WEIGHTED WEAK-TYPE INEQUALITIES FOR GENERALIZED HARDY OPERATORS

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We characterize the pairs of weights (v,w) for which the Hardy-Steklov-type operator $Tf(x) = g(x)\int_{s(x)}^{h(x)} K(x,y)f(y)dy$ applies $L^p(v)$ into weak-$L^q(w)$, $q < p$, assuming certain monotonicity conditions on g, s, h, and K.

1. Introduction

Let us consider the Hardy-Steklov-type operator defined by

$$Tf(x) = g(x)\int_{s(x)}^{h(x)} K(x,y)f(y)dy, \quad f \geq 0, \tag{1.1}$$

where g is a nonnegative measurable function, s and h are continuous and increasing functions ($x < y \Rightarrow s(x) \leq s(y)$, $h(x) \leq h(y)$) defined on an interval (a,b) such that $s(x) \leq h(x)$ for all $x \in (a,b)$, and the kernel $K(x,y)$ defined on $\{(x,y) : x \in (a,b)$ and $s(x) \leq y \leq h(x)\}$ satisfies

(i) $K(x,y) \geq 0$,

(ii) it is increasing and continuous in x and decreasing in y,

(iii) $K(x,z) \leq D[K(x,h(y)) + K(y,z)]$ for $y \leq x$ and $s(x) \leq z \leq h(y)$, where the constant $D > 1$ is independent of x, y, and z.

Gogatishvili and Lang [3] characterized the pairs of weights for the strong- and weak-type (p,q) inequalities for the operator T in the case $p \leq q$. Actually, in [3] the authors deal with Banach functions spaces with some extra condition. On the other hand, Chen and Sinnamon [2] have characterized the weighted strong-type inequality for $1 < p$, $q < \infty$ in terms of a normalizing measure. In both papers, they work with more general functions s, h, and K.

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The goal of this paper is to characterize the weighted weak-type inequalities in the case $q < p$. It is well known that strong-type inequalities for the operator T can be deduced directly from the corresponding ones for $g(x) = 1$, but this is not the case when we work with weak-type inequalities. In [5] it was characterized the weighted weak-type inequality in the case $q < p$ for the operator T when $s \equiv 0$, $h(x) = x$, and $K \equiv 1$. The result was obtained for monotone functions g. In fact, in the proof of the result the authors used the condition

$$\inf_{x \in E} g(x) = \inf_{x \in (\alpha, \beta)} g(x) \tag{1.2}$$

for any bounded set E, where $\alpha = \inf E$ and $\beta = \sup E$. This property clearly holds if g is monotone or if there exists x_0 such that g is increasing in $(a, x_0]$ and decreasing in $[x_0, b)$. In our result, we will assume (1.2) and the same condition for the function $g(x)K(x,y)$, that is, for all y and every bounded set $E_y \subset \{x : s(x) \leq y \leq h(x)\}$,

$$\inf_{x \in E_y} [g(x)K(x,y)] = \inf_{x \in (\alpha_y, \beta_y)} [g(x)K(x,y)], \tag{1.3}$$

where $\alpha_y = \inf E_y$ and $\beta_y = \sup E_y$.

Examples of Hardy-Steklov-type operators are the modified Riemann-Liouville operators defined for $\alpha > 0$ and $\eta \in \mathbb{R}$ as $x^\eta \int_0^x (x-y)^\alpha f(y)dy$ or the more general version $x^\eta \int_{Ax}^{Bx} (x-y)^\alpha f(y)dy$, with $0 < A < B \leq 1$ and $x > 0$; the modified logarithmic kernel operators $x^\eta \int_0^x \log^\beta (x/y) f(y)dy$, with $\beta > 0$ and $\eta \in \mathbb{R}$; the Steklov operator $Tf(x) = \int_{x-1}^{x+1} f$; and the Riemann-Liouville operators, with general variable limits $\int_{s(x)}^{h(x)} (x-y)^\alpha f(y)dy$, with $s(x) \leq h(x) \leq x$. This last operator was studied in [6] in the case $-1 < \alpha < 0$.

