AN APPROXIMATION METHOD FOR CONTINUOUS PSEUDOCONTRACTIVE MAPPINGS

YISHENG SONG AND RUDONG CHEN

Received 20 March 2006; Revised 24 May 2006; Accepted 28 May 2006

Let K be a closed convex subset of a real Banach space E, $T: K \to K$ is continuous pseudocontractive mapping, and $f: K \to K$ is a fixed L-Lipschitzian strongly pseudocontractive mapping. For any $t \in (0,1)$, let x_t be the unique fixed point of $tf + (1-t)T$. We prove that if T has a fixed point and E has uniformly Gâteaux differentiable norm, such that every nonempty closed bounded convex subset of K has the fixed point property for nonexpansive self-mappings, then $\{x_t\}$ converges to a fixed point of T as t approaches to 0. The results presented extend and improve the corresponding results of Morales and Jung (2000) and Hong-Kun Xu (2004).

1. Introduction and preliminaries

Let E be a real Banach space and let J denote the normalized duality mapping from E into 2^{E^*} given by $J(x) = \{f \in E^*, \langle x, f\rangle = \|x\|\|f\|, \|x\| = \|f\|\}$, for all $x \in E$, where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. If E^* is strictly convex, then J is single-valued. In the sequel, we will denote the single-valued duality mapping by j, and denote $F(T) = \{x \in E;\ Tx = x\}$. In Banach space E, the following result *(the subdifferential inequality)* is well known [1, 5]. For all $x, y \in E$, for all $j(x+y) \in J(x+y)$, and for all $j(x) \in J(x)$,

$$\|x\|^2 + 2\langle y, j(x)\rangle \le \|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle. \tag{1.1}$$

Recall that the norm of E is said to be *Gâteaux differentiable* (and E is said to be *smooth*), if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{*}$$

Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2006, Article ID 28950, Pages 1–9
DOI 10.1155/JIA/2006/28950

exists for each x, y on the unit sphere $S(E)$ of E. Moreover, if for each y in $S(E)$ the limit defined by (*) is uniformly attained for x in $S(E)$, we say that the norm of E is *uniformly Gâteaux differentiable*. The norm of E is said to be *Fréchet differentiable*, if for each $x \in S(E)$ the limit (*) is attained uniformly for $y \in S(E)$. The norm of E is said to be *uniformly Fréchet differentiable* (and E is said to be *uniformly smooth*), the limit (*) is attained uniformly for $(x, y) \in S(E) \times S(E)$.

The following results which are found in [1, 4, 5] are well known.

(i) The duality mapping J in smooth Banach space E is single-valued and strong-weak* continuous [5, Lemma 4.3.3].

(ii) If E is a Banach space with a uniformly Gâteaux differentiable norm, then the mapping $J : E \to E^*$ is single-valued and norm-to-weak star uniformly continuous on bounded sets of E [5, Theorem 4.3.6].

(iii) In uniformly smooth Banach space E, the mapping $J : E \to E^*$ is single-valued and norm-to-norm uniformly continuous on bounded sets of E [5, Theorem 4.3.6].

(iv) A uniformly convex Banach space E is reflexive and strictly convex [5, Theorems 4.1.6 and 4.1.2].

(v) If K is a nonempty convex subset of a strictly convex Banach space E and $T : K \to K$ is a nonexpansive mapping, then fixed point set $F(T)$ of T is a closed convex subset of K [5, Theorem 4.5.3].

Let E be a real Banach space and let T be a mapping with domain $D(T)$ and range $R(T)$ in E. T is called *pseudocontractive* (resp., *strongly*) if for any $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\begin{gathered}\langle Tx - Ty, j(x-y)\rangle \le \|x-y\|^2 \\ \left(\text{resp., } \langle Tx - Ty, j(x-y)\rangle \le \beta\|x-y\|^2 \text{ for some } 0<\beta<1\right).\end{gathered} \tag{1.2}$$

