exp\Big(\int_s^x b(m,y)c(m,y)\,dm\Big)ds, \tag{2.42}$$

$$E_1(x,y) := d(x,y)\int_0^x \int_y^\infty e(s,t)\,dt\,ds, \tag{2.43}$$

G and G^{-1} are defined as in Lemma 2.2, φ and φ^{-1} are defined as in Theorem 2.3, and $x_5, y_5 \in \mathbb{R}_+$ are chosen so that

$$G[\varphi^{-1}(p(x,y)a(x,y)) + p(x,y)E_1(x,y)] + p(x,y)d(x,y)\int_0^x \int_y^\infty f(s,t)\,dt\,ds \in \operatorname{Dom}(G^{-1}). \tag{2.44}$$

Proof. Define a function $z(x,y)$ by

$$z(x,y) = a(x,y) + d(x,y)\int_0^x \int_y^\infty \varphi'(u(s,t))[f(s,t)w(u(s,t)) + e(s,t)]\,dt\,ds. \tag{2.45}$$

Then (2.40) can be restated as

$$\varphi(u(x,y)) \le z(x,y) + b(x,y)\int_\alpha^x c(s,y)\varphi(u(s,y))\,ds. \tag{2.46}$$

Obviously, $z(x,y)$ is nonnegative and continuous in $x \in \mathbb{R}_+$. Fixing $y \in \mathbb{R}_+$ in (2.46) and using Lemma 2.1(i) we get

$$\varphi(u(x,y)) \le z(x,y) + b(x,y)\int_\alpha^x z(s,y)c(s,y)\exp\left(\int_s^x b(m,y)c(m,y)\,dm\right)ds. \tag{2.47}$$

Since $z(x,y)$ is nondecreasing in $x \in \mathbb{R}_+$, we obtain from the last inequality that

$$\varphi(u(x,y)) \le z(x,y)p(x,y), \tag{2.48}$$

where $p(x,y)$ is defined by (2.42). From (2.48), we have

$$\varphi(u(x,y)) \le p(x,y)\left(a(x,y) + d(x,y)\int_0^x\int_y^\infty \varphi'(u(s,t))\left[f(s,t)w(u(s,t)) + e(s,t)\right]dt\,ds\right). \tag{2.49}$$

Observe that $p(x,y)$, $a(x,y)$, and $d(x,y)$ are continuous, nondecreasing in x and nonincreasing in y for $x,y \in \mathbb{R}_+$, so also are $p(x,y)a(x,y)$ and $p(x,y)d(x,y)$. Now applying Theorem 2.3 to (2.49), we get the desired bound for $u(x,y)$ appeared in (2.41) directly. □

Theorem 2.6. *Let $u(x,y)$, $f(x,y)$, $e(x,y)$, $\varphi(u)$, and $w(u)$ be defined as in Theorem 2.5. Let $a(x,y)$, $b(x,y)$, $c(x,y)$, and $d(x,y)$ be nonnegative continuous and nonincreasing in each variable $x,y \in \mathbb{R}_+$. If*

$$\begin{aligned}\varphi(u(x,y)) \le{}& a(x,y) + b(x,y)\int_x^\beta c(s,y)\varphi(u(s,y))\,ds \\ &+ d(x,y)\int_x^\infty\int_y^\infty \varphi'(u(s,t))\left[f(s,t)w(u(s,t)) + e(s,t)\right]dt\,ds\end{aligned} \tag{2.50}$$

for all $x,y,\beta \in \mathbb{R}_+$ with $x \le \beta$, then

$$u(x,y) \le G^{-1}\left\{G\left[\varphi^{-1}(\overline{p}(x,y)a(x,y)) + \overline{p}(x,y)\overline{E}_1(x,y)\right] + \overline{p}(x,y)d(x,y)\int_x^\infty\int_y^\infty f(s,t)\,dt\,ds\right\} \tag{2.51}$$

for all $x_6 \le x < \infty$, $y_6 \le y < \infty$, where

$$\overline{p}(x,y) := 1 + b(x,y)\int_x^\beta c(s,y)\exp\left(\int_x^s b(m,y)c(m,y)\,dm\right)ds, \tag{2.52}$$

$$\overline{E}_1(x,y) := d(x,y)\int_x^\infty\int_y^\infty e(s,t)\,dt\,ds, \tag{2.53}$$