As far as we know, our result is new even for the particular cases $Tf(x) = g(x) \int_0^x K(x,y)f(y)dy$ and $Tf(x) = \int_{s(x)}^{h(x)} K(x,y)f(y)dy$. For this last operator, conditions (1.2) and (1.3) hold trivially because $K(x,y)$ is increasing in x.

The notation is standard: $w(E)$ denotes the integral $\int_E w$; if $1 < p < \infty$, then p' denotes the conjugate exponent of p defined by $1/p + 1/p' = 1$, and $L^{q,\infty}(w)$ will denote the space of measurable functions f such that

$$\|f\|_{q,\infty;w} = \sup_{\lambda > 0} \lambda \big(w(\{x : |f(x)| > \lambda\}) \big)^{1/q} < \infty. \tag{1.4}$$

2. Statement and proof of the result

In the next theorem we state the result of this article.

Theorem 2.1. *Let s and h be increasing continuous functions defined on an interval (a,b) satisfying $s(x) \leq h(x)$ for $x \in (a,b)$. Let $K(x,y)$ be defined on $\{(x,y) : x \in (a,b)$ and $s(x) \leq y \leq h(x)\}$ satisfying (i), (ii), (iii) and let g be a nonnegative function defined on (a,b) satisfying (1.2) and (1.3). Let q, p, and r be such that $0 < q < p$, $1 < p < \infty$, and $1/r = 1/q - 1/p$.*

Let w and v be nonnegative measurable functions defined on (a,b) and $(s(a),h(b))$, respectively. The following statements are equivalent.

(i) *There exists a positive constant C such that*

$$[w(\{x\in(a,b):Tf(x)>\lambda\})]^{1/q}\leq\frac{C}{\lambda}\left(\int_{s(a)}^{h(b)}f^{p}v\right)^{1/p}\tag{2.1}$$

for all $f\geq 0$ and all positive real number λ.

(ii) *The functions*

$$\Phi_1(x)=\sup\left\{\inf_{t\in(c,d)}[g(t)K(t,h(\bar{c}))]\left(\int_c^d w\right)^{1/p}\left(\int_{s(d)}^{h(\bar{c})}v^{1-p'}\right)^{1/p'}\right\},\tag{2.2}$$

where the supremum is taken over all the numbers $\bar{c}$, c, and d such that $a\leq\bar{c}\leq c<x<d\leq b$ and $s(d)\leq h(\bar{c})$ and

$$\Phi_2(x)=\sup\left\{\left(\inf_{t\in(c,d)}g(t)\right)\left(\int_c^d w\right)^{1/p}\left(\int_{s(d)}^{h(c)}K^{p'}(c,y)v^{1-p'}(y)dy\right)^{1/p'}\right\},\tag{2.3}$$

where the supremum is taken over all the numbers c and d such that $a\leq c<x<d\leq b$ and $s(d)\leq h(c)$, belong to $L^{r,\infty}(w)$.

Let us observe that if $g\equiv 1$, we get that $\Phi_1\leq\Phi_2$. Then, in this case, the weighted weak-type inequality (i) is equivalent to $\Phi_2\in L^{r,\infty}(w)$. On the other hand, if $K\equiv 1$, then $\Phi_1=\Phi_2$ and we recover [1, Theorem 1.9].

To prove the theorem we will use the following lemma (see [1, Lemma 1.4] for the proof).