If I denotes the identity operator, then (1.2) implies that

$$\langle (I-T)x - (I-T)y,\ j(x-y)\rangle \ge 0, \tag{1.3}$$

$$\langle (I-T)x - (I-T)y,\ j(x-y)\rangle \ge (1-\beta)\|x-y\|^2. \tag{1.4}$$

Let K be a closed convex subset of a uniformly smooth Banach space E, $T : K \to K$ a nonexpansive mapping with $F(T) \neq \emptyset$, $f : K \to K$ a contraction. Then for any $t \in (0,1)$, the mapping

$$T_t^f : x \longmapsto tf(x) + (1-t)Tx \tag{1.5}$$

is also contraction. Let x_t denote the unique fixed point of T_t^f. In [7], Xu proved that as $t \downarrow 0$, $\{x_t\}$ converges to a fixed point p of T that is the unique solution of the variational inequality

$$\langle (I-f)u, j(u-p)\rangle \le 0 \quad \forall p \in F(T). \tag{1.6}$$

Let K be a nonempty closed convex subset of a Banach space E, $T : K \to K$ a continuous pseudocontractive map such that $F(T) \neq \emptyset$, and $f : K \to K$ a fixed Lipschitzian strongly pseudocontractive map. Then for any $t \in (0,1)$, $T_t^f = tf + (1-t)T : K \to K$ is also a continuous strongly pseudocontractive map. Let x_t be the unique fixed point of T_t^f (see [1]), that is,

$$x_t = tf(x_t) + (1-t)Tx_t. \tag{1.7}$$

In this paper, our purpose is to prove that $\{x_t\}$ defined by (1.7) strongly converges to a fixed point of T, which generalizes and improves several recent results. Particularly, it extends and improves [7, Theorems 3.1 and 4.1]. Let μ be a continuous linear functional on l^∞ satisfying $\|\mu\| = 1 = \mu(1)$. Then we know that μ is a mean on $\mathbb{N}$ if and only if

$$\inf\{a_n;\ n \in \mathbb{N}\} \le \mu(a) \le \sup\{a_n; n \in \mathbb{N}\} \tag{1.8}$$

for every $a = (a_1, a_2, \ldots) \in l^\infty$. According to time and circumstances, we use $\mu_n(a_n)$ instead of $\mu(a)$. A mean μ on $\mathbb{N}$ is called a *Banach limit* if

$$\mu_n(a_n) = \mu_n(a_{n+1}) \tag{1.9}$$

for every $a = (a_1, a_2, \ldots) \in l^\infty$. Furthermore, we know the following result [6, Lemma 1] and [5, Lemma 4.5.4].

Lemma 1.1 (see [6, Lemma 1]). *Let C be a nonempty closed convex subset of a Banach space E with a uniformly Gâteaux differentiable norm. Let $\{x_n\}$ be a bounded sequence of E and let μ be a mean on $\mathbb{N}$. let $z \in C$. Then*

$$\mu_n\|x_n - z\|^2 = \min_{y \in C} \mu_n\|x_n - y\|^2 \tag{1.10}$$

if and only if

$$\mu_n\left\langle y - z,\ j(x_n - z)\right\rangle \le 0 \quad \forall y \in C, \tag{1.11}$$

where j is the duality mapping of E.

2. Main results

Lemma 2.1. *Let E be a Banach space and let K be a nonempty closed convex subset of E. Suppose that $T : K \to K$ is a pseudocontractive mapping such that for each fixed strongly pseudocontractive map $f : K \to K$, the equation*

$$x = tf(x) + (1-t)Tx \tag{2.1}$$

has a solution x_t for each $t \in (0,1)$. Suppose that $u \in K$ is a fixed point of T. Then

(i) *$\{x_t\}$ is bounded;*

(ii) $\langle x_t - f(x_t), j(x_t - u)\rangle \le 0$.

