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*Research Article*

# Existence and Asymptotic Behavior of Positive Solutions to $p(x)$-Laplacian Equations with Singular Nonlinearities

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This paper investigates the $p(x)$-Laplacian equations with singular nonlinearities $-\Delta_{p(x)}u = \lambda/u^{\gamma(x)}$ in $\Omega$, $u(x) = 0$ on $\partial\Omega$, where $-\Delta_{p(x)}u = -\text{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian. The existence of positive solutions is given, and the asymptotic behavior of solutions near boundary is discussed.



## 1. Introduction

The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions is a new and interesting topic. We refer to [1, 2], the background of these problems. Many results have been obtained on this kind of problems, for example, [2–13]. In [4, 7], Fan and Zhao give the regularity of weak solutions for differential equations with nonstandard $p(x)$-growth conditions. On the existence of solutions for $p(x)$-Laplacian problems in bounded domain, we refer to [5, 11, 12].

In this paper, we consider the $p(x)$-Laplacian equations with singular nonlinearities:

$$\begin{aligned} -\triangle_{p(x)}u &= \frac{\lambda}{u^{\gamma(x)}} \quad \text{in } \Omega, \\ u(x) &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{$P$}$$

where $-\triangle_{p(x)}u = -\text{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian, $\Omega \subset \mathbb{R}^N$ is a bounded domain with $C^2$ boundary $\partial\Omega$. If $p(x) \equiv p$ (a constant), then $(P)$ is the well-known $p$-Laplacian problem. There are many results on the existence of positive solutions for $p$-Laplacian problems with singular nonlinearities (see [14–18]), but the results on the existence of positive solutions for $p(x)$-Laplacian problems with singular nonlinearities are rare. Our aim is to give the existence of positive solutions for problem $(P)$, and give the asymptotic behavior of positive solutions near boundary.

Throughout the paper, we assume that $0 < \gamma(x) \in C(\overline{\Omega})$ and $p(x)$ satisfy

$(\mathrm{H}_1)$ $p(x) \in C^1(\overline{\Omega})$, $1 < p^- \le p^+ < +\infty$, where $p^- = \inf_\Omega p(x)$, $p^+ = \sup_\Omega p(x)$.

Because of the nonhomogeneity of $p(x)$-Laplacian, $p(x)$-Laplacian problems are more complicated than those of $p$-Laplacian ones, many results and methods for $p$-Laplacian problems are invalid for $p(x)$-Laplacian problems (see [6]), and another difficulty of this paper is that $f(x,u) = 1/u^{\gamma(x)}$ cannot be represented as $h(x)f(u)$. Our results partially generalized the results of [18].

## 2. Preliminary

In order to deal with $p(x)$-Laplacian problems, we need some theories on the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and properties of $p(x)$-Laplacian which we will use later (see [3, 8]). Let

$$L^{p(x)}(\Omega) = \left\{u \mid u \text{ is a measurable real-valued function, } \int_\Omega |u(x)|^{p(x)} dx < \infty\right\},$$
$$C_0^+(\Omega) = \{u \in C(\overline{\Omega}) \mid u > 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega\}. \tag{2.1}$$

We can introduce the norm on $L^{p(x)}(\Omega)$ by

$$|u|_{p(x)} = \inf\left\{\mu > 0 \mid \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1.\right\}. \tag{2.2}$$

The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ becomes a Banach space. We call it generalized Lebesgue space. The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is a separable, reflexive, and uniform convex Banach space (see [3, Theorems 1.10, Theorem 1.14]).

The space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \mid |\nabla u| \in L^{p(x)}(\Omega)\}, \tag{2.3}$$

and it can be equipped with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad \forall u \in W^{1,p(x)}(\Omega). \tag{2.4}$$

$W_0^{1,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable, reflexive, and uniform convex Banach spaces (see [3, Theorem 2.1]).

If $u \in W_{\mathrm{loc}}^{1,p(x)}(\Omega) \cap C_0^+(\Omega)$, $u$ is called a positive solution of $(P)$ if $u(x)$ satisfies

$$\int_Q |\nabla u|^{p(x)-2}\nabla u \nabla q\, dx - \int_Q \frac{\lambda}{u^{\gamma(x)}} q\, dx = 0, \quad \forall q \in W_0^{1,p(x)}(Q), \tag{2.5}$$

for any domain $Q \Subset \Omega$.