Lemma 2.2. *Let a and b be real numbers such that $a<b$. Let $s,h:(a,b)\to\mathbb{R}$ be increasing and continuous functions such that $s(x)\leq h(x)$ for all $x\in(a,b)$. Let $\{(a_j,b_j)\}_j$ be the connected components of the open set $\Omega=\{x\in(a,b):s(x)<h(x)\}$. Then*

(a) *$(s(a_j),h(b_j))\cap(s(a_i),h(b_i))=\emptyset$ for all $j\neq i$,*

(b) *for every j there exists a (finite or infinite) sequence $\{m_k^j\}$ of real numbers such that:*

   (i) *$a_j\leq m_k^j<m_{k+1}^j\leq b_j$ for all k and j;*

   (ii) *$(a_j,b_j)=\bigcup_k(m_k^j,m_{k+1}^j)$ a.e. for all j;*

   (iii) *$s(m_{k+1}^j)\leq h(m_k^j)$ for all k and j and $s(m_{k+1}^j)=h(m_k^j)$ if $a_j<m_k^j<m_{k+1}^j<b_j$.*

Proof of Theorem 2.1. (i)⇒(ii). First, we will prove that $\Phi_1\in L^{r,\infty}(w)$, that is, we will prove that

$$\sup_{\lambda>0}\lambda[w(\{x\in(a,b):\Phi_1(x)>\lambda\})]^{1/r}<\infty.\tag{2.4}$$

Let $\lambda>0$ and $S_\lambda=\{x\in(a,b):\Phi_1(x)>\lambda\}$. For every $z\in S_\lambda$ there exist $\bar{c}_z$, c_z, and d_z, with $a\leq\bar{c}_z\leq c_z<z<d_z\leq b$ such that $s(d_z)\leq h(\bar{c}_z)$ and

$$\lambda<\inf_{t\in(c_z,d_z)}[g(t)K(t,h(\bar{c}_z))]\left(\int_{c_z}^{d_z}w\right)^{1/p}\left(\int_{s(d_z)}^{h(\bar{c}_z)}v^{1-p'}\right)^{1/p'}.\tag{2.5}$$

Let $\mathcal{K} \subset S_\lambda$ be a compact set. Then there exist $(c_{z_1}, d_{z_1}), \dots, (c_{z_k}, d_{z_k})$ which cover $\mathcal{K}$. We may assume without loss of generality that $\sum_{j=1}^{k} \chi_{(c_{z_j}, d_{z_j})} \le 2\chi_{\bigcup_{j=1}^{k}(c_{z_j}, d_{z_j})}$. Let $f : (s(a), h(b)) \to \mathbb{R}$ defined by

$$f(y) = \left(\sum_{j=1}^{k} \frac{v^{-p'}(y)\chi_{(s(d_{z_j}),h(\bar{c}_{z_j}))}(y)}{\left(\inf_{t\in(c_{z_j},d_{z_j})}\left[g(t)K(t,h(\bar{c}_{z_j}))\right]\int_{s(d_{z_j})}^{h(\bar{c}_{z_j})} v^{1-p'}\right)^{p}} \right)^{1/p}. \tag{2.6}$$

If $z \in (c_{z_j}, d_{z_j})$, then $(s(d_{z_j}), h(\bar{c}_{z_j})) \subset (s(z), h(z))$ and since $K(z,y)$ is decreasing in y, we get that

$$Tf(z) \ge g(z)\int_{s(d_{z_j})}^{h(\bar{c}_{z_j})} K(z,y)f(y)dy \ge g(z)K(z,h(\bar{c}_{z_j}))\int_{s(d_{z_j})}^{h(\bar{c}_{z_j})} f(y)dy \ge 1. \tag{2.7}$$

Therefore, $\bigcup_{j=1}^{k}(c_{z_j}, d_{z_j}) \subset \{x \in (a,b) : Tf(x) \ge 1\}$. Applying the weighted weak-type inequality and (2.5) we obtain