Proof. (i) As u is a fixed point of T, we have

$$\begin{aligned}
\|x_t - u\|^2 &= \left\langle t(f(x_t) - u) + (1-t)(Tx_t - u), j(x_t - u)\right\rangle \\
&= t\left\langle f(x_t) - u, j(x_t - u)\right\rangle + (1-t)\left\langle Tx_t - u, j(x_t - u)\right\rangle \\
&= t\left\langle f(x_t) - f(u), j(x_t - u)\right\rangle + t\langle f(u) - u, j(x_t - u)\rangle \\
&\quad + (1-t)\left\langle Tx_t - Tu, j(x_t - u)\right\rangle \\
&\leq t\left\langle f(x_t) - f(u), j(x_t - u)\right\rangle + t\left\langle f(u) - u, j(x_t - u)\right\rangle + (1-t)\|x_t - u\|^2 \\
&\leq \beta t\|x_t - u\|^2 + t\left\langle f(u) - u, j(x_t - u)\right\rangle + (1-t)\|x_t - u\|^2.
\end{aligned} \tag{2.2}$$

Hence

$$(1-\beta)\|x_t - u\|^2 \leq \left\langle f(u) - u, j(x_t - u)\right\rangle \leq \|f(u) - u\| \cdot \|x_t - u\|. \tag{2.3}$$

By (2.3), we get

$$\|x_t - u\| \leq \frac{1}{1-\beta}\|f(u) - u\|, \tag{2.4}$$

so that $\{x_t : 0 < t < 1\}$ is bounded.

(ii) As u is a fixed point of T, from $x_t = tf(x_t) + (1-t)Tx_t$, we get

$$\begin{aligned}
\left\langle x_t - f(x_t), j(x_t - u)\right\rangle &= (1-t)\left\langle Tx_t - f(x_t), j(x_t - u)\right\rangle \\
&= -(1-t)\left\langle (I-T)x_t - (I-T)u, j(x_t - u)\right\rangle \\
&\quad + (1-t)\left\langle x_t - f(x_t), j(x_t - u)\right\rangle \quad \text{(using (1.3))} \\
&\leq (1-t)\left\langle x_t - f(x_t), j(x_t - u)\right\rangle.
\end{aligned} \tag{2.5}$$

Therefore $\langle x_t - f(x_t), j(x_t - u)\rangle \leq 0$. The proof is complete. □

THEOREM 2.2. *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm. Suppose K is a nonempty closed convex subset of E and $T : K \to K$ is a continuous pseudocontractive mapping. Let $f : K \to K$ be a fixed Lipschitzian strongly pseudocontracitve map from K to K. Every nonempty closed bounded convex subset of K has the fixed point property for nonexpansive self-mappings. $\{x_t\}$ (for all $t \in (0,1)$) is defined by (1.7). Then $\{x_t : 0 < t < 1\}$ is bounded if and only if, as $t \to 0$, x_t converges strongly to a fixed point p of T such that p is the unique solution in $F(T)$ to the following variational inequality:*

$$\langle (I-f)p, j(p-u)\rangle \leq 0 \quad \forall u \in F(T). \tag{2.6}$$

Proof. At first, by Lemma 2.1(i), the sufficiency is obvious.

Secondly, we show the necessity. Since $\{x_t : 0 < t < 1\}$ is bounded, f are Lipschitzian mappings, the sets $\{f(x_t) : t \in (0,1)\}$ are bounded. By $x_t = tf(x_t) + (1-t)Tx_t$, we have

$$\begin{aligned} Tx_t &= \frac{1}{1-t}x_t - \frac{t}{1-t}f(x_t), \\ \|Tx_t\| &\leq \frac{1}{1-t}\|x_t\| + \frac{t}{1-t}\|f(x_t)\|. \end{aligned} \tag{2.7}$$

Therefore, the sets $\{Tx_t\}$ are also bounded (using $t \to 0$). This implies that

$$\lim_{t\to 0}\|x_t - Tx_t\| = \lim_{t\to 0} t\|Tx_t - f(x_t)\| = 0. \tag{2.8}$$