G, G^{-1}, φ and φ^{-1} are defined as in Theorem 2.5, and $x_6, y_6 \in \mathbb{R}_+$ are chosen so that

$$G\left[\varphi^{-1}(\overline{p}(x,y)a(x,y)) + \overline{p}(x,y)\overline{E}_1(x,y)\right] + \overline{p}(x,y)d(x,y)\int_x^\infty\int_y^\infty f(s,t)\,dt\,ds \in \operatorname{Dom}(G^{-1}). \tag{2.54}$$

The proof of Theorem 2.6 follows by an argument similar to that in the proof of Theorem 2.5 with suitable modification. We omit the details here.

Remark 1. By choosing suitable functions for φ, some interesting new Gronwall-Ou-Iang type inequalities of two variables can be obtained from Theorems 2.5 and 2.6. For example, the following interesting inequalities are easily obtained.

Corollary 2.7. *Let $u(x,y)$, $a(x,y)$, $b(x,y)$, $c(x,y)$, $d(x,y)$, $e(x,y)$, $f(x,y)$, and $w(u)$ be as defined in Theorem 2.5. Let $k\geq 1$ be a real number. If*

$$\begin{aligned} u^k(x,y) \leq{}& a(x,y)+b(x,y)\int_\alpha^x c(s,y)u^k(s,y)ds \\ &+d(x,y)\int_0^x\int_y^\infty u^{k-1}(s,t)\big[f(s,t)w\big(u(s,t)\big)+e(s,t)\big]dt\,ds \end{aligned} \tag{2.55}$$

for all $x,y,\alpha\in\mathbb{R}_+$ with $\alpha\leq x$, then

$$\begin{aligned} u(x,y) \leq G^{-1}\Big\{&G\Big[p^{1/k}(x,y)a^{1/k}(x,y)+\frac{1}{k}p(x,y)E_1(x,y)\Big] \\ &+\frac{1}{k}p(x,y)d(x,y)\int_0^x\int_y^\infty f(s,t)dt\,ds\Big\} \end{aligned} \tag{2.56}$$

for all $0\leq x\leq x_7$, $y_7\leq y<\infty$, where G, G^{-1}, $p(x,y)$ and $E_1(x,y)$ are as defined in Theorem 2.5, and $x_7,y_7\in\mathbb{R}_+$ are chosen so that

$$G\big[p^{1/k}(x,y)a^{1/k}(x,y)+p(x,y)E_1(x,y)\big]+p(x,y)d(x,y)\int_0^x\int_y^\infty f(s,t)dt\,ds\in \operatorname{Dom}\big(G^{-1}\big). \tag{2.57}$$

Proof. This follows immediately from Theorem 2.5 by setting $\varphi(u)=u^k$. □

Corollary 2.8. *Let $b(x,y)$, $c(x,y)$, $d(x,y)$, $e(x,y)$, $f(x,y)$, and $w(u)$ be as defined in Theorem 2.5. Let $u(x,y),a(x,y)\in C(\mathbb{R}_+,\mathbb{R}_1)$ and $k>0$ be a real number. If*

$$\begin{aligned} u^k(x,y) \leq{}& a(x,y)+b(x,y)\int_\alpha^x c(s,y)u^k(s,y)ds \\ &+d(x,y)\int_0^x\int_y^\infty u^{k}(s,t)\big[f(s,t)w\big(\log u(s,t)\big)+e(s,t)\big]dt\,ds \end{aligned} \tag{2.58}$$

