ON SOME BOUNDARY VALUE PROBLEMS FOR A CLASS OF HYPERBOLIC SYSTEMS OF SECOND ORDER IN CONIC DOMAINS

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For one class of hyperbolic systems of second order, we consider multidimensional versions of the Darboux problem in conic domains. A priori estimates of solutions of these problems are obtained. The existence of a solution of the Darboux problems is proved under the supplementary conditions imposed on the coefficients of the system, when the data support of the problem is of temporary type.

1. Statement of the problem

In the Euclidean space $\mathbb{R}^{n+1}$ of the variables $x = (x_1,\dots,x_n)$ and t we consider a system of linear differential equations of the kind

$$Lu = u_{tt} - \sum_{i,j=1}^{n} A_{ij}u_{x_ix_j} + \sum_{i=1}^{n} B_i u_{x_i} + Cu = F, \tag{1.1}$$

where A_{ij} ($A_{ij} = A_{ji}$), B_i, and C are given real $(m\times m)$-matrices, F is a given and u is an unknown m-dimensional real vector, $n \geq 2$, $m > 1$.

Below, the matrices A_{ij} will be assumed to be symmetric and constant, and for any m-dimensional real vectors η_i, $i = 1,\dots,n$, we have the inequality

$$\sum_{i,j=1}^{n} A_{ij}\eta_i\eta_j \geq c_0 \sum_{i=1}^{n} \left|\eta_i\right|^2, \quad c_0 = \text{const} > 0. \tag{1.2}$$

It can be easily verified that the system (1.1) by virtue of the condition (1.2) is hyperbolic.

Let D be the conic domain $\{(x,t) \in \mathbb{R}^{n+1} : |x|g(x/|x|) < t < +\infty\}$ lying in the half-space $t > 0$, and bounded by the conic manifold $S = \{(x,t) \in \mathbb{R}^{n+1} : t = |x|g(x/|x|)\}$, where g is an entirely definite, positive, continuous, piecewise smooth function given on the unit sphere of the space $\mathbb{R}^n$. For $\tau > 0$, by $D_\tau := \{(x,t) \in \mathbb{R}^{n+1} : |x|g(x/|x|) < t < \tau\}$ we denote the domain lying in the half-space $t > 0$, bounded by the cone S and the hyperplane $t = \tau$.

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Let $S_0 = \partial D_{\tau_0} \cap S$ be the conic portion of the boundary of D_{τ_0} for some $\tau_0 > 0$. Suppose that $S_1,\dots,S_{k_1},S_{k_1+1},\dots,S_{k_1+k_2}$ are nonintersecting smooth conic open hypersurfaces, where $S_1,\dots,S_{k_1}$ are the characteristic manifolds of the system (1.1), and $S_0 = \bigcup_{i=1}^{k_1+k_2} \overline{S}_i$, where $\overline{S}_i$ is the closure of S_i.

Consider the following boundary value problem: find in the domain D_{τ_0} a solution $u(x,t)$ of the system (1.1) satisfying the conditions

$$u|_{S_0} = f_0, \tag{1.3}$$

$$\Gamma^i u_t|_{S_i} = f_i, \quad i = 1,\dots,k_1+k_2, \tag{1.4}$$

where f_i, $i = 0,1,\dots,k_1+k_2$, are given real $\varkappa_i$-dimensional vectors, Γ^i, $i = 1,\dots,k_1+k_2$, are given constant $(\varkappa_i \times m)$-matrices with $\varkappa_0 = m$, $0 \le \varkappa_i \le m$, $i = 1,\dots,k_1+k_2$. Here, the number $\varkappa_i$, $1 \le i \le m$, shows to what extent the part S_i of the boundary ∂D_{τ_0} is occupied; in particular, $\varkappa_i = 0$ denotes that the corresponding part S_i in the boundary condition (1.4) is completely free from the boundary conditions. Below we will see that for the problem (1.1), (1.3), (1.4) to be correct, we must choose the number $\varkappa_i$ in a well-defined way, depending on the geometric properties of the hypersurface S_i.

It will be assumed that the elements of the matrices B_i and C in the system (1.1) are bounded, measurable functions in the domain D_{τ_0}, and the right-hand side of that system $F \in L_2(D_{\tau_0})$.

Note that a particular case of the problem (1.1), (1.3), (1.4) is the Cauchy characteristic problem (or the Goursat problem with data support on a characteristic conoid) [7, 9, 18, 24] and also multidimensional analogues of the first and the second Darboux problems [1, 2, 5, 13, 14, 15, 21, 22, 23, 25]. In the case of a second-order hyperbolic system with the same principal part the question on the unique solvability of Goursat problem with data on a characteristic conoid has been investigated in [6]. In [3, 4] we can find general statement of characteristic problems for second-order hyperbolic systems, as well as examples of systems for which the corresponding homogeneous characteristic problem has nontrivial solutions (a finite set of linearly independent solutions in one cases and an infinite set of these solutions in other cases). The works [11, 12] are worth noticing in which the problem (1.1), (1.3) is considered for the case when the conic hypersurface S_0 is of temporary type. The same problem in a dihedral angle of temporary type has been considered in [16].

2. The methods of selecting the numbers $\varkappa_i$ and matrices Γ^i in the boundary conditions (1.4), depending on geometric properties of S_i

By virtue of the condition (1.2), the symmetric matrix $Q(\xi') = \sum_{i,j=1}^{n} A_{ij}\xi_i\xi_j$, $\xi' = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n \setminus \{(0,\dots,0)\}$ is positive definite. Therefore there exists an orthogonal matrix $T = T(\xi')$ such that the matrix $T^{-1}(\xi')Q(\xi')T(\xi')$ is diagonal, and its elements $\mu_1,\dots,\mu_m$ on the diagonal are positive, that is, $\mu_i = \widetilde{\lambda}_i^2(\xi') > 0$, $\widetilde{\lambda}_i > 0$, $i = 1,\dots,m$. Note that without restriction of generality we may assume that $\widetilde{\lambda}_m(\xi') \ge \cdots \ge \widetilde{\lambda}_1(\xi') > 0$ $\forall \xi' \in \mathbb{R}^n \setminus \{(0,\dots,0)\}$. Below it will be assumed that the multiplicities $\ell_1,\dots,\ell_s$ of these values do not depend on ξ', and we put

$$\begin{gathered}\lambda_1(\xi') = \widetilde{\lambda}_1(\xi') = \cdots = \widetilde{\lambda}_{\ell_1}(\xi') < \lambda_2(\xi') = \widetilde{\lambda}_{\ell_1+1}(\xi') = \cdots = \widetilde{\lambda}_{\ell_1+\ell_2}(\xi') \\ < \lambda_s(\xi') = \widetilde{\lambda}_{m-\ell_s+1}(\xi') = \cdots = \widetilde{\lambda}_m(\xi'), \quad \xi' \in \mathbb{R}^n \setminus \{(0,\dots,0)\}.\end{gathered} \tag{2.1}$$

Note that according to (2.1) and owing to the continuous dependence of roots of the characteristic polynomial of a symmetric matrix on its elements, $\lambda_1(\xi'),\dots,\lambda_s(\xi')$ are continuous homogeneous functions of degree 1 [10, page 434].

It is easily seen that the roots with respect to ξ_{n+1} of the characteristic polynomial $\det(E\xi_{n+1}^2 - Q(\xi'))$ of the system (1.1) are the numbers $\xi_{n+1} = \pm\lambda_i(\xi_1,\dots,\xi_n)$, $i = 1,\dots,s$, with the multiplicities $k_1,\dots,k_s$, where E is the unit $(m \times m)$-matrix. Therefore the cone of normals

$$K = \{\xi = (\xi_1,\dots,\xi_n,\xi_{n+1}) \in \mathbb{R}^{n+1} : \det\left(E\xi_{n+1}^2 - Q(\xi')\right) = 0\} \tag{2.2}$$

of the system (1.1) consists of its separate connected components

$$K_i^{\pm} = \{\xi \in (\xi',\xi_{n+1}) \in \mathbb{R}^{n+1} : \xi_{n+1} \mp \lambda_i(\xi') = 0\}, \quad i = 1,\dots,s. \tag{2.3}$$

Denote by $D_i^- = \{\xi = (\xi',\xi_{n+1}) \in \mathbb{R}^{n+1} : \xi_{n+1} + \lambda_i(\xi') < 0\}$ the conic domain whose boundary is the conic hypersurface K_i^-, $i = 1,\dots,s$. By (2.1) we have $D_1^- \supset D_2^- \supset \cdots \supset D_s^-$. Let $G_i = D_{i-1}^- \setminus \overline{D_i^-}$ for $1 < i \le s$, and $G_1 = \mathbb{R}_-^{n+1} \setminus \overline{D_1^-}$, $\mathbb{R}_-^{n+1} = \{\xi \in \mathbb{R}^{n+1} : \xi_{n+1} < 0\}$, while $G_{s+1} = D_s^-$.

Since for the unit vector of outer normal $\alpha = (\alpha_1,\dots,\alpha_n,\alpha_{n+1})$ at the points of the cone S different from its vertex $O(0,\dots,0)$, we have

$$\alpha_i = \frac{\partial g_0/\partial x_i}{\sqrt{1 + |\nabla_x g_0|^2}}, \quad i = 1,\dots,n, \qquad \alpha_{n+1} = \frac{-1}{\sqrt{1 + |\nabla_x g_0|^2}} \tag{2.4}$$

with $\nabla_x = (\partial/\partial x_1,\dots,\partial/\partial x_n)$, $g_0(x) = |x|g(x/|x|)$, it holds

$$\alpha_{n+1}|_{S\setminus O} < 0. \tag{2.5}$$

According to our supposition, the smooth conic hypersurface S_i for $1 \le i \le k_1$ is a characteristic one. Therefore, taking into account that $S_i \subset S_0 \subset S$ and the inequality (2.5) is fulfilled, for some index m_i, $1 \le m_i \le s$, we have

$$\alpha|_{S_i} \in K_{m_i}^-, \quad i = 1,\dots,k_1. \tag{2.6}$$

Since the hypersurface S_i for $k_1+1 \le i \le k_1+k_2$ at none of its points is characteristic, by virtue of $S_i \subset S_0 \subset S$ and (2.5) as well as by definition of the domains G_j, there exists an index n_i, $1 \le n_i \le s+1$, such that

$$\alpha|_{S_i} \in G_{n_i}, \quad i = k_1+1,\dots,k_1+k_2. \tag{2.7}$$

Without restriction of generality we assume that $m_1 \le \cdots \le m_{k_1}$ and $n_{k_1+1} \le \cdots \le n_{k_1+k_2}$.