Let $W_{0,\mathrm{loc}}^{1,p(x)}(\Omega) = \{u \mid$ there is an open domain $Q \Subset \Omega$ s.t. $u \in W_0^{1,p(x)}(Q)\}$, and define $A : W_{\mathrm{loc}}^{1,p(x)}(\Omega) \cap C_0^+(\Omega) \to (W_{0,\mathrm{loc}}^{1,p(x)}(\Omega))^*$ as

$$\langle Au, \varphi \rangle = \int_\Omega \left( |\nabla u|^{p(x)-2} \nabla u \nabla \varphi - \frac{\lambda}{u^{\gamma(x)}} \varphi \right) dx, \tag{2.6}$$

where $u \in W_{\mathrm{loc}}^{1,p(x)}(\Omega) \cap C_0^+(\Omega)$, $\varphi \in W_{0,\mathrm{loc}}^{1,p(x)}(\Omega)$; then we have the following lemma.

Lemma 2.1 (see [5, Theorem 3.1]). *$A : W_{\mathrm{loc}}^{1,p(x)}(\Omega) \cap C_0^+(\Omega) \to (W_{0,\mathrm{loc}}^{1,p(x)}(\Omega))^*$ is strictly monotone.*

*Let $g \in (W_{0,\mathrm{loc}}^{1,p(x)}(\Omega))^*$, if $\langle g, \varphi \rangle \geq 0$, for all $\varphi \in W_{0,\mathrm{loc}}^{1,p(x)}(\Omega)$, $\varphi \geq 0$ a.e. in $\Omega$, then denote $g \geq 0$ in $(W_{0,\mathrm{loc}}^{1,p(x)}(\Omega))^*$; correspondingly, if $-g \geq 0$ in $(W_{0,\mathrm{loc}}^{1,p(x)}(\Omega))^*$, then denote $g \leq 0$ in $(W_{0,\mathrm{loc}}^{1,p(x)}(\Omega))^*$.*

*Definition 2.2.* Let $u \in W_{\mathrm{loc}}^{1,p(x)}(\Omega) \cap C_0^+(\Omega)$. If $Au \geq 0 (Au \leq 0)$ in $(W_{0,\mathrm{loc}}^{1,p(x)}(\Omega))^*$, then $u$ is called a weak supersolution (weak subsolution) of $(P)$.

Copying the proof of [10], we have the following lemma.

Lemma 2.3 (comparison principle). *Let $u, v \in W_{\mathrm{loc}}^{1,p(x)}(\Omega) \cap C(\Omega)$ be positive and satisfy $Au - Av \geq 0$ in $(W_{0,\mathrm{loc}}^{1,p(x)}(\Omega))^*$. Let $\varphi(x) = \min\{u(x) - v(x), 0\}$. If $\varphi(x) \in W_{0,\mathrm{loc}}^{1,p(x)}(\Omega)$ (i.e., $u \geq v$ on $\partial\Omega$), then $u \geq v$ a.e. in $\Omega$.*

Lemma 2.4 (see [7]). *If $g(x,u)$ is continuous on $\overline{\Omega} \times \mathbb{R}$, $u \in W^{1,p(x)}(\Omega)$ is a bounded weak solution of $-\triangle_{p(x)} u + g(x,u) = 0$ in $\Omega$, $u = w_0$ on $\partial\Omega$, where $w_0 \in W^{1,p(x)}(\Omega)$, then $u \in C_{\mathrm{loc}}^{1,\alpha}(\Omega)$, where $\alpha \in (0,1)$ is a constant.*

## 3. Existence of positive solutions

In order to deal with the existence of positive solutions, let us consider the problem

$$\begin{gathered} -\triangle_{p(x)} u = \frac{\lambda}{(|u| + a_n)^{\gamma(x)}} \quad \text{in } \Omega, \\ u(x) = 0 \quad \text{for } x \in \partial\Omega, \end{gathered} \tag{3.1}$$

where $\{a_n\}$ is a positive strictly decreasing sequence and $\lim_{n \to +\infty} a_n = 0$. We have the following lemma.