for all $x,y,\alpha\in\mathbb{R}^+$ with $\alpha\leq x$, then

$$\begin{aligned} u(x,y) \leq \exp\Big\{G^{-1}\Big[&G\Big(\frac{1}{k}\log\big(p(x,y)a(x,y)\big)+\frac{1}{k}p(x,y)E_1(x,y)\Big) \\ &+\frac{1}{k}p(x,y)d(x,y)\int_0^x\int_y^\infty f(s,t)dt\,ds\Big]\Big\} \end{aligned} \tag{2.59}$$

for all $0 \le x \le x_8$, $y_8 \le y < \infty$, *where* G, G^{-1}, $p(x,y)$ *and* $E_1(x,y)$ *are as defined in Theorem 2.5, and* $x_8, y_8 \in \mathbb{R}_+$ *are chosen so that*

$$G\Big(\frac{1}{k}\log\big(p(x,y)a(x,y)\big) + \frac{1}{k}p(x,y)E_1(x,y)\Big) + \frac{1}{k}p(x,y)d(x,y)\int_0^x\int_y^\infty f(s,t)\,dt\,ds \in \operatorname{Dom}\big(G^{-1}\big). \tag{2.60}$$

Proof. Using the change of variable $v(x,y) = \log u(x,y)$, inequality (2.58) reduces to

$$\begin{aligned} e^{kv(x,y)} \le{}& a(x,y) + b(x,y)\int_\alpha^x c(s,y)e^{kv(s,y)}\,ds \\ &+ d(x,y)\int_0^x\int_y^\infty e^{kv(s,t)}\big[f(s,t)w\big(v(s,t)\big) + e(s,t)\big]\,dt\,ds, \end{aligned} \tag{2.61}$$

which is a special case of inequality (2.40) when $\varphi(v) = \exp(kv)$. By Theorem 2.5, the desired inequality (2.59) follows. □

Theorem 2.9. *Let* $u(x,y)$, $a(x,y)$, $b(x,y)$, $c(x,y)$, $d(x,y)$, $e(x,y)$, $f(x,y)$, *and* $\varphi(u)$ *be as defined in Theorem 2.5, and* $L, M \in C(\mathbb{R}_+^3, \mathbb{R}_+)$ *satisfy*

$$0 \le L(x,y,v) - L(x,y,w) \le M(x,y,w)(v-w) \tag{2.62}$$

for all $x, y, v, w \in \mathbb{R}_+$ *with* $v \le w$. *If*

$$\begin{aligned} \varphi\big(u(x,y)\big) \le{}& a(x,y) + b(x,y)\int_\alpha^x c(s,y)\varphi\big(u(s,y)\big)\,ds \\ &+ d(x,y)\int_0^x\int_y^\infty \varphi'\big(u(s,t)\big)\big[f(s,t)L\big(s,t,u(s,t)\big) + e(s,t)\big]\,dt\,ds \end{aligned} \tag{2.63}$$

for all $x, y, \alpha \in \mathbb{R}_+$ *with* $\alpha \le x$, *then*

$$u(x,y) \le \mathcal{N}_1(x,y) + p(x,y)d(x,y)\mathcal{L}_1(x,y)\exp\big(\mathcal{M}_1(x,y)\big) \tag{2.64}$$

for all $x, y \in \mathbb{R}_+$, *where*

$$\begin{aligned} \mathcal{N}_1(x,y) &:= \varphi^{-1}\big(p(x,y)a(x,y)\big) + p(x,y)E_1(x,y), \\ \mathcal{L}_1(x,y) &:= \int_0^x\int_y^\infty f(s,t)L\big[s,t,\mathcal{N}_1(s,t)\big]\,dt\,ds, \\ \mathcal{M}_1(x,y) &:= \int_0^x\int_y^\infty f(s,t)p(s,t)d(s,t)M\big[s,t,\mathcal{N}_1(s,t)\big]\,dt\,ds, \end{aligned} \tag{2.65}$$

and $p(x,y)$, $E_1(x,y)$ *are defined in (2.42), (2.43), respectively.*

Proof. By similar arguments as those used in the proof of Theorem 2.5, applying Lemma 2.1(i) to (2.63) we get

$$\begin{aligned}\varphi(u(x,y)) &\le p(x,y)a(x,y)\\ &\quad + p(x,y)d(x,y)\int_0^x\int_y^\infty \varphi'(u(s,t))[f(s,t)L(s,t,u(s,t))+e(s,t)]\,dt\,ds\end{aligned} \tag{2.66}$$

for all $x, y \in \mathbb{R}_+$.