By $Q_0(\xi) = E\xi_{n+1}^2 - Q(\xi')$ we denote the characteristic matrix of the system (1.1) and consider the problem on reduction of the quadratic form $(Q_0(\xi)\eta,\eta)$ to the canonic form, when $\xi = \alpha$ is the unit vector of the normal to the hypersurface S_i, $i = 1 \le i \le k_1+k_2$, outer with respect to the domain D_{τ_0}. Here $\eta \in \mathbb{R}^m$ and $(\cdot,\cdot)$ denotes the scalar product in the Euclidean space $\mathbb{R}^m$.

As far as

$$\begin{aligned} &T^{-1}(\alpha')Q(\alpha)T(\alpha') \\ &\quad = \operatorname{diag}\Big(\underbrace{\lambda_1^2(\alpha'),\dots,\lambda_1^2(\alpha')}_{\ell_1},\dots,\underbrace{\lambda_s^2(\alpha'),\dots,\lambda_s^2(\alpha')}_{\ell_s}\Big), \quad \alpha' = (\alpha_1,\dots,\alpha_n), \end{aligned} \tag{2.8}$$

for $\eta = T\zeta$, we have

$$\begin{aligned} (Q_0(\alpha)\eta,\eta) &= ((T^{-1}Q_0T)(\alpha)\zeta,\zeta) = \Big(\big(E\alpha_{n+1}^2 - (T^{-1}QT)(\alpha')\big)\zeta,\zeta\Big) \\ &= \big(\alpha_{n+1}^2 - \lambda_1^2(\alpha')\big)\zeta_1^2 + \cdots + \big(\alpha_{n+1}^2 - \lambda_1^2(\alpha')\big)\zeta_{\ell_1}^2 \\ &\quad + \big(\alpha_{n+1}^2 - \lambda_2^2(\alpha')\big)\zeta_{\ell_1+1}^2 + \cdots + \big(\alpha_{n+1}^2 - \lambda_2^2(\alpha')\big)\zeta_{\ell_1+\ell_2}^2 \\ &\quad + \cdots + \big(\alpha_{n+1}^2 - \lambda_s^2(\alpha')\big)\zeta_{m-\ell_s+1}^2 + \cdots + \big(\alpha_{n+1}^2 - \lambda_s^2(\alpha')\big)\zeta_m^2. \end{aligned} \tag{2.9}$$

For $1 \le i \le k_1$, that is, in (2.6), since $\alpha_{n+1}^2 - \lambda_{m_i}^2(\alpha') = 0$, by virtue of (2.1), we have

$$\begin{gathered} \big[\alpha_{n+1}^2 - \lambda_j^2(\alpha')\big]\big|_{K_{m_i}^-} > 0, \quad j = 1,\dots,m_i-1; \qquad \big[\alpha_{n+1}^2 - \lambda_{m_i}^2(\alpha')\big]\big|_{K_{m_i}^-} = 0, \\ \big[\alpha_{n+1}^2 - \lambda_j^2(\alpha')\big]\big|_{K_{m_i}^-} < 0, \quad j = m_i+1,\dots,s. \end{gathered} \tag{2.10}$$

If $k_1+1 \le i \le k_1+k_2$, that is, in (2.7), by the definition of the domain G_{n_i} it follows from (2.1) that for $n_i \le s$

$$\begin{aligned} &\big[\alpha_{n+1}^2 - \lambda_j^2(\alpha')\big]\big|_{G_{n_i}} > 0, \quad j = 1,\dots,n_i-1, \\ &\big[\alpha_{n+1}^2 - \lambda_j^2(\alpha')\big]\big|_{G_{n_i}} < 0, \quad j = n_i,\dots,s, \text{ and for } n_i = s+1, \\ &\big[\alpha_{n+1}^2 - \lambda_j^2(\alpha')\big]\big|_{G_{n_i}} > 0, \quad j = 1,\dots,s. \end{aligned} \tag{2.11}$$

Denote by $\varkappa_i^+$ and $\varkappa_i^-$ the positive and negative indices of inertia of the quadratic form $(Q_0(\alpha)\eta,\eta)$ for $\alpha \in K_{m_i}^-$ when $1 \le i \le k_1$, and for $\alpha \in G_{n_i}$ when $k_1+1 \le i \le k_1+k_2$. For $1 \le i \le k_1$, by (2.9) and (2.10), we have

$$\varkappa_i^+ = \ell_1 + \cdots + \ell_{m_i-1}, \qquad \varkappa_i^- = \ell_{m_i+1} + \cdots + \ell_s, \qquad (\text{def})_{m_i} = \ell_{m_i}, \tag{2.12}$$

where $(\text{def})_{m_i}$ is the defect of that form, and $\varkappa_i^+ = 0$ for $m_i = 1$. When $k_1+1 \le i \le k_1+k_2$, by virtue of (2.9) and (2.11), we have

$$\varkappa_i^+ = \ell_1 + \cdots + \ell_{n_i-1}, \qquad \varkappa_i^- = \ell_{n_i} + \cdots + \ell_s, \tag{2.13}$$

and $\varkappa_i^+ = 0$ for $n_i = 1$.

If now $\zeta = C^i\eta$ is any nondegenerated linear transformation reducing the quadratic form $(Q_0(\alpha)\eta,\eta)$ in case (2.12) and (2.13) to the canonic form, then owing to the invariance of indices of inertia of a quadratic form with respect to nondegenerated linear transformations, we have

$$(Q_0(\alpha)\eta,\eta) = \sum_{j=1}^{\varkappa_i^+} \left[\Lambda_{ij}^+(\alpha,\eta)\right]^2 - \sum_{j=1}^{\varkappa_i^-} \left[\Lambda_{ij}^-(\alpha,\eta)\right]^2, \quad 1 \le i \le k_1+k_2. \tag{2.14}$$

Here

$$\begin{gathered}
\Lambda_{ij}^+(\alpha,\eta) = \sum_{p=1}^{m} c_{jp}^i(\alpha)\eta_p, \qquad \Lambda_{ij}^-(\alpha,\eta) = \sum_{p=1}^{m} c_{\varkappa_i^+ + j,p}^i(\alpha)\eta_p, \\
C^i = C^i(\alpha) = \left(c_{jp}^i(\alpha)\right)_{j,p=1}^m, \quad 1 \le i \le k_1+k_2.
\end{gathered} \tag{2.15}$$

According to (2.15), in the boundary conditions (1.4) we take as the matrix Γ^i the matrix of the order $(\varkappa_i \times m)$, where $\varkappa_i = \varkappa_i^+$, $1 \le i \le k_1+k_2$, whose elements Γ_{jp}^i are given by the equality

$$\Gamma_{jp}^i = c_{jp}^i(\alpha), \quad j = 1,\ldots,\varkappa_i^+;\ p = 1,\ldots,m, \tag{2.16}$$

where $\alpha \in K_{m_i}^-$ for $1 \le i \le k_1$, and $\alpha \in G_{n_i}$ for $k_1+1 \le i \le k_1+k_2$.

Below it will be assumed that in the boundary conditions (1.4) the elements Γ_{jp}^i of matrices Γ^i on S_i are the bounded measurable functions. It will also be assumed that the domain D_{τ_0} is a Lipschitz one [19, page 68].

3. Derivation of an a priori estimate for a solution of problem (1.1), (1.3), (1.4)

Below, if it will not cause misunderstanding, instead of $u=(u_1,\dots,u_m)\in[W_2^k(D_{\tau_0})]^m$ we will write simply $u\in W_2^k(D_{\tau_0})$. The condition $F=(F_1,\dots,F_m)\in L_2(D_{\tau_0})$ should be understood analogously. Let $u\in W_2^2(D_{\tau_0})$ be a solution of the problem (1.1), (1.3), (1.4). Multiplying both parts of the system of (1.1) scalarly by the vector $2u_t$ and integrating the obtained expression with respect to D_τ, $0<\tau\le\tau_0$, we obtain

$$\begin{aligned}
2\int_{D_\tau}&\left(F-\sum_{i=1}^n B_iu_{x_i}-Cu\right)u_t\,dx\,dt\\
&=\int_{D_\tau}\left[\frac{\partial(u_t,u_t)}{\partial t}+2\sum_{i,j=1}^n A_{ij}u_{x_j}u_{tx_i}\right]dx\,dt-2\int_{S_0\cap\{t\le\tau\}}\sum_{i,j=1}^n A_{ij}u_tu_{x_j}\alpha_i\,ds\\
&=\int_{\partial D_\tau\setminus S_0}\left(u_tu_t+\sum_{i,j=1}^n A_{ij}u_{x_i}u_{x_j}\right)dx\\
&\quad+\int_{S_0\cap\{t\le\tau\}}\left[\left(u_tu_t+\sum_{i,j=1}^n A_{ij}u_{x_i}u_{x_j}\right)\alpha_{n+1}-2\sum_{i,j=1}^n A_{ij}u_tu_{x_j}\alpha_i\right]ds\\
&=\int_{\partial D_\tau\setminus S_0}\left(u_tu_t+\sum_{i,j=1}^n A_{ij}u_{x_i}u_{x_j}\right)dx\\
&\quad+\int_{S_0\cap\{t\le\tau\}}\alpha_{n+1}^{-1}\Bigg[\sum_{i,j=1}^n A_{ij}(\alpha_{n+1}u_{x_i}-\alpha_iu_t)(\alpha_{n+1}u_{x_j}-\alpha_ju_t)\\
&\qquad\qquad+\left(E\alpha_{n+1}^2-\sum_{i,j=1}^n A_{ij}\alpha_i\alpha_j\right)u_tu_t\Bigg]ds\\
&=\int_{\partial D_\tau\setminus S_0}\left(u_tu_t+\sum_{i,j=1}^n A_{ij}u_{x_i}u_{x_j}\right)dx\\
&\quad+\int_{S_0\cap\{t\le\tau\}}\alpha_{n+1}^{-1}\left[\sum_{i,j=1}^n A_{ij}(\alpha_{n+1}u_{x_i}-\alpha_iu_t)(\alpha_{n+1}u_{x_j}-\alpha_ju_t)\right]ds\\
&\quad+\int_{S_0\cap\{t\le\tau\}}\alpha_{n+1}^{-1}(Q_0(\alpha)u_t,u_t)\,ds.
\end{aligned}\tag{3.1}$$