Lemma 3.1. *For any $n = 1, 2, \ldots$, problem (3.1) possesses a weak positive solution $\varpi_n \in C(\overline{\Omega})$.*

*Proof.* The relative functional of (3.1) is

$$\varphi = \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)} dx - \int_\Omega F_n(x,u) dx, \tag{3.2}$$

where $F_n(x,u) = \int_0^u \lambda / ((|t| + a_n)^{\gamma(x)}) dt$. Since $\varphi$ is coercive in $W_0^{1,p(x)}(\Omega)$, then $\varphi$ possesses a nontrivial minimum point $\varpi_n$, then $|\varpi_n|$ is also a nontrivial minimum point of problem (3.1), then (3.1) possesses a weak positive solution. The proof is completed. □

Here and hereafter, we will use the notation $d(x,\partial\Omega)$ to denote the distance of $x\in\Omega$ to the boundary of $\Omega$. Denote $d(x)=d(x,\partial\Omega)$ and $\partial\Omega_\epsilon=\{x\in\Omega\mid d(x)<\epsilon\}$. Since $\partial\Omega$ is $C^2$ regularly, then there exists a positive constant $\sigma$ such that $d(x)\in C^2(\overline{\partial\Omega_{2\sigma}})$, and $|\nabla d(x)|\equiv 1$. Let $\delta\in(0,(1/3)\sigma)$ be a small enough constant. Denote

$$v_1(x)=\begin{cases} d(x), & d(x)<\delta,\\ \delta+\displaystyle\int_\delta^{d(x)}\left(\frac{2\delta-t}{\delta}\right)^{2/(p^--1)}dt, & \delta\le d(x)<2\delta,\\ \delta+\displaystyle\int_\delta^{2\delta}\left(\frac{(2\delta-t)}{\delta}\right)^{2/(p^--1)}dt, & 2\delta\le d(x).\end{cases} \tag{3.3}$$

Obviously, $v_1(x)\in C^1(\overline{\Omega})\cap C_0^+(\Omega)$.

Lemma 3.2. *If $\lambda>0$ is large enough, then $v_1(x)$ is a subsolution of $(P)$.*

*Proof.* Since $|\nabla d(x)|\equiv 1$, when $\lambda>0$ is large enough, we have

$$-\triangle_{p(x)}v_1=-\triangle d(x)\le\frac{\lambda}{[v_1(x)]^{\gamma(x)}},\quad \forall x\in\Omega,\ d(x)<\delta. \tag{3.4}$$

By computation, when $\delta<d(x)<2\delta$, we have

$$\begin{aligned}-\triangle_{p(x)}v_1&=-\operatorname{div}\left\{\left[\left(\frac{2\delta-d(x)}{\delta}\right)^{2/(p^--1)}\right]^{p(x)-1}\nabla d(x)\right\}\\ &=-\left[\left(\frac{2\delta-d(x)}{\delta}\right)^{2/(p^--1)}\right]^{p(x)-1}\triangle d(x)\\ &\quad-\left[\left(\frac{2\delta-d(x)}{\delta}\right)^{2/(p^--1)}\right]^{p(x)-1}[\nabla d(x)\nabla p(x)]\ln\left(\frac{2\delta-d(x)}{\delta}\right)^{2/(p^--1)}\\ &\quad+\frac{2}{\delta}\frac{(p(x)-1)}{p^--1}\left[\frac{2\delta-d(x)}{\delta}\right]^{(2(p(x)-1)/(p^--1))-1}.\end{aligned} \tag{3.5}$$

When $\lambda>0$ is large enough, it is easy to see that

$$\begin{aligned}-\triangle_{p(x)}v_1&\le\frac{\lambda}{[v_1(x)]^{\gamma(x)}},\quad \forall x\in\Omega,\ \delta<d(x)<2\delta,\\ -\triangle_{p(x)}v_1&=0\le\frac{\lambda}{[v_1(x)]^{\gamma(x)}},\quad \forall x\in\Omega,\ 2\delta<d(x).\end{aligned} \tag{3.6}$$

From (3.4) and (3.6), we can conclude that $v_1(x)$ is a subsolution of $(P)$. ☐

Theorem 3.3. *If $\lambda > 0$ is a large enough constant, then problem $(P)$ possesses only one positive solution $u_\lambda$, and $u_\lambda$ is increasing with respect to $\lambda$.*