We first observe that the mapping $2I - T$ has a nonexpansive inverse, denoted by $A = (2I - T)^{-1}$, where I denotes the identity operator, then $F(T) = F(A)$ (see [1, 5]). By [3, Theorem 6], we get that A is a nonexpansive self-mapping on K. Using $A = (2I - T)^{-1}$, we obtain

$$\begin{gathered} x_t - Tx_t = (2I - T)x_t - x_t = A^{-1}x_t - x_t, \quad x_t = AA^{-1}x_t, \\ \|x_t - Ax_t\| = \|AA^{-1}x_t - Ax_t\| \leq \|A^{-1}x_t - x_t\| = \|x_t - Tx_t\|. \end{gathered} \tag{2.9}$$

Since $\lim_{t\to 0}\|x_t - Tx_t\| = 0$, we have

$$\lim_{t\to 0}\|x_t - Ax_t\| = 0. \tag{2.10}$$

We claim that the set $\{x_t : t \in (0,1)\}$ is relatively compact. In fact, let $\{t_n\}$ be a sequence in $(0,1)$ that converges to 0 $(n \to \infty)$, put $x_n := x_{t_n}$,

$$g(x) = \mu_n\|x_n - x\|^2 \quad \forall x \in K, \tag{2.11}$$

where μ is a Banach limit. Define the set

$$K_1 = \left\{x \in K : g(x) = \inf_{y\in K} g(y)\right\}. \tag{2.12}$$

Since E is a reflexive Banach space, K_1 is a nonempty bounded closed convex subset of E (for more details, see [5]), and since

$$\lim_{n\to\infty}\|x_n - Ax_n\| = 0, \tag{2.13}$$

for all $x \in K_1$, we get

$$g(Ax) = \mu_n\|x_n - Ax\|^2 = \mu_n\|Ax_n - Ax\|^2 \leq \mu_n\|x_n - x\|^2 = g(x). \tag{2.14}$$

Hence, $Ax \in K_1$, that is, K_1 is invariant under A. Since every nonempty closed bounded convex subset of K has the fixed point property for nonexpansive self-mappings, there is a fixed point $p \in K_1$ of A. By $F(T) = F(A)$, p is also a fixed point of T. Using Lemma 1.1, we get

$$\mu_n \left\langle x - p, j(x_n - p) \right\rangle \leq 0 \quad \forall x \in K. \tag{2.15}$$

By (2.3), and taking $x = f(p)$, we have

$$\mu_n \|x_n - p\|^2 \leq \frac{1}{1-\beta} \mu_n \left\langle f(p) - p, j(x_n - p) \right\rangle \leq 0, \tag{2.16}$$

that is,

$$\mu_n \|x_n - p\|^2 = 0. \tag{2.17}$$

We have proved that for any sequence $\{x_{t_n}\}$ in $\{x_t : t \in (0,1)\}$, there exists a subsequence which is still denoted by $\{x_{t_n}\}$ that converges to some fixed point p of T. To prove that the entire net $\{x_t\}$ converges to p, supposed that there exists another sequence $\{x_{s_k}\} \subset \{x_t\}$ such that $x_{s_k} \to q$, as $s_k \to 0$, then we also have $q \in F(T)$ (using $\lim_{t \to 0} \|x_t - Tx_t\| = 0$). Next we show that $p = q$ and p is the unique solution in $F(T)$ to the following variational inequality:

$$\langle (I - f)p, j(p - u) \rangle \leq 0 \quad \forall u \in F(T). \tag{2.18}$$

Since the sets $\{x_t - u\}$ and $\{x_t - f(x_t)\}$ are bounded and the duality map J is single-valued and norm to weak* uniformly continuous on bounded sets of a Banach space E with uniformly Gâteaux differentiable norm, for any $u \in F(T)$, by $x_{s_k} \to q (s_k \to 0)$, we have

$$\begin{aligned}
&\|(I - f)x_{s_k} - (I - f)q\| \longrightarrow 0 \quad (s_k \longrightarrow 0), \\
&\left| \langle x_{s_k} - f(x_{s_k}), j(x_{s_k} - u) \rangle - \langle (I - f)q, j(q - u) \rangle \right| \\
&\quad = \left| \langle (I - f)x_{s_k} - (I - f)q, j(x_{s_k} - u) \rangle + \langle (I - f)q, j(x_{s_k} - u) - j(q - u) \rangle \right| \\
&\quad \leq \|(I - f)x_{s_k} - (I - f)q\| \|x_{s_k} - u\| \\
&\qquad + \left| \langle (I - f)q, j(x_{s_k} - u) - j(q - u) \rangle \right| \longrightarrow 0 \quad \text{as } s_k \longrightarrow 0.
\end{aligned} \tag{2.19}$$