Since $(\alpha_{n+1}(\partial/\partial x_i)-\alpha_i(\partial/\partial t))$ is an inner differential operator on the conic hypersurface S_0, according to (1.3) and the boundedness of $|\alpha_{n+1}^{-1}|$ on S_0, we have

$$\begin{aligned}
&\left|\int_{S_0\cap\{t\le\tau\}}\alpha_{n+1}^{-1}\left[\sum_{i,j=1}^n A_{ij}(\alpha_{n+1}u_{x_i}-\alpha_iu_t)(\alpha_{n+1}u_{x_j}-\alpha_ju_t)\right]ds\right|\\
&\qquad\le c_1\|f_0\|^2_{W_0^1(S_0\cap\{t\le\tau\})},\quad c_1=\text{const}>0.
\end{aligned}\tag{3.2}$$

On the other hand, by virtue of (2.14), (2.15), (2.16), and (1.4), (2.5), we get

$$\begin{aligned}
&\int_{S_0\cap\{t\le\tau\}} \alpha_{n+1}^{-1}(Q_0(\alpha)u_t,u_t)\,ds \\
&\quad= -\sum_{i=1}^{k_1+k_2}\int_{S_i\cap\{t\le\tau\}}\left\{|\alpha_{n+1}^{-1}|\sum_{j=1}^{\varkappa_i^+}[\Lambda_{ij}^+(\alpha,u_t)]^2\right\}ds \\
&\qquad+\sum_{i=1}^{k_1+k_2}\int_{S_i\cap\{t\le\tau\}}\left\{|\alpha_{n+1}^{-1}|\sum_{j=1}^{\varkappa_i^-}[\Lambda_{ij}^-(\alpha,u_t)]^2\right\}ds \\
&\quad\ge -\sum_{i=1}^{k_1+k_2}\int_{S_i\cap\{t\le\tau\}}\left\{|\alpha_{n+1}^{-1}|\sum_{j=1}^{\varkappa_i^+}[\Lambda_{ij}^+(\alpha,u_t)]^2\right\}ds \\
&\quad\ge -c_2\sum_{i=1}^{k_1+k_2}\int_{S_i\cap\{t\le\tau\}}\left\{\sum_{j=1}^{\varkappa_i^+}[\Lambda_{ij}(\alpha,u_t)]^2\right\}ds = -c_2\sum_{i=1}^{k_1+k_2}\|f_i\|_{L_2(S_i\cap\{t\le\tau\})},
\end{aligned} \tag{3.3}$$

where $0 < c_2 = \sup_{S_0}|\alpha_{n+1}^{-1}| < +\infty$.

Suppose

$$w(\tau) = \int_{\partial D_\tau\setminus S_0}\left(u_t u_t + \sum_{i,j=1}^{n} A_{ij}u_{x_i}u_{x_j}\right)dx, \qquad \widetilde{u}_i = \alpha_{n+1}u_{x_i} - \alpha_i u_t. \tag{3.4}$$

Then by the boundedness and measurability of elements of the matrices B_i and C in the system (1.1), as well as by (3.1), (3.2), and (3.3), we obtain

$$\begin{aligned}
w(\tau) \le{}& c_3\int_0^\tau w(t)\,dt + c_4\int_{D_\tau} uu\,dx\,dt + c_5\|f_0\|^2_{W_2^1(S_0\cap\{t\le\tau\})} \\
&+ c_6\sum_{i=1}^{k_1+k_2}\|f_i\|^2_{L_2(S_i\cap\{t\le\tau\})} + c_7\|F\|^2_{L_2(D_\tau)}.
\end{aligned} \tag{3.5}$$

Here and in what follows, all the encountered values c_i, $i \ge 1$, are positive constants, independent of u.

Let (x,τ_x) be the point of intersection of the conic hypersurface S and the line parallel to the t-axis and passing through the point $(x,0)$. We have

$$u(x,\tau) = u(x,\tau_x) + \int_{\tau_x}^{\tau} u_t(x,t)\,dt, \quad \tau \ge \tau_x, \tag{3.6}$$

whence with regard for (1.3) we find that

$$\begin{aligned}
&\int_{\partial D_\tau\setminus S_0} u(x,\tau)u(x,\tau)\,dx \\
&\quad\le 2\int_{\partial D_\tau\setminus S_0} u(x,\tau_x)u(x,\tau_x)\,dx + 2|\tau-\tau_x|\int_{\partial D_\tau\setminus S_0} dx\int_{\tau_x}^{\tau} u_t(x,t)u_t(x,t)\,dt \\
&\quad\le c_8\int_{S_0\cap\{t\le\tau\}} uu\,ds + c_9\int_0^\tau w(t)\,dt = c_8\|f_0\|^2_{L_2(S_0\cap\{t\le\tau\})} + c_9\int_0^\tau w(t)\,dt.
\end{aligned} \tag{3.7}$$

Introduce the notation

$$w_0(\tau)=\int_{\partial D_\tau\setminus S_0}\left(uu+u_tu_t+\sum_{i,j=1}^{n}A_{ij}u_{x_i}u_{x_j}\right)dx. \tag{3.8}$$

Summing up the inequalities (3.5) and (3.7), we arrive at

$$w_0(\tau)\leq c_{10}\left[\int_0^\tau w_0(t)dt+\|f_0\|^2_{W_2^1(S_0\cap\{t\leq\tau\})}+\sum_{i=1}^{k_1+k_2}\|f_i\|^2_{L_2(S_i\cap\{t\leq\tau\})}+\|F\|^2_{L_2(D_\tau)}\right], \tag{3.9}$$

from which by Gronwall's lemma we find that

$$w_0(\tau)\leq c_{11}\left(\|f_0\|^2_{W_2^1(S_0\cap\{t\leq\tau\})}+\sum_{i=1}^{k_1+k_2}\|f_i\|^2_{L_2(S_i\cap\{t\leq\tau\})}+\|F\|^2_{L_2(D_\tau)}\right). \tag{3.10}$$

Integrating both parts of the inequality (3.10) with respect to τ, we can easily get the following a priori estimate for the solution $u\in W_2^2(D_{\tau_0})$ of the problem (1.1), (1.3), (1.4):

$$\|u\|_{W_2^1(D_{\tau_0})}\leq c\left(\|f_0\|_{W_2^1(S_0)}+\sum_{i=1}^{k_1+k_2}\|f_i\|_{L_2(S_i)}+\|F\|_{L_2(D_{\tau_0})}\right) \tag{3.11}$$

with a positive constant c independent of u.

Here we introduce the notion of a strong generalized solution of the problem (1.1), (1.3), (1.4) of the class W_2^1.

Definition 3.1. Let $f_0\in W_2^1(S_0)$, $f_i\in L_2(S_i)$, $i=1,\dots,k_1+k_2$, and $F\in L_2(D_{\tau_0})$. The vector function $u=(u_1,\dots,u_m)$ is said to be a strong generalized solution of the problem (1.1), (1.3), (1.4) of the class W_2^1 if $u\in W_2^1(D_{\tau_0})$ and there exists a sequence of vector functions $\{u_k\}_{k=1}^\infty$ from the space $W_2^2(D_{\tau_0})$ such that

$$\begin{gathered}\lim_{k\to\infty}\|u_k-u\|_{W_2^1(D_\tau)}=0,\qquad \lim_{k\to\infty}\|u_k|_{S_0}-f_0\|_{W_2^1(S_0)}=0,\\ \lim_{k\to\infty}\left\|\Gamma^i\frac{\partial u_k}{\partial t}\bigg|_{S_i}-f_i\right\|_{L_2(S_i)}=0,\quad i=1,\dots,k_1+k_2,\\ \lim_{k\to\infty}\|Lu_k-F\|_{L_2(D_{\tau_0})}=0.\end{gathered} \tag{3.12}$$

Below we will prove the existence of a strong generalized solution of the problem (1.1), (1.3), (1.4) of the class W_2^1 in case the conic hypersurface S_0 is of temporary type, that is, when the characteristic matrix of the system (1.1) is negative definite on $S_0\setminus 0$. The latter can be written as follows:

$$\left(\left[E\alpha_{n+1}^2-\sum_{i,j=1}^{n}A_{ij}\alpha_i\alpha_j\right]\eta,\eta\right)<0\quad\forall\eta\in\mathbb{R}^n\setminus\{(0,\dots,0)\}, \tag{3.13}$$

where the vector $\alpha=(\alpha_1,\dots,\alpha_n,\alpha_{n+1})$ is the outer unit normal to the cone S_0 at the points different from its vertex 0.