$$\begin{aligned}
\int_{\bigcup_{j=1}^{k}(c_{z_j},d_{z_j})} w &\le C\left(\sum_{j=1}^{k} \frac{\int_{s(d_{z_j})}^{h(\bar{c}_{z_j})} v^{1-p'}}{\left(\inf_{t\in(c_{z_j},d_{z_j})}\left[g(t)K(t,h(\bar{c}_{z_j}))\right]\int_{s(d_{z_j})}^{h(\bar{c}_{z_j})} v^{1-p'}\right)^{p}} \right)^{q/p} \\
&= C\left(\sum_{j=1}^{k} \frac{1}{\inf_{t\in(c_{z_j},d_{z_j})}\left[g(t)K(t,h(\bar{c}_{z_j}))\right]^{p}\left(\int_{s(d_{z_j})}^{h(\bar{c}_{z_j})} v^{1-p'}\right)^{p-1}} \right)^{q/p} \\
&\le \frac{C}{\lambda^{q}}\left(\sum_{j=1}^{k} \int_{c_{z_j}}^{d_{z_j}} w \right)^{q/p} \\
&\le \frac{C}{\lambda^{q}}\left(\int_{\bigcup_{j=1}^{k}(c_{z_j},d_{z_j})} w \right)^{q/p}.
\end{aligned} \tag{2.8}$$

The last inequality implies that $\lambda(\int_{\mathcal{K}} w)^{1/r} \le C$ for any compact set $\mathcal{K} \subset S_\lambda$ which implies (2.4). The proof of (2.4) for the function Φ_2 follows in a similar way applying (i) to the function

$$f(y) = \left(\sum_{j=1}^{k} \frac{K^{p'}(c_{z_j},y)v^{-p'}(y)\chi_{(s(d_{z_j}),h(c_{z_j}))}(y)}{\left(\inf_{t\in(c_{z_j},d_{z_j})} g(t)\int_{s(d_{z_j})}^{h(c_{z_j})} K^{p'}(c_{z_j},t)v^{1-p'}(t)dt\right)^{p}} \right)^{1/p}. \tag{2.9}$$

(ii)⇒(i). Let $\{a^N\}_{N=1}^{\infty}$ and $\{b^N\}_{N=1}^{\infty}$ be sequences in (a,b) such that

$$\lim_{N\to\infty} a^N = a, \qquad \lim_{N\to\infty} b^N = b. \tag{2.10}$$

In order to prove (i) it will suffice to show that

$$w(\{x \in (a^N, b^N) : Tf(x) > \lambda\}) \leq \frac{C}{\lambda^q} \tag{2.11}$$

for all nonnegative function f bounded with compact support such that $\int_{s(a)}^{h(b)} f^p v = 1$ and with a constant C independent of N, λ, and f.

Let us fix $N \in \mathbb{N}$. Observe that if $O_\lambda = \{x \in (a^N, b^N) : Tf(x) > \lambda\}$ and $U = \{x \in (a,b) : \Phi_1(x) \leq \lambda^{q/r}, \Phi_2(x) \leq \lambda^{q/r}\}$, then

$$\begin{aligned} w(O_\lambda) &\leq w(O_\lambda \cap U) + w(\{x \in (a,b) : \Phi_1(x) > \lambda^{q/r}\}) \\ &\quad + w(\{x \in (a,b) : \Phi_2(x) > \lambda^{q/r}\}) \\ &\leq w(O_\lambda \cap U) + \frac{\|\Phi_1\|_{r,\infty;w}^r}{\lambda^q} + \frac{\|\Phi_2\|_{r,\infty,;w}^r}{\lambda^q}. \end{aligned} \tag{2.12}$$

Therefore, the implication will be proved if we establish that $w(O_\lambda \cap U) \leq C/\lambda^q$. Let (a_j, b_j) and $\{m_k^j\}$ be the sequences given by the lemma for the set $\Omega_N = \{x \in (a^N, b^N) : s(x) < h(x)\}$. Then, for fixed j,

$$w(O_\lambda \cap U \cap (a_j, b_j)) = \sum_k w(O_\lambda \cap U \cap (m_k^j, m_{k+1}^j)). \tag{2.13}$$