Defining a nonnegative continuous function $z(x,y)$ as the right-hand side of (2.66), then using similar procedures as in the proof of Theorem 2.3, we can derive from (2.66) that

$$\begin{aligned} u(x,y) &\le \varphi^{-1}(z(x,y)),\\ \varphi^{-1}(z(x,y)) &\le \mathcal{N}_1(x,y) + p(x,y)d(x,y)\int_0^x\int_y^\infty f(s,t)L[s,t,\varphi^{-1}(z(s,t))]\,dt\,ds\end{aligned} \tag{2.67}$$

for all $x, y \in \mathbb{R}_+$, where $\mathcal{N}_1(x,y)$ is defined in (2.65).

Setting

$$\xi(x,y) = \int_0^x\int_y^\infty f(s,t)L[s,t,\varphi^{-1}(z(s,t))]\,dt\,ds, \tag{2.68}$$

then from the last inequality we have

$$\varphi^{-1}(z(x,y)) \le \mathcal{N}_1(x,y) + p(x,y)d(x,y)\xi(x,y) \tag{2.69}$$

for all $x, y \in \mathbb{R}_+$. Since $L(x,y,v)$ is nondecreasing with respect to v for fixed x, y, by (2.68) and (2.69) with condition (2.62), we obtain

$$\begin{aligned}\xi(x,y) &\le \int_0^x\int_y^\infty f(s,t)L[s,t,\mathcal{N}_1(s,t)+p(s,t)d(s,t)\xi(s,t)]\,dt\,ds\\ &\le \int_0^x\int_y^\infty f(s,t)L[s,t,\mathcal{N}_1(s,t)]\,dt\,ds\\ &\quad + \int_0^x\int_y^\infty f(s,t)p(s,t)d(s,t)M[s,t,\mathcal{N}_1(s,t)]\xi(s,t)\,dt\,ds.\end{aligned} \tag{2.70}$$

Applying Lemma 2.2(i) (the case when $w(u) = u$, $c(x,y) \equiv 1$) to the last inequality we obtain

$$\begin{aligned}\xi(x,y) &\le \left(\int_0^x\int_y^\infty f(s,t)L[s,t,\mathcal{N}_1(s,t)]\,dt\,ds\right)\cdot\\ &\exp\left(\int_0^x\int_y^\infty f(s,t)p(s,t)d(s,t)M[s,t,\mathcal{N}_1(s,t)]\,dt\,ds\right)\\ &= \mathcal{L}_1(x,y)\exp(\mathcal{M}_1(x,y)),\end{aligned} \tag{2.71}$$

where $\mathcal{L}_1(x,y)$ and $\mathcal{M}_1(x,y)$ are defined in (2.65). The required inequality (2.64) now follows from (2.67), (2.69) and the last inequality. □

Theorem 2.10. *Let $u(x,y)$, $a(x,y)$, $b(x,y)$, $c(x,y)$, $d(x,y)$, $f(x,y)$, and $\varphi(u)$ be as defined in Theorem 2.6, and $L(x,y,v)$ and $M(x,y,v)$ as defined in Theorem 2.9. If*

$$\begin{aligned}\varphi(u(x,y)) \leq a(x,y) + b(x,y)\int_x^\beta c(s,y)\varphi(u(s,y))\,ds \\ + d(x,y)\int_x^\infty\int_y^\infty \varphi'(u(s,t))\big[f(s,t)L(s,t,u(s,t)) + e(s,t)\big]dt\,ds\end{aligned} \tag{2.72}$$