In the case of the inequality (3.13), by virtue of (2.12), (2.13), (2.14), (2.15), and (2.16) and by our choice of matrices Γ^i, $i = 1,\dots,k_1+k_2$, the numbers in (1.4) $\varkappa_i = 0$, $i = 1,\dots,k_1+k_2$, that is, the boundary conditions (1.4) in the problem (1.1), (1.3), (1.4) are missing, and the a priori estimate (3.11) of the solution $u \in W_2^2(D_\tau)$ of the problem (1.1), (1.3) takes the form

$$\|u\|_{W_2^1(D_{\tau_0})} \le c\Big(\|f_0\|_{W_2^1(S_0)} + \|F\|_{L_2(D_{\tau_0})}\Big). \tag{3.14}$$

Note that in [11] we have elucidated the geometric meaning of the condition (3.13), and for the solution $u \in W_2^2(D_\tau)$ of the problem (1.1), (1.3) in that case we have obtained the a priori estimate (3.14), although in the above-mentioned work we have not proved the existence of a strong generalized solution of the problem (1.1), (1.3) of the class W_2^1 whose uniqueness follows directly from the estimate (3.14).

4. Proof of the existence of a strong generalized solution of the problem (1.1), (1.3) of the class W_2^1

Let us consider the problem on the solvability of the above-mentioned problem, when the conic hypersurface S_0 is of temporal type. For the sake of simplicity we restrict ourselves to the case where the boundary condition (1.3) is homogeneous, that is,

$$u|_{S_0} = 0. \tag{4.1}$$

After the change of variables

$$y = \frac{x}{t}, \quad z = t \quad \text{or} \quad x = zy, \quad t = z \tag{4.2}$$

with respect to the unknown vector function $v(y,z) = u(zy,z)$, the system (1.1) takes the form

$$L_1 v = v_{zz} - \frac{1}{z^2}\sum_{i,j=1}^{n} \widetilde{A}_{ij} v_{y_i y_j} - \frac{2}{z}\sum_{i=1}^{n} y_i v_{z y_i} + \frac{1}{z}\sum_{i=1}^{n} \widetilde{B}_i v_{y_i} + \widetilde{C} v = \widetilde{F}. \tag{4.3}$$

Here

$$\widetilde{A}_{ij} = E y_i y_j + A_{ij}, \qquad \widetilde{B}_i = B_i(zy,z), \qquad \widetilde{C} = C(zy,z), \qquad \widetilde{F} = F(zy,z). \tag{4.4}$$

By G we denote an n-dimensional domain which is the intersection of the conic domain $D : t > |x| g(x/|x|)$ and the hyperplane $t = 1$ in which the variable x is replaced by y. Obviously, $\partial G = \{y \in \mathbb{R}^n : 1 = |y| g(y/|y|)\}$. Under the transformation $(x,t) \to (y,z)$ in accordance with (4.2), the domain D_τ transforms into the cylindrical domain $\Omega_\tau = G \times (0,\tau) = \{(y,z) \in \mathbb{R}^{n+1} : y \in G,\ z \in (0,\tau)\}$ in the space of the variables y, z. Denote by $\Gamma_\tau = \partial G \times [0,\tau]$ the lateral surface of the cylinder Ω_τ. The boundary condition (4.1) with respect to the vector function v takes the form

$$v|_{\Gamma_{\tau_0}} = 0. \tag{4.5}$$

We divide the exposition of the proof of existence a strong generalized solution of problem (1.1), (1.3) of the class W_2^1 into several items.

(1^0) In this item we will derive an a priori estimate for the solution $v = (v_1,\dots,v_m)$ of the problem (4.3), (4.5) from the space $W_2^2(\Omega_{\tau_0})$, which is equal to zero in the domain Ω_δ.

Let v be a solution of the problem (4.3), (4.5) from the space $W_2^2(\Omega_{\tau_0})$, such that for some positive δ

$$v|_{\Omega_\delta} = 0, \quad 0 < \delta < \tau. \tag{4.6}$$

Under the assumption that $(0,\dots,0) \in G$ and $\operatorname{diam} G$ is sufficiently small, by virtue of (1.2) and (4.4), for any m-dimensional real vectors η_i, $i = 1,\dots,n$, the inequality

$$\sum_{i,j=1}^{n} \widetilde{A}_{ij}(y)\eta_i\eta_j \geq \widetilde{c}_0 \sum_{i=1}^{n} |\eta_i|^2, \quad \widetilde{c}_0 = \text{const} > 0,\ \forall y \in G, \tag{4.7}$$

is valid.

If $\nu = (\nu_1,\dots,\nu_n,\nu_{n+1})$ is the unit vector of the outer normal to the boundary $\partial\Omega_{\tau_0}$ of the cylinder Ω_{τ_0} at the points (y,z), where it exists, then taking into account (4.6) we can easily see that

$$\nu_{n+1}|_{\Gamma_{\tau_0}} = 0, \qquad \nu_i|_{\partial\Omega_{\tau_0}\cap\{z=\tau_0\}} = 0, \quad i = 1,\dots,n, \qquad v_z|_{\Gamma_{\tau_0}} = 0. \tag{4.8}$$

Suppose $G_\tau = \Omega_{\tau_0} \cap \{z = \tau\}$.

Multiplying both parts of the system (4.3) scalarly by the vector $2v_z$ and integrating the obtained expression with respect to Ω_τ, $\delta < \tau \leq \tau_0$, with regard for (4.4), (4.5), (4.6), and (4.8) we obtain

$$\begin{aligned}
2\int_{\Omega_\tau} &\left(\widetilde{F} - \frac{1}{z}\sum_{i=1}^{n}\widetilde{B}_i v_{y_i} - \widetilde{C}v\right) v_z\,dy\,dz \\
&= \int_{\Omega_\tau}\left[2v_{zz}v_z - \frac{2}{z^2}\sum_{i,j=1}^{n}\widetilde{A}_{ij}(y)v_{y_iy_j}v_z - \frac{4}{z}\sum_{i=1}^{n} y_i v_{zy_i}v_z\right]dy\,dz \\
&= \int_{\Omega_\tau}\left[\frac{\partial(v_zv_z)}{\partial z} + \frac{2}{z^2}\sum_{i,j=1}^{n}\widetilde{A}_{ij}(y)v_{y_i}v_{zy_j} + \frac{2}{z^2}\sum_{i,j=1}^{n}\frac{\widetilde{A}_{ij}(y)}{\partial y_j}v_{y_i}v_z - \frac{2}{z}\sum_{i=1}^{n} y_i\frac{\partial(v_zv_z)}{\partial y_i}\right]dy\,dz \\
&= \int_{\Omega_\tau}\left[\frac{\partial(v_zv_z)}{\partial z} + \frac{1}{z^2}\frac{\partial}{\partial z}\left(\sum_{i,j=1}^{n}\widetilde{A}_{ij}(y)v_{y_i}v_{y_j}\right) + \frac{2}{z^2}\sum_{i,j=1}^{n} Ey_i v_{y_i}v_z + \frac{2}{z}\sum_{i=1}^{n}\frac{\partial y_i}{\partial y_i}v_zv_z\right]dy\,dz \\
&= \int_{G_\tau}\left[v_zv_z + \frac{1}{\tau^2}\sum_{i,j=1}^{n}\widetilde{A}_{ij}(y)v_{y_i}v_{y_j}\right]dy \\
&\quad + \int_{\Omega_\tau\setminus\Omega_\delta}\left[\frac{2}{z^3}\sum_{i,j=1}^{n}\widetilde{A}_{ij}(y)v_{y_i}v_{y_j} + \frac{2}{z^2}\sum_{i,j=1}^{n} Ey_i v_{y_i}v_z + \frac{2n}{z}v_zv_z\right]dy\,dz.
\end{aligned} \tag{4.9}$$

Since the domains of variation of the variables y_i in G are bounded, that is, $\sup_G |y_i| \leq d$, $i = 1,\ldots,n$, by virtue of (4.4) and (4.7) for some $\widetilde{c}_1 = \text{const} > 0$ the inequality

$$\sum_{i,j=1}^{n} \widetilde{A}_{ij}(y)\eta_i\eta_j \leq \widetilde{c}_1 \sum_{i=1}^{n} |\eta_i|^2 \quad \forall \eta_i \in \mathbb{R}^n,\ \forall y \in G \tag{4.10}$$

holds.

Denoting

$$\begin{gathered}\widetilde{w}(\tau) = \int_{G_\tau} \left[v_z v_z + \sum_{i=1}^{n} v_{y_i} v_{y_i} \right] dy, \\ \widetilde{w}_0(\tau) = \int_{G_\tau} \left[vv + v_z v_z + \sum_{i=1}^{n} v_{y_i} v_{y_i} \right] dy,\end{gathered} \tag{4.11}$$

due to (4.7), (4.9), and (4.10) we have

$$\begin{aligned}
&\min\left(1, \frac{\widetilde{c}_0}{\tau^2}\right)\widetilde{w}(\tau) \\
&\quad \leq \int_{\Omega_\tau \setminus \Omega_\delta} \left[\frac{2\widetilde{c}_1}{z^3} \sum_{i=1}^{n} v_{y_i} v_{y_i} + \frac{dn}{z^2} \sum_{i=1}^{n} (v_{y_i} v_{y_i} + v_z v_z) + \frac{2n}{z} v_z v_z \right] dy\, dz \\
&\qquad + \int_{\Omega_\tau \setminus \Omega_\delta} \left[\widetilde{F}\widetilde{F} + v_z v_z + \frac{1}{z} \sum_{i=1}^{n} \|\widetilde{B}_i\|_{L_\infty} (v_{y_i} v_{y_i} + v_z v_z) + \|\widetilde{C}\|_{L_\infty} (vv + v_z v_z) \right] dy\, dz \\
&\quad \leq \left(\frac{2\widetilde{c}_1}{\delta^3} + \frac{dn}{\delta^2} + \frac{1}{\delta} \max_{1 \leq i \leq n} \|\widetilde{B}_i\|_{L_\infty} \right) \int_{\Omega_\tau \setminus \Omega_\delta} \left(\sum_{i=1}^{n} v_{y_i} v_{y_i} \right) dy\, dz \\
&\qquad + \left(\frac{dn^2}{\delta^2} + \frac{2n}{\delta} + 1 + \|\widetilde{C}\|_{L_\infty} \right) \int_{\Omega_\tau \setminus \Omega_\delta} v_z v_z\, dy\, dz \\
&\qquad + \|\widetilde{C}\|_{L_\infty} \int_{\Omega_\tau \setminus \Omega_\delta} vv\, dy\, dz + \int_{\Omega_\tau \setminus \Omega_\delta} \widetilde{F}\widetilde{F}\, dy\, dz \\
&\quad \leq c_2(\delta) \int_{\Omega_\tau} \left[vv + v_{zz} + \sum_{i=1}^{n} v_{y_i} v_{y_i} \right] dy\, dz + \int_{\Omega_\tau} \widetilde{F}\widetilde{F}\, dy\, dz \\
&\quad = c_2(\delta) \int_0^\tau \widetilde{w}_0(\sigma)\, d\sigma + \int_{\Omega_\tau} \widetilde{F}\widetilde{F}\, dy\, dz,
\end{aligned} \tag{4.12}$$

where $c_2(\delta) = \text{const} > 0$, $\delta < \tau \leq \tau_0$, while $\|\widetilde{B}_i\|_{L_\infty}$ and $\|\widetilde{C}\|_{L_\infty}$ are the upper bounds of norms of the matrices $\widetilde{B}_i$ and $\widetilde{C}$ in the domain Ω_{τ_0}.