If $x \in (m_k^j, m_{k+1}^j)$, since $s(m_{k+1}^j) \leq h(m_k^j)$, we get that

$$\begin{aligned} Tf(x) &= g(x) \int_{s(x)}^{s(m_{k+1}^j)} K(x,y) f(y)dy + g(x) \int_{s(m_{k+1}^j)}^{h(m_k^j)} K(x,y) f(y)dy \\ &\quad + g(x) \int_{h(m_k^j)}^{h(x)} K(x,y) f(y)dy = T_{j,k}^1 f(x) + T_{j,k}^2 f(x) + T_{j,k}^3 f(x). \end{aligned} \tag{2.14}$$

It is clear that

$$w(O_\lambda \cap U \cap (m_k^j, m_{k+1}^j)) \leq w(E^1) + w(E^2) + w(E^3), \tag{2.15}$$

where $E^\ell = \{x \in (m_k^j, m_{k+1}^j) \cap U : T_{j,k}^\ell f(x) > \lambda/3\}$, $\ell = 1,2,3$.

First, notice that the property (*iii*) of the kernel K implies

$$K(x,y) \leq D[K(x, h(m_k^j)) + K(m_k^j, y)] \tag{2.16}$$

for $x \in (m_k^j, m_{k+1}^j)$ and $y \in (s(m_{k+1}^j), h(m_k^j))$.

In order to estimate $w(E^1)$ let us observe that

$$\begin{aligned} T^1_{j,k}f(x) &\le Dg(x)K\big(x,h(m^j_k)\big)\int_{s(x)}^{s(m^j_{k+1})} f(y)dy \\ &\quad + Dg(x)\int_{s(x)}^{s(m^j_{k+1})} K\big(m^j_k,y\big)f(y)dy = DT^{1,1}_{j,k}f(x)+DT^{1,2}_{j,k}f(x). \end{aligned} \tag{2.17}$$

Then, $w(E^1)\le w(E^{1,1})+w(E^{1,2})$, where

$$E^{1,\ell}=\Big\{x\in (m^j_k,m^j_{k+1})\cap U : T^{1,\ell}_{j,k}f(x)>\frac{\lambda}{6D}\Big\},\quad \ell=1,2. \tag{2.18}$$

Let us select an increasing sequence $\{x_i\}_i$, $x_i\in(m^j_k,m^j_{k+1})$, such that $x_0=m^j_k$ and

$$\int_{s(x_i)}^{s(m^j_{k+1})} f=\int_{s(x_{i-1})}^{s(x_i)} f. \tag{2.19}$$

Let $E^{1,1}_i=E^{1,1}\cap(x_i,x_{i+1})$, $\alpha^1_i=\inf E^{1,1}_i$, and $\beta^1_i=\sup E^{1,1}_i$. If $E^{1,1}_i\ne\emptyset$, let $t\in E^{1,1}_i$. Using the property of the sequence $\{x_i\}_i$ we have

$$\frac{\lambda}{6D}\le 4g(t)K\big(t,h(m^j_k)\big)\int_{s(x_{i+1})}^{s(x_{i+2})} f. \tag{2.20}$$

Now, by using (1.3) and Hölder inequality we get

$$\frac{\lambda}{6D}\le 4\inf_{t\in(\alpha^1_i,\beta^1_i)}\big[g(t)K\big(t,h(m^j_k)\big)\big]\bigg(\int_{s(x_{i+1})}^{s(x_{i+2})} v^{1-p'}\bigg)^{1/p'}\bigg(\int_{s(x_{i+1})}^{s(x_{i+2})} f^p v\bigg)^{1/p}. \tag{2.21}$$

Now, multiplying by $(\int_{\alpha^1_i}^{\beta^1_i} w)^{1/p}$ and using the inequalities $s(\beta^1_i)\le s(x_{i+1})$ and $s(x_{i+2})\le s(m^j_{k+1})\le h(m^j_k)$ we get that

$$\frac{\lambda}{6D}\bigg(\int_{\alpha^1_i}^{\beta^1_i} w\bigg)^{1/p}\le 4\Phi_1(x)\bigg(\int_{s(x_{i+1})}^{s(x_{i+2})} f^p v\bigg)^{1/p}\le 4\lambda^{q/r}\bigg(\int_{s(x_{i+1})}^{s(x_{i+2})} f^p v\bigg)^{1/p}, \tag{2.22}$$

where x is any element of $E^{1,1}_i$; and summing up in i we obtain

$$w(E^{1,1})\le\frac{C}{\lambda^q}\int_{s(m^j_k)}^{s(m^j_{k+1})} f^p v. \tag{2.23}$$