*Proof.* Denote $u_n = \varpi_n + a_n$, where $\varpi_n$ is a solution of (3.1). Since $\{u_n\}$ is a sequence of positive solutions of

$$-\triangle_{p(x)} u = \frac{\lambda}{u^{\gamma(x)}} \quad \text{in } \Omega, \qquad u(x) = a_n \quad \text{for } x \in \partial\Omega, \tag{II}$$

then every $u_n$ is subsolution and supersolution of $-\triangle_{p(x)} u = \lambda / u^{\gamma(x)}$ in $\Omega$. According to comparison principle, we have that $u_n \geq u_{n+1}$ for $n = 1,2,\ldots$. Since $v_1(x)$ is a subsolution of $(P)$ and $v_1(x) = 0$ on $\partial\Omega$, then $u_n \geq u_{n+1} \geq v_1$ for $n = 1,2,\ldots$. According to Lemma 2.4, we have that $\{u_n\}$ has uniform $C^{1,\alpha}$ local regularity property, and hence we can choose a subsequence, which we denoted by $\{u_n^1\}$, such that $u_n^1 \to w$ and $\nabla u_n^1 \to h$ in $\Omega$. In fact, $h = \nabla w$ in $\Omega$.

For any domain $D \Subset \Omega$, for any $\varphi \in W_0^{1,p(x)}(D)$. The $C^{1,\alpha}$ regularity result implies that the sequences $\{u_n\}$ and $\{\nabla u_n\}$ are equicontinuous in $D$; from the $C^{1,\alpha}$ estimate we conclude that $\nabla w \in C^{\alpha}(D)$ for some $0 < \alpha < 1$. Thus $w \in W^{1,p(x)}(D) \cap C^{1,\alpha}(D)$. From the $C^{1,\alpha}$ regularity result, we see that $|\nabla u_n^1|^{p-1} |\nabla \varphi| \leq C |\nabla \varphi|$ on $D$, and since the function $\xi \to |\xi|^{p-2}\xi$ is continuous on $\mathbb{R}^n$, it follows that $|\nabla u_n^1(x)|^{p-2} \nabla u_n^1(x) \cdot \nabla\varphi(x) \to |\nabla w(x)|^{p-2} \nabla w(x) \cdot \nabla \varphi(x)$ for $x \in D$. Thus, by the dominated convergence theorem, for any $\varphi \in W_0^{1,p(x)}(D)$, we can see that

$$\int_D \left| \nabla u_n^1(x) \right|^{p-2} \nabla u_n^1(x) \cdot \nabla \varphi(x) dx \longrightarrow \int_D \left| \nabla w(x) \right|^{p-2} \nabla w(x) \cdot \nabla \varphi(x) dx. \tag{3.7}$$

Furthermore, since $0 \leq \lambda / ([u_n^1(x)]^{\gamma(x)}) \leq \lambda / ([u_{n+1}^1(x)]^{\gamma(x)})$, and $\lambda / ([u_n^1(x)]^{\gamma(x)}) \to \lambda / ([w(x)]^{\gamma(x)})$ for each $x \in D$, by the monotone convergence theorem we obtain

$$\int_D \frac{\lambda}{\left[u_n^1(x)\right]^{\gamma(x)}} \varphi dx \longrightarrow \int_D \frac{\lambda}{\left[w(x)\right]^{\gamma(x)}} \varphi dx, \quad \forall \varphi \in W_0^{1,p(x)}(D). \tag{3.8}$$

Therefore, it follows that

$$\int_D \left| \nabla w(x) \right|^{p-2} \nabla w(x) \cdot \nabla \varphi(x) dx - \int_D \frac{\lambda}{\left[w(x)\right]^{\gamma(x)}} \varphi dx = 0, \quad \forall \varphi \in W_0^{1,p(x)}(D), \tag{3.9}$$

and hence $w$ is a weak solution of $-\Delta_{p(x)} w = \lambda / ([w(x)]^{\gamma(x)})$ on $D$.

Obviously, $w$ is a solution of $(P)$, and satisfies $w \geq v_1$. According to comparison principle, it is easy to see that $(P)$ possesses only one positive solution, and $u_\lambda$ is increasing with respect to $\lambda$. □

## 4. Asymptotic behavior of positive solutions

In the following, we will use $C_i$ to denote positive constants.

Theorem 4.1. *If $u$ is a positive weak solution of problem $(P)$, then $C_2 d(x) \le u(x)$ as $x \to \partial\Omega$.*

*Proof.* Similar to the proof of Lemma 3.2, there exists a positive constant $C_2$ such that when $\delta > 0$ is small enough, then $v_2(x) = C_2 d(x)$ is a subsolution of $(P)$ on $\overline{\partial\Omega_\delta}$. Thus $u(x) \ge v_2(x) = C_2 d(x)$ on $\overline{\partial\Omega_\delta}$. The proof is completed. □

Denote $\gamma^* = \max_{x\in\overline{\partial\Omega_{2\sigma}}} \gamma(x)$ and $\gamma_* = \min_{x\in\overline{\partial\Omega_{2\sigma}}} \gamma(x)$.