By (4.6) we have

$$v(y,\tau) = \int_0^\tau v_z(y,\sigma)\, d\sigma, \tag{4.13}$$

whence

$$\begin{aligned}\int_{G_\tau} v(y,\tau)v(y,\tau)dy &\le \int_G\left[\int_0^\tau |v_z(y,\sigma)|d\sigma\right]^2 dy\\ &\le \int_G\left[\left(\int_0^\tau 1^2 d\sigma\right)^{1/2}\left(\int_0^\tau |v_z(y,\sigma)|^2 d\sigma\right)^{1/2}\right]^2 dy\\ &\le \tau\int_G\int_0^\tau v_z^2(y,\sigma)d\sigma\, dy = \tau\int_{\Omega_\tau} v_z^2\, dy\, dz.\end{aligned} \tag{4.14}$$

Taking into account (4.14), from (4.12), we have

$$\widetilde{w}_0(\tau) \le c_3(\delta)\int_0^\tau \widetilde{w}_0(\sigma)d\sigma + c_4(\delta)\int_{\Omega_\tau}\widetilde{F}\widetilde{F}\, dy\, dz, \tag{4.15}$$

where $c_i(\delta) = \text{const} > 0$, $i = 3,4$. This, on the basis of Gronwall's lemma, enables one to conclude that

$$\widetilde{w}_0(\tau) \le c(\delta)\int_{\Omega_\tau}\widetilde{F}\widetilde{F}\, dy\, dz, \quad 0 < \tau \le \tau_0, \tag{4.16}$$

with the constant $c(\delta) > 0$.

In its turn, from (4.16) it follows that

$$\|v\|_{W_2^1(\Omega_{\tau_0})} \le \widetilde{c}(\delta)\|\widetilde{F}\|_{L_2(\Omega_{\tau_0})}, \quad \widetilde{c}(\delta) = \text{const} > 0. \tag{4.17}$$

Remark 4.1. To construct a solution v of the problem (4.3), (4.5) from the space $W_2^2(\Omega_{\tau_0})$ for

$$\widetilde{F}|_{\Omega_\delta} = 0, \quad 0 < \delta < \tau_0, \tag{4.18}$$

satisfying automatically according to (4.16) and (4.18) the condition (4.6), we take advantage of Galerkin's method (see [17, pages 213–220]). Note that unlike the equations and systems of hyperbolic type considered in [17], in the system (4.3) we have terms involving mixed derivatives v_{zy_i}.

(2^0) First we present the proof of the existence of a weak generalized solution of the problem (4.3), (4.5), (4.6) of the class W_2^1. Let $\{\varphi_k(y)\}_{k=1}^\infty$ be an orthogonal basis in the separable Hilbert space $[\overset{\circ}{W}{}_2^1(G)]^m$. As elements of the basis $\{\varphi_k(y)\}_{k=1}^\infty$ in the space $[\overset{\circ}{W}{}_2^1(G)]^m$ we can take the proper vector functions of the Laplace operator: $\Delta\varphi_k = \lambda_k\varphi_k$, $\varphi_k|_{\partial G} = 0$ (see [17, pages 110, 248]). Note that in the space $[\overset{\circ}{W}{}_2^1(G)]^m$ we can, as an equivalent norm, take

$$\begin{gathered}\|v\|^2_{\overset{\circ}{W}{}_2^1(G)} = \int_G\left(\sum_{i=1}^n v_{y_i}v_{y_i}\right)dy,\\ v = (v_1,\dots,v_m), \qquad v_i \in \overset{\circ}{W}{}_2^1(G), \quad i = 1,\dots,m.\end{gathered} \tag{4.19}$$

An approximate solution $v^N(y,z)$ of the problem (4.3), (4.5) will be sought in the form of the sum

$$v^N(y,z) = \sum_{k=1}^{N} C_k^N(z)\varphi_k(y), \tag{4.20}$$

in which the coefficients $C_k^N(z)$ are defined from the following relations:

$$\begin{aligned}
&\left(\frac{\partial^2 v^N}{\partial z^2}, \varphi_\ell\right)_{L_2(G)} + \frac{1}{z^2}\int_G \left\{\sum_{i,j=1}^{n}\left[\widetilde{A}_{ij}(y)v^N_{y_i}\varphi_{\ell y_j} + \frac{\partial \widetilde{A}_{ij}}{\partial y_j}v^N_{y_i}\varphi_\ell\right]\right\}dy \\
&\quad + \frac{2}{z}\int_G \left\{\sum_{i=1}^{n}\left[y_i v^N_z \varphi_{\ell y_i} + v^N_z\varphi_\ell\right]\right\}dy + \int_G \left[\frac{1}{z}\sum_{i=1}^{n}\widetilde{B}_i v^N_{y_i}\varphi_\ell + \widetilde{C}v^N\varphi_\ell\right]dy \\
&= (\widetilde{F}, \varphi_\ell)_{L_2(G)}, \quad \delta \le z \le \tau_0,\ \ell = 1,\dots,N,
\end{aligned} \tag{4.21}$$

$$\left.\frac{d}{dz}C_k^N(z)\right|_{z=\delta} = 0, \quad C_k^N(z)|_{z=\delta} = 0,\ k = 1,\dots,N, \tag{4.22}$$

$$C_k^N(z) = 0, \quad 0 \le z < \delta,\ k = 1,\dots,N. \tag{4.23}$$

The equalities (4.21) make a system of linear ordinary differential equations of second order with respect to z and to unknown functions C_k^N, $k = 1,\dots,N$, with constant matrix elements, which in their turn are the coefficients of the derivatives of second order $d^2C_k^N(z)/dz^2$, and with the different from zero determinant, since by itself it represents Gram's determinant with respect to the scalar product in $L_2(G)$ of the linearly independent system of vector functions $\varphi_1(y),\dots,\varphi_N(y)$. The coefficients of every equation of that system are bounded measurable functions, and the right-hand sides $g_\ell(z) = (\widetilde{F},\varphi_\ell)_{L_2(G)} \in L_1((0,\tau_0))$.

As is known (see [17, page 214]), the system (4.21) has a unique solution which satisfies the initial conditions (4.22), as well as the condition (4.23) by (4.18), where $d^2C_k^N(z)/dz^2 \in L_1((0,\tau_0))$.

Let us now show that for $v = v^N$ the estimates (4.16) and (4.17) are valid. Indeed, multiplying each of the inequalities (4.21) by the corresponding $(d/dz)C_\ell^N(z)$ and summing up with respect to ℓ from 1 to N, we obtain the equality

$$\begin{aligned}
&\left(\frac{\partial^2 v^N}{\partial z^2}, \frac{\partial v^N}{\partial z}\right)_{L_2(G)} + \frac{1}{z^2}\int_G \left\{\sum_{i,j=1}^{n}\left[\widetilde{A}_{ij}(y)v^N_{y_i}v^N_{zy_j} + \frac{\partial \widetilde{A}_{ij}}{\partial y_j}v^N_{y_i}v^N_z\right]\right\}dy \\
&\quad + \frac{1}{z}\int_G \left\{\sum_{i=1}^{n}\left[y_i v^N_z v^N_{zy_i} + v^N_z v^N_z\right]\right\}dy \\
&\quad + \int_G \left[\frac{1}{z^2}\sum_{i=1}^{n}\widetilde{B}_i v^N_{y_i}v^N_z + \widetilde{C}v^N v^N_z\right]dy = (\widetilde{F}, v^N_z)_{L_2(G)},
\end{aligned} \tag{4.24}$$

which after integration with respect to z from 0 to τ_0, with regard for (4.23) and further transformations allow us to derive the inequalities (4.16) and (4.17). Note that by (4.23) it becomes obvious that

$$v^N|_{\Omega_\delta} = 0, \quad N = 1,2,\dots. \tag{4.25}$$

Thus the estimate

$$\begin{gathered}\int_{G_{\tau_0}}\left[v^N v^N + v_z^N v_z^N + \sum_{i=1}^{n} v_{y_i}^N v_{y_i}^N\right]dy \le c_5(\delta)\|\widetilde{F}\|^2_{L_2(\Omega_\tau)}, \quad 0<\tau\le\tau_0,\ N\ge 1,\\ \|v^N\|_{W_2^1(\Omega_{\tau_0})} \le c_6(\delta)\|\widetilde{F}\|_{L_2(\Omega_{\tau_0})}, \quad N\ge 1,\end{gathered} \tag{4.26}$$

is valid, where the positive constants $c_5(\delta)$ and $c_6(\delta)$ do not depend on N.