To estimate $w(E^{1,2})$, we select an increasing sequence $\{z_i\}_i$, $z_i\in(m^j_k,m^j_{k+1})$ such that $z_0=m^j_k$ and

$$\int_{s(z_i)}^{s(m^j_{k+1})} K\big(m^j_k,y\big)f(y)dy=\int_{s(z_{i-1})}^{s(z_i)} K\big(m^j_k,y\big)f(y)dy. \tag{2.24}$$

As before, let $E_i^{1,2} = E^{1,2} \cap (z_i, z_{i+1})$, $\alpha_i^2 = \inf E_i^{1,2}$, and $\beta_i^2 = \sup E_i^{1,2}$. If $E_i^{1,2} \neq \emptyset$, then Hölder inequality and (1.2) give

$$\frac{\lambda}{6D} \leq 4 \inf_{t \in (\alpha_i^2, \beta_i^2)} g(t) \left(\int_{s(z_{i+1})}^{s(z_{i+2})} K^{p'}(m_k^j, t) v^{1-p'}(t) dt \right)^{1/p'} \left(\int_{s(z_{i+1})}^{s(z_{i+2})} f^p v \right)^{1/p}. \tag{2.25}$$

Notice that $s(\beta_i^2) \leq s(z_{i+1})$, $m_k^j \leq \alpha_i^2$, and $s(z_{i+2}) \leq s(m_{k+1}^j) \leq h(m_k^j) \leq h(\alpha_i^2)$. Then multiplying by $(\int_{\alpha_i^2}^{\beta_i^2} w)^{1/p}$ both members of the above inequality we get

$$\frac{\lambda}{6D} \left(\int_{\alpha_i^2}^{\beta_i^2} w \right)^{1/p} \leq 4\Phi_2(x) \left(\int_{s(z_{i+1})}^{s(z_{i+2})} f^p v \right)^{1/p} \leq 4\lambda^{q/r} \left(\int_{s(z_{i+1})}^{s(z_{i+2})} f^p v \right)^{1/p}, \tag{2.26}$$

where x is any element of $E_i^{1,2}$. Now, summing up in i and putting together with (2.23) we obtain

$$w(E^1) \leq \frac{C}{\lambda^q} \int_{s(m_k^j)}^{s(m_{k+1}^j)} f^p v. \tag{2.27}$$

To estimate $w(E^2)$ we proceed in a similar way. In fact, by using (2.16) we get that

$$\begin{aligned} T_{j,k}^2 f(x) &\leq Dg(x) K(x, h(m_k^j)) \int_{s(m_{k+1}^j)}^{h(m_k^j)} f(y) dy \\ &\quad + Dg(x) \int_{s(m_{k+1}^j)}^{h(m_k^j)} K(m_k^j, y) f(y) dy = DT_{j,k}^{2,1} f(x) + DT_{j,k}^{2,2} f(x), \end{aligned} \tag{2.28}$$

which implies that $w(E^2) \leq w(E^{2,1}) + w(E^{2,2})$, where the sets $E^{2,\ell}$, $\ell = 1,2$ are defined as the sets $E^{1,\ell}$ with $T_{j,k}^{2,\ell} f$ instead of $T_{j,k}^{1,\ell} f$. Now, the estimates of $w(E^{2,1})$ and $w(E^{2,2})$ follow as in the previous cases obtaining

$$w(E^2) \leq \frac{C}{\lambda^q} \int_{s(m_{k+1}^j)}^{h(m_k^j)} f^p v. \tag{2.29}$$