for all $\beta, x, y \in \mathbb{R}_+$ with $x \leq \beta$, then

$$u(x,y) \leq \overline{\mathcal{N}}_1(x,y) + \overline{p}(x,y)d(x,y)\overline{\mathcal{L}_1}(x,y)\exp\left(\overline{\mathcal{M}_1}(x,y)\right) \tag{2.73}$$

for all $x, y \in \mathbb{R}_+$, where

$$\begin{aligned}\overline{\mathcal{N}}_1(x,y) &:= \varphi^{-1}\left(\overline{p}(x,y)a(x,y)\right) + \overline{p}(x,y)\overline{E}_1(x,y), \\ \overline{\mathcal{L}}_1(x,y) &:= \int_x^\infty\int_y^\infty f(s,t)L\big[s,t,\overline{\mathcal{N}}_1(s,t)\big]dt\,ds, \\ \overline{\mathcal{M}}_1(x,y) &:= \int_x^\infty\int_y^\infty f(s,t)\overline{p}(s,t)d(s,t)M\big[s,t,\overline{\mathcal{N}}_1(s,t)\big]dt\,ds,\end{aligned} \tag{2.74}$$

and $\overline{p}(x,y)$, $\overline{E}_1(x,y)$ are defined in (2.52), (2.53), respectively.

The proof of Theorem 2.10 follows by an argument similar to that in the proof of Theorem 2.9 with suitable modification. We omit the details here.

Remark 2. As in Corollaries 2.7 and 2.8, other new Gronwall-Ou-Iang type integral inequalities of two variables can be obtained from Theorems 2.9 and 2.10 by choosing suitable functions for φ. Details are omitted here.

3. Applications

(a) Consider the partial differential equation

$$D_1D_2u^\ell(x,y) = h_1(x,y,u(x,y)) + r(x,y), \tag{3.1}$$

$$u^\ell(x,\infty) = \sigma_\infty(x), \qquad u^\ell(0,y) = \tau(y), \qquad u^\ell(0,\infty) = k, \tag{3.2}$$

where $h_1 \in C(\mathbb{R}_+^2 \times \mathbb{R}, \mathbb{R})$, $r \in C(\mathbb{R}_+^2, \mathbb{R})$, $\sigma_\infty, \tau \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\ell \geq 1$ and k are real constants.

Assume that

$$\begin{aligned}\left|h_1(x,y,u)\right| &\leq |u|^{\ell-1}\left(d(x,y)w(|u|) + e(x,y)\right), \\ \left|\sigma_\infty(x) + \tau(y) - k\right| &\leq a(x,y),\end{aligned} \tag{3.3}$$

where $a(x,y)$, $d(x,y)$, $e(x,y)$ and $w(u)$ are defined as in Theorem 2.3. If $u(x,y)$ is a solution of (3.1) with condition (3.2), then it can be written as (see [1, page 80]):

$$u^\ell(x,y) = \sigma_\infty(x) + \tau(y) - k - \int_0^x\int_y^\infty r(s,t)\,dt\,ds - \int_0^x\int_y^\infty h_1(s,t,u(s,t))\,dt\,ds \tag{3.4}$$

for all $x, y \in \mathbb{R}_+$. Applying (3.3) to (3.4), we get

$$\begin{aligned} |u(x,y)|^{\ell} \le a(x,y) &+ \int_0^x \int_y^\infty |r(s,t)|\,dt\,ds \\ &+ \int_0^x \int_y^\infty |u(s,t)|^{\ell-1}\big[d(s,t)w(|u(s,t)|)+e(s,t)\big]dt\,ds \end{aligned} \tag{3.5}$$

for all $x, y \in \mathbb{R}_+$. An application of Theorem 2.3 to (3.5) yields

$$u(x,y) \le G^{-1}\left\{G\left[\left(a(x,y)+\int_0^x\int_y^\infty r(s,t)dt\,ds\right)^{1/\ell}+E_\ell(x,y)\right]+\frac{1}{\ell}\int_0^x\int_y^\infty d(s,t)dt\,ds\right\} \tag{3.6}$$

for all $0 \le x \le \tilde{x}_1$, $\tilde{y}_1 \le y < \infty$, where

$$E_\ell(x,y) = \frac{1}{\ell}\int_0^x\int_y^\infty e(s,t)dt\,ds, \tag{3.7}$$

G and G^{-1} are defined as in Theorem 2.3, and $\tilde{x}_1, \tilde{y}_1 \in \mathbb{R}_+$ are chosen so that the quantity inside the curly brackets in (3.6) is in the range of G.