Owing to (4.26) and to weak compactness of the closed ball in the Hilbert space $W_2^1(\Omega_{\tau_0})$ we can choose from the sequence $\{v^N\}$ the subsequence (denoted as above), converging weakly in $W_2^1(\Omega_{\tau_0})$ to some element $v \in W_2^1(\Omega_{\tau_0})$. Note that by virtue of (4.25), the equality (4.6) will be valid for that element v. It should be also noted that since $v^N|_{\Gamma_{\tau_0}} = 0$, $N \ge 1$, by the compactness of taking the trace: $v \to v|_{\Gamma_{\tau_0}}$ from the space $W_2^1(\Omega_{\tau_0})$ into $L_2(\Gamma_{\tau_0})$, the element v satisfies the homogeneous boundary condition (4.5) (see [17, page 71]).

Let us now show that v is a weak generalized solution of the system (4.3), that is, the identity

$$\begin{aligned}\int_{\Omega_{\tau_0}}\Bigg[&-v_z w_z + \frac{1}{z^2}\sum_{i,j=1}^{n} v_{y_i}\big(\widetilde{A}_{ij}w\big)_{y_j} + \frac{2}{z}\sum_{i=1}^{n} v_z\big(y_i w\big)_{y_i} + \frac{1}{z}\sum_{i=1}^{n}\widetilde{B}_i v_{y_i} w + \widetilde{C} v w\Bigg]dy\,dz\\ &= \int_{\Omega_{\tau_0}}\widetilde{F} w\,dy\,dz\end{aligned} \tag{4.27}$$

holds for any $w \in V$, where V is the closure with respect to the norm of the space $W_2^1(\Omega_{\tau_0})$ of vector functions $\omega = (\omega_1,\dots,\omega_m)$ of the class $C^2(\overline{\Omega}_{\tau_0})$, satisfying the following homogeneous boundary conditions:

$$w|_{\Gamma_{\tau_0}} = 0, \qquad w|_{z=\tau_0} = 0. \tag{4.28}$$

Towards this end, we first multiply each of the equalities (4.21) by its own function $d_\ell(z) \in C^2[0,\tau_0]$, $d_\ell(\tau_0) = 0$, then sum the obtained equality with respect to ℓ from 1 to N and integrate with respect to z from 0 to τ_0. Next, integration by parts in the first term results in the identity

$$\begin{aligned}\int_{\Omega_{\tau_0}}\Bigg[&-v_z^N w_z + \frac{1}{z^2}\sum_{i,j=1}^{n} v_{y_i}^N\big(\widetilde{A}_{ij}w\big)_{y_j} + \frac{2}{z}\sum_{i=1}^{n} v_z^N\big(y_i w\big)_{y_i} + \frac{1}{z}\sum_{i=1}^{n}\widetilde{B}_i v_{y_i}^N w + \widetilde{C} v^N w\Bigg]dy\,dz\\ &= \int_{\Omega_{\tau_0}}\widetilde{F} w\,dy\,dz,\end{aligned} \tag{4.29}$$

which is valid for any w of the type $\sum_{\ell=1}^{N} d_\ell(z)\varphi_\ell(y)$. The union of such v is denoted by V_N. If we pass in (4.29) to the limit by the above-chosen subsequence for fixed w

of the class V_N, then we will arrive at the identity (4.27) for the limiting function $v \in W_2^1(\Omega_{\tau_0})$, valid for any $w \in \bigcup_{N=1}^{\infty} V_N$. Now we can show that $\bigcup_{N=1}^{\infty} V_N$ is dense in V.

Indeed, let $w \in C^2(\overline{\Omega}_{\tau_0})$ and let the equalities (4.28) be fulfilled. Then there exists an extension w_0 of the vector function w to the greater cylinder $\Omega_* = \{(y,z) \in \mathbb{R}^{n+1} : y \in G,\ z \in (-\tau_0, \tau_0)\}$ of the class $C^2(\overline{\Omega}_*)$, such that $w_0|_{\partial\Omega_*} = 0$, $w_0|_{\Omega_{\tau_0}} = w$ (see [8, page 591]). Consequently, $w_0 \in \overset{\circ}{W}{}_2^1(\Omega_*)$, and since the system of functions

$$\left\{\varphi_\ell(y)\sin\frac{\pi k(z+\tau_0)}{2\tau_0}\right\}_{k,\ell=1}^{\infty} \tag{4.30}$$

is fundamental in the space $\overset{\circ}{W}{}_2^1(\Omega_*)$ (see [20, pages 112, 165]), for any $\varepsilon > 0$ there exists a linear combination $\sum_{i=1}^{k} \alpha_i \widetilde{w}_i$ of vector functions from the system (4.30) such that

$$\left\| w_0 - \sum_{i=1}^{k} \alpha_i \widetilde{w}_i \right\|_{W_2^1(\Omega_*)} < \varepsilon, \tag{4.31}$$

because $\|\widetilde{w}\|_{\overset{\circ}{W}{}_2^1(\Omega_*)} = \|\widetilde{w}\|_{W_2^1(\Omega_*)}$. By virtue of (4.28) and the fact that $w_0|_{\Omega_{\tau_0}} = w$, we have

$$\left\| w - \sum_{i=1}^{k} \alpha_i \widetilde{w}_i \right\|_{V} = \left\| w - \sum_{i=1}^{k} \alpha_i \widetilde{w}_i \right\|_{W_2^1(\Omega_{\tau_0})} \le \left\| w_0 - \sum_{i=1}^{n} \alpha_i \widetilde{w}_i \right\|_{W_2^1(\Omega_*)} < \varepsilon. \tag{4.32}$$

But $\sum_{i=1}^{k} \alpha_i \widetilde{w}_i \in \bigcup_{N=1}^{\infty} V_N$. Therefore from (4.32) and the fact that the set $\{w \in C^2(\overline{\Omega}_{\tau_0}) : w|_{\Gamma_{\tau_0}} = 0,\ w|_{z=\tau_0} = 0\}$ is dense in the space V, we find that $\bigcup_{N=1}^{\infty} V_N$ is dense in V. Since $v \in W_2^1(\Omega_{\tau_0})$, this in its turn implies that the identity (4.27), which is valid for any $w \in \bigcup_{N=1}^{\infty} V_N$, will be valid for any $w \in V$ as well. Thus the limiting vector function $v = v(y,z)$ is a weak generalized solution of (4.3) satisfying the equalities (4.5) and (4.6).

(3^0) Let us show that if the following additional conditions

$$\partial G \in C^2; \qquad B_{ix_j}, B_{it}, C_{x_j}, C_t \in L_\infty(D_{\tau_0}), \quad i,j = 1,\dots,n, \tag{4.33}$$

$$F \in W_2^1(D_{\tau_0}), \quad F|_{D_\delta} = 0 \tag{4.34}$$

are fulfilled, then the above-obtained limiting function v is a solution of the problem (4.3), (4.5), (4.6) from the space $W_2^2(\Omega_{\tau_0})$, where $L_\infty(D_{\tau_0})$ is the space of measurable bounded on D_{τ_0} functions.

We multiply by $(d^2/dz^2)C_\ell^N(z)$ the expression obtained after differentiation of the equality (4.21) with respect to z and then sum with respect to ℓ from 1 to N. As a result we obtain

$$
\begin{aligned}
&\left(v_{zzz}^N, v_{zz}^N\right)_{L_2(G)} - \frac{2}{z^3}\int_G \left\{\sum_{i,j=1}^n \left[\widetilde{A}_{ij}(y)v_{y_i}^N v_{zzy_j}^N + \frac{\partial \widetilde{A}_{ij}}{\partial y_j} v_{y_i}^N v_{zz}^N\right]\right\} dy \\
&\quad + \frac{1}{z^2}\int_G \left\{\sum_{i,j=1}^n \left[\widetilde{A}_{ij}(y)v_{zy_i}^N v_{zzy_j}^N + \frac{\partial \widetilde{A}_{ij}}{\partial y_j} v_{zy_i}^N v_{zz}^N\right]\right\} dy \\
&\quad - \frac{2}{z^2}\int_G \left\{\sum_{i=1}^n \left[y_i v_z^N v_{zzy_j}^N + v_z^N v_{zz}^N\right]\right\} dy + \frac{2}{z}\int_G \left\{\sum_{i=1}^n \left[y_i v_{zz}^N v_{zzy_i}^N + v_{zz}^N v_{zz}^N\right]\right\} dy \\
&\quad + \int_G \left[-\frac{1}{z^2}\sum_{i=1}^n \widetilde{B}_i v_{y_i}^N v_{zz}^N + \frac{1}{z}\sum_{i=1}^n \frac{\partial \widetilde{B}_i}{\partial z} v_{y_i}^N v_{zz}^N + \frac{1}{z}\sum_{i=1}^n \widetilde{B}_i v_{zy_i}^N v_{zz}^N + \frac{\partial \widetilde{C}}{\partial z} v^N v_{zz}^N + \widetilde{C} v_z^N v_{zz}^N\right] dy \\
&= \left(\widetilde{F}_z, v_{zz}^N\right)_{L_2(G)}.
\end{aligned}
\tag{4.35}
$$