Actually, the estimations are easier because we do not need to split the sets $E^{2,\ell}$. For the estimation of $w(E^3)$ let us define the function

$$H(x) = \int_{h(m_k^j)}^{h(x)} K(x, y) f(y) dy. \tag{2.30}$$

Since h is continuous and K is continuous in the first variable, we may select a decreasing sequence $\{x_i\}_i$ in (m_k^j, m_{k+1}^j) such that $x_0 = m_{k+1}^j$ and $H(x_i) = \int_{h(m_k^j)}^{h(x_i)} K(x_i,y)f(y)dy = (D+1)^{-i}H(m_{k+1}^j)$. We claim that

$$H(x_i) \le (D+1)^4 \left(K(x_{i+2}, h(x_{i+3})) \int_{h(m_k^j)}^{h(x_{i+3})} f(y)dy + \int_{h(x_{i+3})}^{h(x_{i+2})} K(x_{i+2},y)f(y)dy \right). \tag{2.31}$$

In fact, first notice that

$$\begin{aligned} H(x_i) &= (D+1)^2 \int_{h(m_k^j)}^{h(x_{i+2})} K(x_{i+2},y)f(y)dy \\ &= (D+1)^2 \left[\int_{h(m_k^j)}^{h(x_{i+3})} K(x_{i+2},y)f(y)dy + \int_{h(x_{i+3})}^{h(x_{i+2})} K(x_{i+2},y)f(y)dy \right]. \end{aligned} \tag{2.32}$$

Now, applying property (iii) of K we get that

$$\begin{aligned} H(x_i) &\le D(D+1)^2 \left[K(x_{i+2}, h(x_{i+3})) \int_{h(m_k^j)}^{h(x_{i+3})} f(y)dy + \int_{h(m_k^j)}^{h(x_{i+3})} K(x_{i+3},y)f(y)dy \right] \\ &\quad + (D+1)^2 \int_{h(x_{i+3})}^{h(x_{i+2})} K(x_{i+2},y)f(y)dy \\ &\le (D+1)^3 \left[K(x_{i+2}, h(x_{i+3})) \int_{h(m_k^j)}^{h(x_{i+3})} f(y)dy + \int_{h(x_{i+3})}^{h(x_{i+2})} K(x_{i+2},y)f(y)dy \right] \\ &\quad + \frac{D}{D+1} H(x_i), \end{aligned} \tag{2.33}$$

and the claim follows. Now, we have

$$w(E^3) \le \sum_{i \ge 0} [w(E_i^{3,1}) + w(E_i^{3,2})], \tag{2.34}$$

where

$$\begin{aligned} E_i^{3,1} &= \left\{ x \in (x_{i+1}, x_i) \cap U : g(x) K(x_{i+2}, h(x_{i+3})) \int_{h(m_k^j)}^{h(x_{i+3})} f(y)dy > \frac{\lambda}{6(D+1)^4} \right\}, \\ E_i^{3,2} &= \left\{ x \in (x_{i+1}, x_i) \cap U : g(x) \int_{h(x_{i+3})}^{h(x_{i+2})} K(x_{i+2},y)f(y)dy > \frac{\lambda}{6(D+1)^4} \right\}. \end{aligned} \tag{2.35}$$

Working as in previous cases we have

$$\sum_{i \ge 0} w(E_i^{3,2}) \le \frac{C}{\lambda^q} \int_{h(m_k^j)}^{h(m_{k+1}^j)} f^p v. \tag{2.36}$$