(b) Consider the partial differential equation

$$D_1D_2u^\ell(x,y) = h_2(x,y,u(x,y),\log u(x,y)) + D_2g(x,y,u(x,y)), \tag{3.8}$$

$$u^\ell(x,\infty) = \sigma_\infty(x), \qquad u^\ell(0,y) = \tau(y), \qquad u^\ell(0,\infty) = k, \tag{3.9}$$

where $h_2 \in C(\mathbb{R}_+^3 \times \mathbb{R},\mathbb{R})$, $g \in C(\mathbb{R}_+^3,\mathbb{R})$, $\sigma_\infty, \tau \in C(\mathbb{R}_+,\mathbb{R}_+)$, $\ell, k > 0$ are constants. Assume that

$$\begin{aligned} |h_2(x,y,u,\log u)| &\le u^\ell\big[f(x,y)w(|\log u|)+e(x,y)\big], \\ |g(x,y,u)| &\le c(x,y)u^\ell, \\ \left|\sigma_\infty(x)+\tau(y)-k-\int_0^x g(s,\infty,\sigma_\infty(s))ds\right| &\le a(x,y) \end{aligned} \tag{3.10}$$

for all $x, y \in \mathbb{R}_+$ and all $u > 0$, where $a(x,y)$, $c(x,y)$, $e(x,y)$, $f(x,y)$, and $w(u)$ are as defined in Corollary 2.8. If $u(x,y) \in C(\mathbb{R}_+,\mathbb{R}_1)$ is a solution of (3.8) with condition (3.9), then it can be written as (see [1, page 80]):

$$\begin{aligned} u^\ell(x,y) = \sigma_\infty(x)+\tau(y)-k &- \int_0^x g(s,\infty,\sigma_\infty(s))ds \\ &+ \int_0^x g(s,y,u(s,y))ds - \int_0^x\int_y^\infty h_2(s,t,u(s,t),\log u(s,t))dt\,ds \end{aligned} \tag{3.11}$$

for all $x, y \in \mathbb{R}_+$.

Applying (3.10) to (3.11), we obtain

$$\begin{aligned} u^{\ell}(x,y) \le a(x,y) &+ \int_0^x c(s,y)u^{\ell}(s,y)\,ds \\ &+ \int_0^x \int_y^{\infty} u^{\ell}(s,t)\big[f(s,t)w\big(\log u(s,t)\big) + e(s,t)\big]\,dt\,ds \end{aligned} \tag{3.12}$$

for all $x, y \in \mathbb{R}_+$. An application of Corollary 2.8 to (3.12) yields

$$\begin{aligned} u(x,y) \le \exp\Big(G^{-1}\Big\{G\Big[\frac{1}{\ell}\log\big(p^*(x,y)a(x,y)\big) + E_{\ell}(x,y)\Big] \\ + \frac{1}{\ell}p^*(x,y)\int_0^x\int_y^{\infty} f(s,t)\,dt\,ds\Big\}\Big) \end{aligned} \tag{3.13}$$

for all $0 \le x \le \tilde{x}_2$, $\tilde{y}_2 \le y < \infty$, where

$$p^*(x,y) = 1 + \int_0^x c(s,y)\exp\Big(\int_s^x c(m,y)\,dm\Big)ds, \tag{3.14}$$

G, G^{-1} are as defined in Corollary 2.8, $E_{\ell}(x,y)$ is as defined in Application (a) above, and $\tilde{x}_2, \tilde{y}_2 \in \mathbb{R}_+$ are chosen so that the quantity inside the curly brackets in (3.13) is in the range of G.

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