Therefore, noting Lemma 2.1(ii), for any $u \in F(T)$, we get

$$\langle (I - f)q, j(q - u) \rangle = \lim_{s_k \to 0} \left\langle x_{s_k} - f(x_{s_k}), j(x_{s_k} - u) \right\rangle \leq 0. \tag{2.20}$$

Similarly, we also can show that

$$\langle (I - f)p, j(p - u) \rangle = \lim_{n \to \infty} \left\langle x_{t_n} - f(x_{t_n}), j(x_{t_n} - u) \right\rangle \leq 0. \tag{2.21}$$

Interchange p and u to obtain

$$\langle (I-f)q, j(q-p)\rangle \le 0. \tag{2.22}$$

Interchange q and u to obtain

$$\langle (I-f)p, j(p-q)\rangle \le 0. \tag{2.23}$$

This implies that (using (1.4))

$$(1-\beta)\|p-q\|^2 \le \langle (I-f)p-(I-f)q, j(p-q)\rangle \le 0. \tag{2.24}$$

We must have $p=q$. The proof is complete. □

Since every bounded nonempty closed convex subset with normal structure of the reflexive Banach space has the fixed point property for nonexpansive self-mappings [1, 5], and if $F(T)\neq\emptyset$, by Lemma 2.1(i), we have $\{x_t : 0<t<1\}$ is bounded, so that we can obtain the following corollary.

Corollary 2.3. *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm. Suppose K is a nonempty closed convex subset of E with normal structure and $T : K \to K$ is a continuous pseudocontractive mapping such that $F(T)\neq\emptyset$. Let $f : K \to K$ be a fixed Lipschitzian strongly pseudocontracitve map from K to K. Then, as $t\to 0$, $\{x_t\}$ $(t\in(0,1))$ defined by (1.7) converges strongly to a fixed point p of T such that p is the unique solution in $F(T)$ to the following variational inequality:*

$$\langle (I-f)p, j(p-u)\rangle \le 0 \quad \forall u\in F(T). \tag{2.25}$$

Since every nonempty closed convex subset of a uniformly convex Banach space has normal structure [1, 5], we can also obtain the following corollary.

Corollary 2.4. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Suppose K is a nonempty closed convex subset of E and $T : K \to K$ is a continuous pseudocontractive mapping such that $F(T)\neq\emptyset$. Let $f : K \to K$ be a fixed Lipschitzian strongly pseudocontracitve map from K to K. Then, as $t\to 0$, $\{x_t\}$ $(t\in(0,1))$ defined by (1.7) converges strongly to a fixed point p of T such that p is the unique solution in $F(T)$ to the following variational inequality:*

$$\langle (I-f)p, j(p-u)\rangle \le 0 \quad \forall u\in F(T). \tag{2.26}$$

Every bounded nonempty closed convex subset of uniformly smooth Banach space has normal structure [2, Lemma 8], and every uniformly smooth Banach space is a reflexive Banach space with uniformly Gâteaux differentiable norm. So that we can also obtain the following corollary.

Corollary 2.5. *Let E be a uniformly smooth Banach space. Suppose K is a nonempty closed convex subset of E and $T : K \to K$ is a continuous pseudocontractive mapping such that $F(T)\neq\emptyset$. Let $f : K \to K$ be a fixed Lipschitzian strongly pseudocontracitve map from*

K to K. Then, as $t \to 0$, $\{x_t\}$ $(t \in (0,1))$ defined by (1.7) converges strongly to a fixed point p of T such that p is the unique solution in $F(T)$ to the following variational inequality:

$$\langle (I-f)p, j(p-u)\rangle \leq 0 \quad \forall u \in F(T). \tag{2.27}$$

Recall the set A of M is a *Chebyshev set*, if for all $x \in M$, there exactly exists unique element $y \in A$ such that $d(x,y) = d(x,A)$, where (M,d) is a metric space and $d(x,A) = \inf_{y\in A} d(x,y)$. Every nonempty closed convex subsets of a strictly convex and reflexive Banach space E is a *Chebyshev set* [4, Corollary 5.1.19].