It can be easily verified that

$$
\left(v_{zzz}^N, v_{zz}^N\right)_{L_2(G)} = \frac{1}{2}\frac{d}{dz}\left(v_{zz}^N v_{zz}^N\right)_{L_2(G)},
\tag{4.36}
$$

$$
\begin{aligned}
&-\frac{2}{z^3}\int_G \left[\sum_{i,j=1}^n \widetilde{A}_{ij}(y)v_{y_i}^N v_{zzy_j}^N\right] dy \\
&\quad = \frac{d}{dz}\left\{-\frac{2}{z^3}\int_G \left[\sum_{i,j=1}^n \widetilde{A}_{ij}(y)v_{y_i}^N v_{zy_j}^N\right] dy\right\} - \frac{6}{z^4}\int_G \left[\sum_{i,j=1}^n \widetilde{A}_{ij}(y)v_{y_i}^N v_{zy_j}^N\right] dy \\
&\qquad + \frac{2}{z^3}\int_G \left[\sum_{i,j=1}^n \widetilde{A}_{ij}(y)v_{zy_i}^N v_{zy_j}^N\right] dy,
\end{aligned}
\tag{4.37}
$$

$$
\begin{aligned}
&\frac{1}{z^2}\int_G \left[\sum_{i,j=1}^n \widetilde{A}_{ij}(y)v_{zy_i}^N v_{zzy_j}^N\right] dy \\
&\quad = \frac{1}{2}\frac{d}{dz}\left\{\frac{1}{z^2}\int_G \left[\sum_{i,j=1}^n \widetilde{A}_{ij}(y)v_{zy_i}^N v_{zy_j}^N\right] dy\right\} + \frac{1}{z^3}\int_G \left[\sum_{i,j=1}^n \widetilde{A}_{ij}(y)v_{zy_i}^N v_{zy_j}^N\right] dy,
\end{aligned}
\tag{4.38}
$$

$$
\begin{aligned}
&-\frac{2}{z^2}\int_G \left[\sum_{i=1}^n y_i v_z^N v_{zzy_i}^N\right] dy \\
&\quad = \frac{d}{dz}\left\{-\frac{2}{z^2}\int_G \left[\sum_{i=1}^n y_i v_z^N v_{zy_i}^N\right] dy\right\} - \frac{4}{z^3}\int_G \left[\sum_{i=1}^n y_i v_z^N v_{zy_i}^N\right] dy \\
&\qquad + \frac{2}{z^2}\int_G \left[\sum_{i=1}^n y_i v_{zz}^N v_{zy_i}^N\right] dy,
\end{aligned}
\tag{4.39}
$$

$$
\begin{aligned}
&\frac{2}{z}\int_G \left[\sum_{i=1}^n y_i v_{zz}^N v_{zzy_i}^N\right] dy \\
&\quad = \frac{2}{z}\int_G \left[\sum_{i=1}^n \left\{\frac{1}{2}\frac{\partial}{\partial y_i}\left(y_i v_{zz}^N v_{zz}^N\right) - \frac{1}{2}\left(v_{zz}^N v_{zz}^N\right)\right\}\right] dy \\
&\quad = \frac{2}{z}\int_{\partial G} \left[\sum_{i=1}^n \frac{1}{2} y_i v_{zz}^N v_{zz}^N\right] \nu_i \, ds - \frac{n}{z}\int_G v_{zz}^N v_{zz}^N \, dy = -\frac{n}{z}\int_G v_{zz}^N v_{zz}^N \, dy,
\end{aligned}
\tag{4.40}
$$

where $\nu = (\nu_1, \dots, \nu_n)$ is the unit vector of the outer normal to ∂G. In deriving (4.40), we have taken into account that by the construction $v^N_{zz}|_{\partial G} = 0$.

Substituting (4.36), (4.37), (4.38), (4.39), and (4.40) into (4.35), integrating the latter with respect to z from 0 to τ and taking into account that $v^N|_{\Omega_\delta} = 0$, $\delta < \tau$ by (4.25), we have

$$\begin{aligned}
&\frac{1}{2}\left(v^N_{zz}, v^N_{zz}\right)_{L_2(G_\tau)} \\
&\quad - \frac{2}{\tau^3}\int_{G_\tau}\left[\sum_{i,j=1}^n \widetilde{A}_{ij}(y)v^N_{y_i}v^N_{zy_j}\right]dy - 6\int_{\Omega_\tau\setminus\Omega_\delta}\frac{1}{z^4}\left[\sum_{i,j=1}^n \widetilde{A}_{ij}(y)v^N_{y_i}v^N_{zy_j}\right]dy\,dz \\
&\quad + 2\int_{\Omega_\tau\setminus\Omega_\delta}\frac{1}{z^3}\left[\sum_{i,j=1}^n \widetilde{A}_{ij}(y)v^N_{zy_i}v^N_{zy_j}\right]dy\,dz \\
&\quad - 2\int_{\Omega_\tau\setminus\Omega_\delta}\frac{1}{z^3}\left[\sum_{i,j=1}^n \frac{\partial\widetilde{A}_{ij}}{\partial y_j}v^N_{y_i}v^N_{zz}\right]dy\,dz + \frac{1}{2\tau^2}\int_{G_\tau}\left[\sum_{i,j=1}^n \widetilde{A}_{ij}(y)v^N_{zy_i}v^N_{zy_j}\right]dy \\
&\quad + \int_{\Omega_\tau\setminus\Omega_\delta}\frac{1}{z^3}\left[\sum_{i,j=1}^n \widetilde{A}_{ij}(y)v^N_{zy_i}v^N_{zy_j}\right]dy\,dz + \int_{\Omega_\tau\setminus\Omega_\delta}\frac{1}{z^2}\left[\sum_{i,j=1}^n \frac{\partial\widetilde{A}_{ij}}{\partial y_j}v^N_{zy_i}v^N_{zz}\right]dy\,dz \\
&\quad - \frac{2}{\tau^2}\int_{G_\tau}\left[\sum_{i=1}^n y_i v^N_z v^N_{zy_i}\right]dy - 4\int_{\Omega_\tau\setminus\Omega_\delta}\frac{1}{z^3}\left[\sum_{i=1}^n y_i v^N_z v^N_{zy_i}\right]dy\,dz \\
&\quad + 2\int_{\Omega_\tau\setminus\Omega_\delta}\frac{1}{z^2}\left[\sum_{i=1}^n y_i v^N_{zz}v^N_{zy_i}\right]dy\,dz - 2\int_{\Omega_\tau\setminus\Omega_\delta}\frac{n}{z^2}v^N_z v^N_{zz}\,dy\,dz \\
&\quad - n\int_{\Omega_\tau\setminus\Omega_\delta}\frac{1}{z}v^N_{zz}v^N_{zz}\,dy\,dz + 2\int_{\Omega_\tau\setminus\Omega_\delta}\frac{n}{z}v^N_{zz}v^N_{zz}\,dy\,dz \\
&\quad + \int_{\Omega_\tau\setminus\Omega_\delta}\left[-\frac{1}{z^2}\sum_{i=1}^n \widetilde{B}_i v^N_{y_i}v^N_{zz} + \frac{1}{z}\sum_{i=1}^n \frac{\partial\widetilde{B}_i}{\partial z}v^N_{y_i}v^N_{zz} + \frac{1}{z}\sum_{i=1}^n \widetilde{B}_i v^N_{zy_i}v^N_{zz} + \frac{\partial\widetilde{C}}{\partial z}v^N v^N_{zz} + \widetilde{C}v^N_z v^N_{zz}\right]dy\,dz \\
&= \left(\widetilde{F}_z, v^N_{zz}\right)_{L_2(\Omega_\tau)}.
\end{aligned} \tag{4.41}$$

Owing to the well-known inequalities

$$\begin{gathered}
\left|\int_{G_\tau}\varphi(y)\psi(y)dy\right| \le \left(\int_{G_\tau}\varphi^2\,dy\right)^{1/2}\left(\int_{G_\tau}\psi^2\,dy\right)^{1/2}, \\
|ab| \le \varepsilon|a|^2 + \frac{1}{4\varepsilon}|b|^2, \quad \varepsilon = \text{const} > 0,
\end{gathered} \tag{4.42}$$

the two summands

$$I_1 = -\frac{2}{\tau^3}\int_{G_\tau}\left[\sum_{i,j=1}^n \widetilde{A}_{ij}(y)v^N_{y_i}v^N_{zy_j}\right]dy, \qquad I_2 = -\frac{2}{\tau^2}\int_{G_\tau}\left[\sum_{i=1}^n y_i v^N_z v^N_{zy_i}\right]dy \tag{4.43}$$

from the left-hand side of (4.41) admit the following estimates:

$$|I_1| \le \varepsilon\frac{\widetilde{c}_6}{\tau^3}\int_{G_\tau} v^N_{zy_j}v^N_{zy_j}\,dy+\frac{c_7(\varepsilon)}{\tau^3}\int_{G_\tau} v^N_{y_i}v^N_{y_i}\,dy,$$
$$|I_2| \le \varepsilon\frac{\widetilde{c}_8}{\tau^2}\int_{G_\tau} v^N_{zy_i}v^N_{zy_i}\,dy+\frac{c_9(\varepsilon)}{\tau^2}\int_{G_\tau} v^N_{z}v^N_{z}\,dy, \tag{4.44}$$

in which the positive constants $\widetilde{c}_6$ and $\widetilde{c}_8$ depend only on the coefficients A_{ij} of the system (1.1) and on the finite domain G, while $c_7, c_9 = \text{const} > 0$ depend only on ε.

By (4.44) and (4.25), for a sufficiently small positive $\varepsilon = \varepsilon(\delta)$, reasoning just in the same way as for the inequality (4.16) and using the estimate (4.16), from (4.41) we get

$$\widetilde{w}_0(\tau) \le c_{10}(\delta)\int_{\Omega_\tau}\left(\widetilde{F}\widetilde{F}+\widetilde{F}_z\widetilde{F}_z\right)dy\,dz, \quad c_{10}(\delta) = \text{const} > 0, \tag{4.45}$$

where $\widetilde{w}_0(\tau) = \int_{G_\tau}[v^N_{zz}v^N_{zz}+\sum_{i=1}^n v^N_{zy_i}v^N_{zy_i}]dy$, which in its turn results in

$$||v^N_{zz}||_{L_2(\Omega_{\tau_0})}+\sum_{i=1}^n||v^N_{zy_i}||_{L_2(\Omega_{\tau_0})} \le c_{11}(\delta)\left[||\widetilde{F}||_{L_2(\Omega_{\tau_0})}+||\widetilde{F}_z||_{L_2(\Omega_{\tau_0})}\right], \tag{4.46}$$

where $c_{11}(\delta) = \text{const} > 0$.