Theorem 4.2. *If $1 \le \gamma_* < \gamma^*$, for any weak solution $u$ of problem $(P)$, we have*

$$C_3\left[d(x)\right]^{\theta_1} \le u(x) \le C_4\left[d(x)\right]^{\theta_2} \quad \text{as } x \longrightarrow \partial\Omega, \tag{4.1}$$

*where $\theta_1 = \max_{d(x)\le\sigma}(p(x)/(p(x)-1+\gamma(x)))$, $\theta_2 = \min_{d(x)\le\sigma}(p(x)/(p(x)-1+\gamma(x)))$.*

*Proof.* From Theorem 4.1 we only consider $(P)$ in the case of $1 < \gamma_* < \gamma^*$. Denote

$$v_3(x) = \begin{cases} a\left(d(x)\right)^{\theta}, & d(x) < \delta, \\ a\delta^{\theta} + \displaystyle\int_{\delta}^{d(x)} a\theta\delta^{\theta-1}\left(\frac{2\delta - t}{\delta}\right)^{2/(p^- -1)} dt, & \delta \le d(x) < 2\delta, \\ a\delta^{\theta} + \displaystyle\int_{\delta}^{2\delta} a\theta\delta^{\theta-1}\left(\frac{2\delta - t}{\delta}\right)^{2/(p^- -1)} dt, & 2\delta \le d(x), \end{cases} \tag{4.2}$$

where $a$ and $\theta$ are positive constants and satisfy $\theta \in (0,1)$, $0 < \delta$ is small enough.

Obviously, $v_3(x) \in C^1(\Omega) \cap C_0^+(\Omega)$. By computation,

$$-\triangle_{p(x)} v_3(x) = -(a\theta)^{p(x)-1}(\theta-1)\left(p(x)-1\right)\left(d(x)\right)^{(\theta-1)(p(x)-1)-1}\left(1+\Pi(x)\right), \quad d(x) < \delta, \tag{4.3}$$

where

$$\Pi(x) = d\frac{(\nabla p \nabla d)\ln a\theta}{(\theta-1)\left(p(x)-1\right)} + d\frac{(\nabla p \nabla d)\ln d}{\left(p(x)-1\right)} + d\frac{\triangle d}{(\theta-1)\left(p(x)-1\right)}. \tag{4.4}$$

Obviously $|\Pi(x)| \le 1/2$, when $\delta > 0$ is small enough. Let $\theta = \theta_1$ and $a \in (0,1)$ is small enough, when $\delta \in (0,a)$ is small enough, we can conclude that

$$-\triangle_{p(x)} v_3(x) \le \frac{\lambda}{\left[v_2(x)\right]^{\gamma(x)}}, \quad d(x) < \delta. \tag{4.5}$$

By computation, when $\delta < d(x) < 2\delta$, we have

$$\begin{aligned}
-\triangle_{p(x)} v_3 &= -\operatorname{div}\left\{\left[a\theta\delta^{\theta-1}\left(\frac{2\delta - d(x)}{\delta}\right)^{2/(p^- -1)}\right]^{p(x)-1} \nabla d(x)\right\} \\
&= -\left[a\theta\delta^{\theta-1}\left(\frac{2\delta - d(x)}{\delta}\right)^{2/(p^- -1)}\right]^{p(x)-1} \\
&\quad \times [\nabla d(x) \nabla p(x)] \ln a\theta\delta^{\theta-1}\left(\frac{2\delta - d(x)}{\delta}\right)^{2/(p^- -1)} \\
&\quad -\left[a\theta\delta^{\theta-1}\left(\frac{2\delta - d(x)}{\delta}\right)^{2/(p^- -1)}\right]^{p(x)-1} \triangle d(x) \\
&\quad + \frac{2}{\delta}\frac{(p(x)-1)}{p^- - 1}\left(a\theta\delta^{\theta-1}\right)^{p(x)-1}\left[\frac{2\delta - d(x)}{\delta}\right]^{(2(p(x)-1)/(p^- -1))-1}.
\end{aligned} \tag{4.6}$$