Theorem 2.6. *Let E be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. Suppose K is a nonempty closed convex subset of E and $T : K \to K$ is a continuous pseudocontractive mapping such that $F(T) \neq \emptyset$. Let $f : K \to K$ be a fixed Lipschitzian strongly pseudocontracitve map from K to K. Then, as $t \to 0$, $\{x_t\}$ $(t \in (0,1))$ defined by (1.7) converges strongly to a fixed point p of T such that p is the unique solution in $F(T)$ to the following variational inequality:*

$$\langle (I-f)p,\ j(p-u)\rangle \leq 0 \quad \forall u \in F(T). \tag{2.28}$$

Proof. By $F(T) \neq \emptyset$ and Lemma 2.1(i), we have that $\{x_t : 0 < t < 1\}$ is bounded. Using the same proof for the necessity of Theorem 2.2, we can find

$$K_1 = \left\{x \in K : g(x) = \inf_{y\in K} g(y)\right\}, \tag{2.29}$$

K_1 is a nonempty bounded closed convex subset of E, and K_1 is invariant under A. Now we just need to show that the set K_1 contains a fixed point of A. Since $F(T) = F(A) \neq \emptyset$, let u be one of those. Since every nonempty closed convex subsets of a strictly convex and reflexive Banach space E is a *Chebyshev set*, there exists a unique $p \in K_1$ such that

$$\|u - p\| = \inf_{x\in K_1} \|u - x\|. \tag{2.30}$$

Next we show that $p = Ap = Tp$. By $u = Au$ and $Ap \in K_1$,

$$\|u - Ap\| = \|Au - Ap\| \leq \|u - p\|. \tag{2.31}$$

Hence $p = Ap$. The rest of the proof follows from Theorem 2.2. The proof is complete. □

Remark 2.7. We remark that Theorem 2.6 appears to be independent of Theorem 2.2. On the one hand, it is easy to find examples of spaces which satisfy the fixed point property for nonexpansive self-mappings, which are not strictly convex. On the other hand, it appears to be unknown whether a reflexive and strictly convex Banach space satisfies the fixed point property for nonexpansive self-mappings.

Acknowledgment

This work is supported by the National Natural Science Foundation of China Grant no. 10471033.

References

[1] S.-S. Chang, Y. J. Cho, and H. Zhou, *Iterative Methods for Nonlinear Operator Equations in Banach Spaces*, Nova Science, New York, 2002.

[2] J. Gao, *Modulus of convexity in Banach spaces*, Applied Mathematics Letters **16** (2003), no. 3, 273–278.

[3] R. H. Martin Jr., *Differential equations on closed subsets of a Banach space*, Transactions of the American Mathematical Society **179** (1973), 399–414.

[4] R. E. Megginson, *An Introduction to Banach Space Theory*, Graduate Texts in Mathematics, vol. 183, Springer, New York, 1998.

[5] W. Takahashi, *Nonlinear Functional Analysis. Fixed Point Theory and Its Applications*, Yokohama Publishers, Yokohama, 2000.

[6] W. Takahashi and Y. Ueda, *On Reich's strong convergence theorems for resolvents of accretive operators*, Journal of Mathematical Analysis and Applications **104** (1984), no. 2, 546–553.

[7] H.-K. Xu, *Viscosity approximation methods for nonexpansive mappings*, Journal of Mathematical Analysis and Applications **298** (2004), no. 1, 279–291.

Yisheng Song: College of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, China
E-mail address: songyisheng123@yahoo.com.cn

Rudong Chen: Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China
E-mail address: chenrd@tjpu.edu.cn