By the estimates (4.26) and (4.46), some subsequence $\{v^{N_k}\}$ converges weakly in L_2 together with the first-order derivatives $v^{N_k}_z$, $v^{N_k}_{y_i}$, $i = 1,\dots,n$, and the derivatives $v^{N_k}_{zz}$, $v^{N_k}_{zy_i}$, $i = 1,\dots,n$, to the above-constructed solution v and, respectively, to v_z, v_{y_i}, v_{zz}, v_{zy_i}, $i = 1,\dots,n$. It should be noted that for v the inequality

$$||v_{zz}||_{L_2(\Omega_{\tau_0})}+\sum_{i=1}^n||v_{zy_i}||_{L_2(\Omega_{\tau_0})} \le c_{12}(\delta)\left[||\widetilde{F}||_{L_2(\Omega_{\tau_0})}+||\widetilde{F}_z||_{L_2(\Omega_{\tau_0})}\right], \tag{4.47}$$

where $c_{12}(\delta) = \text{const} > 0$, is valid.

By (4.17) and (4.47), the vector function v will belong to the space $W^2_2(\Omega_{\tau_0})$, if we show that v has generalized derivatives $v_{y_iy_j}$ from $L_2(\Omega_{\tau_0})$, $i, j = 1,\dots,n$.

Denote by $\widetilde{V}$ the space of all vector functions $w = (w_1,\dots,w_m) \in L_2(\Omega_{\tau_0})$ having generalized derivatives $w_{y_iy_j}$, $i, j = 1,\dots,n$, from $L_2(\Omega_{\tau_0})$ and satisfying the homogeneous boundary condition (4.5), that is, $w|_{\Gamma_{\tau_0}} = 0$.

Just in the same way as we obtained (4.27) from (4.21), it follows from (4.21) that the above-constructed vector function v satisfies the integral identity

$$\int_{\Omega_{\tau_0}}\left[v_{zz}w+\frac{1}{z^2}\sum_{i,j=1}^n v_{y_i}\left(\widetilde{A}_{ij}w\right)_{y_j}-\frac{2}{z}\sum_{i=1}^n v_{zy_i}y_iw+\frac{1}{z}\sum_{i=1}^n\widetilde{B}_iv_{y_i}w+\widetilde{C}vw\right]dy\,dz$$
$$=\int_{\Omega_{\tau_0}}\widetilde{F}w\,dy\,dz \quad \forall w \in \widetilde{V}. \tag{4.48}$$

If in (4.48) we take as $w \in \widetilde{V}$ the vector function $w(y,z) = \psi(z)\Psi(y)$, where the scalar function $\psi(t)$ and the vector function $\Psi(y)$ are arbitrary elements respectively from $L_2((0,\tau_0))$ and $\overset{\circ}{W}{}_2^1(G)$, then the equality (4.48) by Fubini's theorem can be rewritten in the form

$$\int_0^{\tau_0} \psi(z)\left\{\int_{G_z}\left[v_{zz}\Psi + \frac{1}{z^2}\sum_{i,j=1}^n v_{y_i}(\widetilde{A}_{ij}\Psi)_{y_j} - \frac{2}{z}\sum_{i=1}^n v_{zy_i}y_i\Psi + \frac{1}{z}\sum_{i=1}^n \widetilde{B}_i v_{y_i}\Psi + \widetilde{C}v\Psi\right]dy\right\}dz$$
$$= \int_0^{\tau_0}\psi(z)\left[\int_{G_z}\widetilde{F}\Psi\,dy\right]dz, \tag{4.49}$$

which, due to arbitrariness in choice of $\psi(z) \in L_2((0,\tau_0))$, for almost all $z \in (0,\tau_0)$ yields

$$\begin{aligned}\int_{G_z}\left[\sum_{i,j=1}^n v_{y_i}(\widetilde{A}_{ij}\Psi)_{y_j} + z\sum_{i=1}^n \widetilde{B}_i v_{y_i}\Psi + z^2\widetilde{C}v\Psi\right]dy \\ = \int_{G_z}\left(-z^2 v_{zz} + 2z\sum_{i=1}^n v_{zy_i}y_i + z^2\widetilde{F}\right)\Psi\,dy \quad \forall\Psi \in \overset{\circ}{W}{}_2^1(G).\end{aligned} \tag{4.50}$$

Since for such $z \in (0,\tau_0)$ the vector function

$$\widehat{F} = \left[-z^2 v_{zz} + 2z\sum_{i=1}^n v_{zy_i}y_i + z^2\widetilde{F}\right] \in L_2(G), \tag{4.51}$$

the identity (4.50) implies that the vector function $v = (v_1,\dots,v_m)$ is the generalized solution from the space $\overset{\circ}{W}{}_2^1(G)$ for the following elliptic system of equations:

$$-\sum_{i,j=1}^n \widetilde{A}_{ij}v_{y_iy_j} + z\sum_{i=1}^n \widetilde{B}_i v_{y_i} + z^2\widetilde{C}v = \widehat{F}. \tag{4.52}$$

According to (4.7), the system (4.52) is strongly elliptic. Therefore under the assumption that $\partial G \in C^2$, that is, the appearing in the definition of conic domain D function $g \in C^2$, in the system (1.1) the coefficients $B_i, C \in C^1(\overline{D}_{\tau_0})$ and thus in the system (4.52) the coefficients $\widetilde{B}_i, \widetilde{C} \in C^1(\overline{\Omega}_{\tau_0})$, the generalized solution v of the system (4.52) from the space $\overset{\circ}{W}{}_2^1(G)$ belongs also to the space $W_2^2(G)$ for such $z \in (0,\tau_0)$ (see [19, page 109]), and

$$\begin{aligned}\|v\|_{W_2^2(G_z)} &\le c_{13}\|\widehat{F}\|_{L_2(G_z)} \\ &\le c_{14}\left[\|v_{zz}\|_{L_2(G_z)} + \sum_{i=1}^n \|v_{zy_i}\|_{L_2(G_z)} + \|\widetilde{F}\|_{L_2(G_z)}\right], \quad c_{13}, c_{14} = \text{const} > 0.\end{aligned} \tag{4.53}$$

Thus for such $z \in (0,\tau_0)$ the vector function v has generalized derivatives $v_{y_iy_j}$, $i,j = 1,\dots,n$, from $L_2(G_z)$, and by (4.47) and (4.53) we have $\widetilde{g}_{ij}(z) = \|v_{y_iy_j}\|_{L_2(G_\tau)} \in L_2((0,\tau_0))$. Hence it remains to notice that the function $\widehat{g}(y,z) \in L_2(\Omega_{\tau_0})$ has the generalized derivative $\widehat{g}_{y_i}(y,z) \in L_2(\Omega_{\tau_0})$, $1 \le i \le n$, if and only if for almost all $z \in (0,\tau_0)$ the function $\widehat{g}$ has the generalized derivative $\widehat{g}_{y_i} \in L_2(G_z)$ and $\widehat{\varphi}_i(z) = \|\widehat{g}_{y_i}\|_{L_2(G_z)} \in L_2((0,\tau_0))$.

Getting back from y, z to the initial variables x, t, we see that by the equalities (4.2) the vector function $u(x,t) = v(x/t,t)$ is a solution of the system (1.1) from the space $W_2^2(D_{\tau_0})$, satisfying the homogeneous boundary condition (4.1) and by virtue of (4.34) we have $u|_{D_\delta} = 0$.

Thus we have proved the following.

Lemma 4.2. *Let $g \in C^2$, $B_i, C \in C^1(\overline{D}_{\tau_0})$, $i = 1,\dots,n$, $F \in W_2^1(D_{\tau_0})$, $F|_{D_\delta} = 0$, $0 < \delta < \tau_0$, and let the condition (4.7) be fulfilled. Then the problem (1.1), (4.1) has a unique solution from the space $W_2^2(D_{\tau_0})$, and $u|_{D_\delta} = 0$.*

In the case where $F \in L_2(D_{\tau_0})$, since the space of infinitely differentiable finite functions $C_0^\infty(D)$ is dense in $L_2(D_{\tau_0})$, there exists a sequence of vector functions $F_k \in C_0^\infty(D_{\tau_0})$ such that $F_k \to F$ in $L_2(D_{\tau_0})$. Since $F_k \in C_0^\infty(D_{\tau_0})$, we have $F_k \in W_2^1(D_{\tau_0})$, and for sufficiently small positive δ_k, $\delta_k < \tau_0$, we have $F_k|_{D_{\delta_k}} = 0$.

Therefore by Lemma 4.2, there exists the unique solution $u_k \in W_2^2(D_{\tau_0})$ of the problem (1.1), (4.1). By (4.1), from the inequality (3.14) we find that

$$||u_k - u_p||_{W_2^1(D_{\tau_0})} \le c||F_k - F_p||_{L_2(D_{\tau_0})}, \tag{4.54}$$

from which it follows that the sequence $\{u_k\}_{k=1}^\infty$ is fundamental in $W_2^1(D_{\tau_0})$, since $F_k \to F$ in $L_2(D_{\tau_0})$. Due to the fact that the space $W_2^1(D_{\tau_0})$ is complete, there exists a vector function $u \in W_2^1(D_{\tau_0})$ such that $u_k \to u$ in $W_2^1(D_{\tau_0})$ and $Lu_k = F_k \to F$ in $L_2(D_{\tau_0})$. Consequently, u is a strong generalized solution of the problem (1.1), (4.1) of the class W_2^1 for which by (3.14) we have the estimate

$$\|u\|_{W_2^1(D_{\tau_0})} \le c\|F\|_{L_2(D_{\tau_0})}. \tag{4.55}$$

Thus the following theorem is valid.

Theorem 4.3. *Let $g \in C^2$, $B_i, C \in C^1(\overline{D}_{\tau_0})$, $i = 1,\dots,n$, and let the condition (4.7) be fulfilled. Then for any $F \in L_2(D_{\tau_0})$ there exists a unique strong generalized solution of the problem (1.1), (4.1) of the class W_2^1 for which the estimate (4.55) is valid.*

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