Thus, there exists a positive constant $C^*$ such that

$$\left| -\triangle_{p(x)} v_3 \right| \leq C^* \delta^{(\theta-1)(p(x)-1)-1}, \quad \delta < d(x) < 2\delta. \tag{4.7}$$

Obviously,

$$v_3(x) \leq a(\theta+1)\delta^{\theta}, \quad \delta < d(x) < 2\delta. \tag{4.8}$$

Let $\theta = \theta_1$, when $a \in (0,1)$ is small enough, $\delta \in (0,a)$ is small enough, then

$$-\triangle_{p(x)} v_3(x) \leq \frac{\lambda}{[v_2(x)]^{\gamma(x)}}, \quad \delta < d(x) < 2\delta. \tag{4.9}$$

It is easy to see that

$$-\triangle_{p(x)} v_3(x) = 0 \leq \frac{\lambda}{[v_2(x)]^{\gamma(x)}}, \quad 2\delta < d(x). \tag{4.10}$$

Combining (4.5), (4.9), and (4.10), it is easy to see that when $\theta = \theta_1$, $a \in (0,1)$ is small enough and $\delta \in (0,a)$ is small enough, then $v(x)$ is a subsolution of $(P)$, then $u(x) \geq C_3[d(x)]^{\theta_1}$ on $\overline{\partial\Omega_\delta}$.

Similarly, when $\delta > 0$ is small enough, $\theta = \theta_2$, and $a \geq \max_{x\in\overline{\partial\Omega_\delta}}(u(x)/\delta^{\theta})$ is large enough, we can see that $v(x)$ is a supersolution of $(P)$ on $\overline{\partial\Omega_\delta}$, and $u(x) \leq a[d(x)]^{\theta_2}$ on $\overline{\partial\Omega_\delta}$. The proof is completed. □

Theorem 4.3. *If* $\lim_{d(x)\to 0} p(x) = p_0$ *and* $\lim_{d(x)\to 0} p(x)/(p(x)-1+\gamma(x)) = s$, *where* $s \leq 1$ *is a positive constant, u is a solution of (P), then*

$$\lim_{d(x)\to 0} \frac{u(x)}{C(d(x))^s} = 1, \qquad C = \lim_{d(x)\to 0}\left[\frac{\lambda}{\theta^{p(x)-1}(1-\theta)(p(x)-1)}\right]^{1/(p(x)-1+\gamma(x))}. \tag{4.11}$$

*Proof.* It can be obtained easily from Theorem 4.2. □

THEOREM 4.4. *If* $1 \geq \gamma^*$, *for any positive constant* $\theta \in (0,1)$, *u is a weak solution of problem (P), then there exists a positive constant* $C_5$ *such that* $C_1 d(x) \leq u(x) \leq C_5(d(x))^\theta$ *as* $x \to \partial\Omega$.

*Proof.* According to Theorem 4.1, it only needs to prove $u(x) \leq C_5(d(x))^\theta$ as $x \to \partial\Omega$. Define a function on $\overline{\partial\Omega_\delta}$ as $v_4(x) = C_5(d(x))^\theta$, where $C_5 \geq (1/\delta^\theta)\max_{x\in\overline{\partial\Omega_\delta}} u(x)$. Similar to the proof of Theorem 4.2, when $\delta > 0$ is small enough, then $v_4(x)$ is a supersolution of $(P)$ on $\overline{\partial\Omega_\delta}$, then $u(x) \leq v_4(x) = C_5(d(x))^\theta$ on $\overline{\partial\Omega_\delta}$. The proof is completed. □

THEOREM 4.5. *If* $\gamma_* < 1 < \gamma^*$, *u is a weak solution of problem (P), then there exists a positive constant* $C_6$ *such that* $C_1 d(x) \leq u(x) \leq C_6(d(x))^\theta$ *as* $x \to \partial\Omega$, *where* $\theta = \min_{d(x)\leq\delta}(p(x)/(p(x)-1+\gamma(x)))$.

*Proof.* According to Theorem 4.1, it only needs to prove $u(x) \leq C_6(d(x))^\theta$ as $x \to \partial\Omega$. Similar to the proof of Theorem 4.2, when $\delta > 0$ is small enough, then $v_5(x) = C_6(d(x))^\theta$ is a supersolution of $(P)$ on $\overline{\partial\Omega_\delta}$, then $u(x) \leq v_5(x) = C_6(d(x))^\theta$ on $\overline{\partial\Omega_\delta}$. The proof is completed. □

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