In order to estimate $\sum_{i\geq 0} w(E_i^{3,1})$ we will apply the ideas of [4, Lemma 1]. Let $\{u_s'\}$ be the decreasing sequence in (m_k^j, m_{k+1}^j) defined by $u_0' = m_{k+1}^j$ and

$$\int_{h(m_k^j)}^{h(u_s')} f = 2^{-s}\int_{h(m_k^j)}^{h(m_{k+1}^j)} f, \tag{2.37}$$

and let $\{u_n\}$ be the subsequence of $\{u_s'\}$ defined by $u_0 = u_0'$ and if $[u_{s+1}', u_s') \cap \{x_i\} = \emptyset$, then we delete the term u_{s+1}' of $\{u_s'\}$. Let $\widetilde{E}_n^{3,1} = \bigcup_{\{i\geq 0:\ u_{n+1}\leq x_{i+3}<u_n\}} E_i^{3,1}$, $\widetilde{\alpha}_n = \inf \widetilde{E}_n^{3,1}$, and $\widetilde{\beta}_n = \sup \widetilde{E}_n^{3,1}$. If $u_{s+1}' = u_{n+1} \leq x_{i+3} < u_n$, by the construction of the sequences we get that $x_{i+3} \leq u_s'$ and $u_{n+2} \leq u_{s+2}'$, then

$$\int_{h(m_k^j)}^{h(x_{i+3})} f \leq \int_{h(m_k^j)}^{h(u_s')} f = 4\int_{h(u_{s+2}')}^{h(u_{s+1}')} f \leq 4\int_{h(u_{n+2})}^{h(u_{n+1})} f. \tag{2.38}$$

Let us assume that $\widetilde{E}_n^{3,1} \neq \emptyset$. By the above inequalities and the monotonicity of K we have for all $t \in \widetilde{E}_n^{3,1}$,

$$\frac{\lambda}{6(D+1)^4} \leq 4g(t)K\big(t, h(x_{i+3})\big)\int_{h(u_{s+2}')}^{h(u_{s+1}')} f \leq 4g(t)K\big(t, h(u_{n+1})\big)\int_{h(u_{n+2})}^{h(u_{n+1})} f. \tag{2.39}$$

Now, multiplying by $(\int_{\widetilde{\alpha}_n}^{\widetilde{\beta}_n} w)^{1/p}$, applying Hölder inequality, and using that $s(\widetilde{\beta}_n) \leq h(u_{n+2})$ we get that

$$\frac{\lambda}{6(D+1)^4}\left(\int_{\widetilde{\alpha}_n}^{\widetilde{\beta}_n} w\right)^{1/p} \leq 4\Phi_1(x)\left(\int_{h(u_{n+2})}^{h(u_{n+1})} f^p v\right)^{1/p} \leq 4\lambda^{q/r}\left(\int_{h(u_{n+2})}^{h(u_{n+1})} f^p v\right)^{1/p}, \tag{2.40}$$

where x is any point in $\widetilde{E}_n^{3,1}$. Then

$$\begin{aligned}\sum_{i\geq 0} w(E_i^{3,1}) &= \sum_n \sum_{\{i\geq 0:\ u_{n+1}\leq x_{i+3}<u_n\}} w(E_i^{3,1}) \\ &\leq \sum_n w(\widetilde{E}_n^{3,1}) \leq \sum_n \int_{\widetilde{\alpha}_n}^{\widetilde{\beta}_n} w \\ &\leq \frac{C}{\lambda^q}\sum_n \int_{h(u_{n+2})}^{h(u_{n+1})} f^p v \leq \frac{C}{\lambda^q}\int_{h(m_k^j)}^{h(m_{k+1}^j)} f^p v.\end{aligned} \tag{2.41}$$

Putting together the estimations of $w(E^1)$, $w(E^2)$, and $w(E^3)$ we have

$$w\big(O_\lambda \cap U \cap (m_k^j, m_{k+1}^j)\big) \leq \frac{C}{\lambda^q}\int_{s(m_k^j)}^{h(m_{k+1}^j)} f^p v. \tag{2.42}$$

Summing up in k in the above inequality and by (2.13) we get that

$$w(O_\lambda \cap U \cap (a_j, b_j)) \le \frac{C}{\lambda^q} \int_{s(a_j)}^{h(b_j)} f^p v. \tag{2.43}$$

Keeping in mind the lemma and summing up in j we obtain the desired inequality. □

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